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Cauchy Problem and Modified Lacunary Interpolations for Solving Initial Value Problems

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Abstract

The purpose of this paper is to obtain approximate solution of Cauchy problem and solving initial values problems by modified spline functions of degree six which interpolate the lacunary data third and fifth derivatives. Other purpose of this construction is to study the convergence analysis and the upper bounds of errors of the approximate solution is investigated, also we compared with the exact solution to demonstrate the prescribed lacunary spline function interpolation.

Keywords: *Cauchy Problem, Lacunary Interpolation, Convergence Analysis.*

AMS subject classifications: 65D05, 65D07 and 65D32.

1 Introduction

Consider the Cauchy's initial value problem:

$$y''(x) = f(x, y(x), y'(x)) \ , \ x_0 \in [0, 1], \ \ y(x_0) = y_1 \ , y'(x_0) = y'_2 \tag{1}$$

With the help of lacunary spline functions of type (0, 3, 5) see Al Bayati (2009) [1], using that $f \in C^{n-1}([0,1]\times R^2)$, $n \geq 2$ and that it satisfies the Lipschitz continuous

$$\left| f^{(q)}(x, y_1, y_1') - f^{(q)}(x, y_2, y_2') \right| \leq L\left\{ \left| y_1 - y_2 \right| + \left| y_1' - y_2' \right| \right\},\ q=0,1,\ldots,n\text{-}1 \qquad (2)$$

for all $x \in [0,1]$ and all real y_1, y_2, y_1', y_2'. These conditions ensure the existence of unique solution of the problem (1).

The present paper is concerned with the spline approximation method, to solve the problems of type (1). It is known that the most classical methods fail when ε is small relative to the mesh width h that is use partitions the knots. Also to show that this modified lacunary interpolation can furnish accurate numerical approximations of (1) when ε is either small or large as compared to h. The Cauchy problem has been used by many authors for solving these problems, Gyovari solved Cauchy problem by sing modified lacunary spline function which interpolating the lacunary data (0, 2, 3). (Saxena, 1982) used deficient lacunary spline for solving Cauchy problem also. In 1994, and also see (Saeed, 2000), (Jwamer, 2005), and their references.

In this paper we have shown that by making use of the continuity of the third and fifth derivative of the spline function, the resulting for two spline differences scheme gives a new type can be solved efficiently by the well-known algorithm to the Cauchy's problem.

In section 2, we give a brief description of the method. The derivation of the difference schemes spline function has been given in Section 3, and also, we have shown the second-order accuracy method and convergence analysis are studied. We have solved two numerical examples to demonstrate the applicability of the methods in section 4. In the last section, the discussion on the results is given in Section 5.

2 Construct of approximate values $\overline{Y}_k^{(q)}$

Let $x_k = \dfrac{k}{m}; k = 0,1, \ldots, n$, $h = \dfrac{1}{m}$, $\omega(h, y^{(r)}) = \underset{|x-x_1|\le h}{Max}\left\{\left|y^{(r)}(x) - y^{(r)}(x)\right|\right\}$, $r = 0,1, \ldots,6.$

And let $\overline{Y}_k^{(q)}$: $\bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \bar{y}_2^{(q)}, \ldots, \bar{y}_n^{(q)}$; $q = 0,1, \ldots,6,$ be approximate to the exact values $Y_k^{(q)}$: $y_0^{(q)}, y_1^{(q)}, y_2^{(q)}, \ldots, y_n^{(q)}$; $q = 0,1, \ldots,6.$

Now from these approximate values we construct a spline function $\overline{S}_\Delta(x)$ wich interpolates to the set $\overline{Y}$ on the mesh Δ and approximate the solution y(x) of equation (1). The set $\overline{Y}^{(q)}$ is defined as:

$\bar{y}_0 = y_0$, $\bar{y}'_0 = y'_0$, $\bar{y}_0^{(2+q)} = f^{(q)}(x_0, y_0\ y'_0)$ where $q = 0\ 1\ldots,r$.

