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Numerical Solution of Second Kind Linear Fredholm Integral Equations via Quarter-Sweep Arithmetic Mean Iterative Method

Mohana Sundaram Muthuvalu and Jumat Sulaiman

School of Science and Technology, Universiti Malaysia Sabah,
Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia
e-mail: sundaram_at2@yahoo.com
School of Science and Technology, Universiti Malaysia Sabah,
Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia
e-mail: jumat@ums.edu.my

Abstract

In previous studies, the concept of quarter-sweep iteration has been pointed out to accelerate the convergence rate in solving any system of linear equations generated by using approximation equations. Based on the same concept, the essential aim of this paper is to investigate the effectiveness of the Quarter-Sweep Arithmetic Mean (QSAM) method by using the quarter-sweep approximation equation based on repeated Simpson scheme in solving second kind linear integral equations of Fredholm type. Furthermore, the formulation and implementation of the proposed method are also presented. Some numerical tests were also conducted to verify the efficiency of the proposed method.

Keywords: *Linear Fredholm equations, Quarter-sweep iteration, Repeated Simpson, Arithmetic Mean*

1 Introduction

The theory and application of the integral equations have been one of the principal tools in various areas of science such as applied mathematics, physics, biology, chemistry and engineering. On the other hand, integral equations are encountered in numerous applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radiative equilibrium, particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, and radiative heat transfer problems [21]. Basically, integral equations can be classified according to the integration domain. Integral equations in which the integration domain varies with the independent variable in the equation are known as Volterra equations and those with fixed integration domain are Fredholm equations. In this paper, second kind linear integral equations type of Fredholm is considered.

Generally, second kind linear Fredholm integral equations can be written as follows

$$\lambda y(x) - \int_\Gamma K(x,t)y(t)dt = f(x), \ \Gamma = [a,b] \ \lambda \neq 0 \tag{1}$$

where the parameter λ, kernel $K \in L^2(\Gamma \times \Gamma)$ and free term $f \in L(\Gamma)$ are given, and $y \in L(\Gamma)$ is the unknown function to be determined. The kernel function $K(x,t)$ is assumed to be absolutely integrable and satisfy other properties that are sufficient to imply the Fredholm alternative theorem. Meanwhile, Eq. (1) also can be rewrite in the equivalent operator form

$$(\lambda - \kappa)y = f \ . \tag{2}$$

where the integral operator define as follows

$$\kappa \, y(t) = \int_\Gamma K(x,t)y(t)dt \, . \tag{3}$$

Theorem (Fredholm Alternative) [3]
Let χ be a Banach space and let $\kappa : \chi \to \chi$ be compact. Then the equation $(\lambda - \kappa)y = f$, $\lambda \neq 0$ has a unique solution $x \in \chi$ if and only if the homogeneous equation $(\lambda - \kappa)z = 0$ has only the trivial solution $z = 0$. In such a case, the operator $\lambda - \kappa : \chi \underset{onto}{\overset{1-1}{\to}} \chi$ has a bounded inverse $(\lambda - \kappa)^{-1}$.

Definition (Compact operators) [3]
Let χ and Y be normed vector space and let $\kappa : \chi \to \mathrm{Y}$ be linear. Then κ is compact if the set $\{\kappa\, x \mid \|x\| \ x \leq 1\}$ has compact closure in Y . This is equivalent to saying that for every bounded sequence $\{x_n\} \subset \chi$, the sequences $\{\kappa x_n\}$ has a subsequence that is convergent to some points in Y . Compact operators are also called completely continuous operators.

A numerical approach to the solution of integral equations is an essential branch of scientific inquiry. As a matter of fact, some valid methods of solving linear Fredholm integral equations have been developed in recent years. To solve Eq. (1) numerically, we either seek to determine an approximate solution by using the quadrature method [9-11, 16], or use the projection method [4, 5, 7, 8]. Such discretizations of integral equations lead to dense linear system and can be prohibitively expensive to solve using direct methods as the order of the