$$\bar{y}_{k+1} = \bar{y}_k + h\bar{y}'_k + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} f[u, y_k^*(u), y_k^{**}{}'(u)]du\,dt,$$

$$\bar{y}'_{k+1} = \bar{y}'_k + \int_{x_k}^{x_{k+1}} f[t, y_k^*(t), y_k^{**}{}'(t)]dt,$$

$$\bar{y}_{k+1}^{(q+2)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}'_{k+1}),\ q = 0,3,5.,,\ k = 0, 1, 2, \ldots, m-1$$

and for $x_k \le x \le x_{k+1}$

$$y_k^*(x) = \sum_{j=0}^{r+2} (x - x_k)^j \frac{\bar{y}_k^{(j)}}{j!}, \qquad y_k^*{}'(x) = \sum_{j=0}^{r+1} (x - x_k)^j \frac{\bar{y}_k^{(j+1)}}{j!}$$

and $$y'^{**}_{k+1}(x) = \bar{y}'_k + \int_{x_k}^{x_1} f[t, y_k^*(t), y_k^*{}'(t)]dt$$

Using these approximate values $\overline{Y}_k^{(q)}$ $(q = 0,3,5.,\ k = 0, 1, 2, \ldots, m)$ and $\bar{y}'_0$, $\bar{y}'_m$ on the bases of [1], we construct the lacunary spline function $\overline{S}_\Delta(x)$ of the type (0, 3, 5) ($\overline{S}_\Delta(x) = \overline{S}_k(x)$ if $x_k \le x \le x_{k+1}$) and denote by $\overline{S}_{n,6}^5$ the class of six degree splines $\overline{S}(x)$ as the following:

$$G(x)=\begin{cases}\bar{S}_{\Delta}(x_k)=\bar{y}_k\\ \bar{S}_{\Delta}^{(q)}(x_k)=\bar{y}_k^{(q)}\end{cases} \tag{3}$$

Where $q=3,5$ and $k=0,1,2,\ldots,m$, the existence and uniqueness of the above spline function have been shown in [1],

$$\begin{aligned}\bar{S}_0&=\bar{y}_0+(x-x_0)\overline{y'}_0+(x-x_0)^2\bar{a}_{0,2}+\frac{(x-x_0)^3}{6}\bar{y}_o^{(3)}+(x-x_0)^4\bar{a}_{0,4}+\\&\quad\frac{(x-x_0)^5}{120}\bar{y}_0^{(5)}+(x-x_0)^6\bar{a}_{0,6}\end{aligned} \tag{4}$$

Let us examine now intervals $[x_i,x_{i+1}]$, i=1,2,…, n-2., Defined $\bar{S}_i(x)$ as:

$$\begin{aligned}\bar{S}_i(x)&=\bar{y}_i+(x-x_i)\bar{a}_{i,1}+(x-x_i)^2\bar{a}_{i,2}+\frac{(x-x_i)^3}{6}\bar{y}_i^{(3)}+(x-x_i)^4\bar{a}_{i,4}+\\&\quad\frac{(x-x_i)^5}{120}\bar{y}_i^{(5)}+(x-x_i)^6\bar{a}_{i,6}\end{aligned} \tag{5}$$

Here

$$\bar{a}_{0,2}=h^{-2}[\bar{y}_1-\bar{y}_0-h\,\bar{y}'_0]-\frac{h}{24}[\bar{y}_1^{(3)}+3\bar{y}_0^{(3)}]+\frac{h^3}{720}[4\bar{y}_1^{(5)}+5\bar{y}_0^{(5)}],$$

$$\bar{a}_{0,4}=\frac{h^{-1}}{24}[\bar{y}_1^{(3)}-\bar{y}_0^{(3)}]-\frac{h}{144}[\bar{y}_1^{(5)}+2\bar{y}_0^{(5)}]\ \text{and}\ \bar{a}_{0,6}=\frac{h^{-1}}{720}[\bar{y}_1^{(5)}+2\bar{y}_0^{(5)}]$$

Also

$$\bar{a}_{i+1,1}+\bar{a}_{i,1}=2h^{-1}[\bar{y}_{i+1}-\bar{y}_i]+\frac{h^2}{12}[\bar{y}_{i+1}^{(3)}+\bar{y}_i^{(3)}]-\frac{h^3}{120}[\bar{y}_{i+1}^{(5)}+\bar{y}_i^{(5)}],$$

$$\bar{a}_{i,4}=\frac{h^{-1}}{24}[\bar{y}_{i+1}^{(3)}-\bar{y}_i^{(3)}]-\frac{h}{144}[\bar{y}_{i+1}^{(5)}+2\bar{y}_i^{(5)}]\ \text{and}\ \bar{a}_{i,6}=\frac{h^{-1}}{720}[\bar{y}_{i+1}^{(5)}-\bar{y}_i^{(5)}].$$

Similarly for the last interval $[x_n,\ x_{n-1}]$, we can define approximate values of $\bar{S}_n(x)$.

3 Convergence of a spline functions to a solution

In this section, we find the order of convergence for the new approximate spline function $\bar{S}_{\Delta}(x)$ given in the section before to the exact solution of the

Cauchy problem (1) and corresponding to the values of $y_k\,(k = 0,1,2,\ldots,m)$ of a problem (1), and prove the following theorem:

Theorem 1: Let $\bar{y}_k^{(q)}\ (q = 0,3,5; k = 0,1,2\ldots,m)$ be the approximate values defined above. Then the following estimates of spline function $\bar{S}_\Delta(x)$ are valid:

(i) $\left|S_k^{(q)}(x) - \bar{S}_k^{(q)}(x)\right| \le CI_q\, h^{6-q}\, \omega_6(h)$; for $q = 0,1,\ldots,6,\ k = 0,1,\ldots,m-2$

where C_q denote the difference constants independent of h.

(ii) $\left|y_k^{(q)}(x) - \bar{S}_k^{(q)}(x)\right| \le D_q\, h^{6-q}\, \omega_6(h)$; for $q = 0,1,\ldots,6$, where $y(x)$ is a solution of problem (1) and D_q denote the difference constants independent of h.

Proof: (i) From theorem 1 of [1] and equation (3), we have

$$S_0(x) - \bar{S}_0(x) = (x - x_0)^2(a_{0,2} - \bar{a}_{0,2}) + (x - x_0)^4(a_{0,4} - \bar{a}_{0,4}) + (x - x_0)^6(a_{0,6} - \bar{a}_{0,6}) \quad (6)$$

Where

$$a_{0,2} - \bar{a}_{0,2} = \frac{1}{h^2}(y_1 - \bar{y}_1) + \frac{h}{24}[y_1^{(3)} - \bar{y}_1^{(3)}] + \frac{h^3}{180}[y_1^{(5)} - \bar{y}_1^{(5)}] \text{ implies that}$$

$$\left|a_{0,2} - \bar{a}_{0,2}\right| \le \frac{1}{360}(2k_1 + 15k_2 + 360k_3)\ \omega_6(h) = \frac{1}{360} I_1\ \omega_6(h)$$

where $I_1 = 2k_1 + 15k_2 + 360k_3$ and k_1 , k_2 and k_3 are constants dependent of h.

Similarly

$$\left|a_{0,4} - \bar{a}_{0,4}\right| \le \frac{1}{24h}\left|y_1^{(3)} - \bar{y}_1^{(3)}\right| + \frac{h}{144}\left|y_1^{(5)} - \bar{y}_1^{(5)}\right|$$

$$\le \frac{1}{144}(k_4 + 6k_5)\omega_6(h) = \frac{1}{144} I_2 \omega_6(h)$$

where $I_2 = k_4 + 6k_5$ and k_4 , k_5 are constants dependent of h.

and

$$\left|a_{0,6} - \bar{a}_{0,6}\right| \le \frac{1}{720\,h}\left|y_1^{(5)} - \bar{y}_1^{(5)}\right| = \frac{1}{720} I_3 \omega_6(h)$$

And hence

$$\left|S_0(x)-\bar{S}_0(x)\right| \le h^2\left|a_{0,2}-\bar{a}_{0,2}\right| + h^4\left|a_{0,4}-\bar{a}_{0,4}\right| + h^6\left|a_{0,6}-\bar{a}_{0,6}\right|$$
$$\le I\ \omega_6(h)$$

Where $I = I_1 + I_2 + I_3$, independent of h.

By taking the first derivative of equation (5), we have

$$\left|S_0'(x)-\bar{S}_0'(x)\right| \le \frac{2}{h}\left|y_1-\bar{y}_1\right| + \frac{h^2}{4}\left|y_1'''-\bar{y}_1'''\right| + \frac{17}{360}\left|y_1^{(5)}-\bar{y}_1^{(5)}\right|$$
$$\le \frac{1}{360}(17\bar{k}_1 + 90\bar{k}_2 + \bar{k}_3 180)\,\omega_6(h) = \frac{1}{360}\bar{I}_1\omega_6(h)$$

and by successive differentiations obtain

$\left|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)\right| \le I_q\, h^{6-q}\,\omega_6(h)$; for $q = 0,1,\ldots,6$. Which proves (i) for $k=0$ and $x \in [x_0, x_1]$. Further more in the interval $[x_{k-1}, x_k]$

$$S_k(x)-\bar{S}_k(x) = (x-x_k)^2(a_{k,2}-\bar{a}_{k,2}) + (x-x_k)^4(a_{k,4}-\bar{a}_{k,4}) + (x-x_k)^6(a_{k,6}-\bar{a}_{k,6})$$

$$a_{k,2}-\bar{a}_{k,2} = \frac{1}{h}(a_{k,1}-\bar{a}_{k,1})\frac{1}{h^2}(y_{k+1}-\bar{y}_{k+1}) + \frac{h}{24}[y_{k+1}^{(3)}-\bar{y}_{k+1}^{(3)}] + \frac{h^3}{180}[y_{k+1}^{(5)}-\bar{y}_{k+1}^{(5)}]$$

implies that

$$\left|a_{k,2}-\bar{a}_{k,2}\right| \le \frac{1}{360}(2C_1 + 15C_2 + 360C_3 + 360C_4)\ \omega_6(h) = \frac{1}{360}I_1^*\ \omega_6(h)$$

where I_1^* and C_1, C_2, C_3 and C_4 be a constants dependent of h.

Similarly

$\left|a_{k,4}-\bar{a}_{k,4}\right| \le I_2^*\ \omega_6(h)$ and $\left|a_{k,6}-\bar{a}_{k,6}\right| \le I_3^*\ \omega_6(h)$ also I_2^* and I_3^* are dependent of h.

and by taking the successive differentiation, we obtain

$\left|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)\right| \le I_q\, h^{6-q}\,\omega_6(h)$; for $q = 0,1,\ldots,6$. Which proves (i) for $k = 0, 1, \ldots, m-2$.

We can repeat the same steps for $k = m-1$.

Proof of theorem 1 (ii):

$$\left|y^{(q)}(x)-\bar{S}_{\Delta}^{(q)}(x)\right| \le \left|y^{(q)}(x)-S_{\Delta}^{(q)}(x)\right| + \left|S_{\Delta}^{(q)}(x)-\bar{S}_{\Delta}^{(q)}(x)\right|$$

From theorem 2 of [1], the following estimates are valid

$$\left|y^{(q)}(x)-\bar{S}_{\Delta}^{(q)}(x)\right| \le C_q h^{6-q}\omega_6(h)$$

(7)

Using equation (7) and estimate in (i), we have

$$\begin{aligned}\left|y^{(q)}(x)-\bar{S}_{\Delta}^{(q)}(x)\right| &\le C_q h^{6-q}\omega_6(h) + I_q h^{6-q}\omega_6(h)\\ &= (C_q + I_q)\, h^{6-q}\omega_6(h)\\ &= D_q h^{6-q}\omega_6(h)\end{aligned}$$

Which is proves (ii).

Theorem 2: If the function f in Cauchy's problem (1) satisfies conditions (2) and (3), then the following inequalities are hold:

$\left|\bar{S}_0''(x) - f[x, \bar{S}_0(x), \bar{S}_0'(x)]\right| \le I_{0,2}^{*}\,\omega_6(h)$ where $I_{0,2}^{*}$ is constants dependent of h and $x \in [x_0, x_1]$,

$\left|\bar{S}_k''(x) - f[x, \bar{S}_k(x), \bar{S}_k'(x)]\right| \le I_{k,2}^{*}\,\omega_6(h)$ where $I_{k,2}^{*}$ is constants dependent of h and $x \in [x_{k-1}, x_k]$,

$\left|\bar{S}_{m-1}''(x) - f[x, \bar{S}_{m-1}(x), \bar{S}_{m-1}'(x)]\right| \le I_{m-1,2}^{*}\,\omega_6(h)$ where $I_{m-1,2}^{*}$ is constants dependent of h and $x \in [x_{m-1}, x_m]$.

Proof: Using condition (1), (2) and (3), we have

$$\begin{aligned}\left|\bar{S}_{\Delta}''(x) - f[x, \bar{S}_{\Delta}(x), \bar{S}_{\Delta}'(x)]\right| &\le \left|\bar{S}_{\Delta}''(x) + y''(x) - y''(x) - f[x, \bar{S}_{\Delta}(x), \bar{S}_{\Delta}'(x)]\right|\\ &\le \left|\bar{S}_{\Delta}''(x) - y''(x)\right| + \left|y''(x) - f[x, \bar{S}_{\Delta}(x), \bar{S}_{\Delta}'(x)]\right|\\ &\le \left\|\bar{S}_{\Delta}''(x) - y''(x)\right\| + L\left\{\left|\bar{S}_{\Delta}(x) - y(x)\right| + \left|\bar{S}_{\Delta}'(x) - y'(x)\right|\right\}\end{aligned}$$

That proves theorem 2 with the help theorem 1.

Note: Similar manner to theorem 2 was proved under different conditions by Saxena (1987) [5], and Janos Gyorvari (1984) [2, 3].

4 Numerical results

In this section, the methods discussed in section 2 and 3 were tested on two problems, and the absolute errors in the analytical solution were calculated. Our results confirm the theoretical analysis of the methods with the Cauchy problem. For the sake of comparisons we also tabulated the results seen that the present method is better than method Albayati (2009),

Problem (1): we consider that the second order initial value problem $y'' + 4y = \cos(x)$ where $x \in [0,1]$ and $y(0) = 1,\ y'(0) = 0$ with the exact solution $y(x) = \frac{1}{3}(2 * \cos(2x) + \cos(x))$, see [7].

Problem (2): Let $y'' - y = x$ where $y(0) = y'(0) = 0$, see [1].

It turns out that the six degree spline which presented in this paper, yield approximate solution that is O(h^6) as stated in Theorem 1. The results are shown in the Table 1 and Table 2 for different step sizes h.

Table 1 Absolute maximum error for the derivatives $\bar{S}(x)$.

H	$\lVert \bar{s}'(x) - y'(x) \rVert_\infty$	$\lVert \bar{s}''(x) - y''(x) \rVert_\infty$	$\lVert \bar{s}^{(4)}(x) - y^{(4)}(x) \rVert_\infty$	$\lVert \bar{s}^{(6)}(x) - y^{(6)}(x) \rVert_\infty$
0.1	18×10^{-4}	17.5×10^{-3}	14.4×10^{-2}	85.22×10^{-2}
0.01	18.33×10^{-7}	18.325×10^{-5}	14×10^{-4}	85×10^{-4}
0.001	18.333×10^{-10}	18.333×10^{-7}	14.333×10^{-6}	85.5×10^{-6}

Table 2 Absolute maximum error for the derivatives $\bar{S}(x)$.

h	$\lVert \bar{s}'(x) - y'(x) \rVert_\infty$	$\lVert \bar{s}''(x) - y''(x) \rVert_\infty$	$\lVert \bar{s}^{(4)}(x) - y^{(4)}(x) \rVert_\infty$	$\lVert \bar{s}^{(6)}(x) - y^{(6)}(x) \rVert_\infty$
0.1	17×10^{-4}	66.6×10^{-3}	66.8×10^{-3}	10.02×10^{-2}
0.01	16.66×10^{-6}	67×10^{-4}	67×10^{-4}	10×10^{-3}
0.001	16.667×10^{-8}	66.66×10^{-5}	66.667×10^{-5}	10×10^{-4}

5 Conclusion

In this paper**,** we have developed a new numerical method for solving Cauchy problem based on lacunary interpolation polynomial of six degree splines. The results of the two tables above obtained are very encouraging, and the numerical results shows that the accuracy improves with increasing the h, than for better results, using the larger h is recommended.

6 Open Problem

We can develop the idea for boundary value problems of the higher order and higher order initial values problems with difference type of lacunary interpolations.

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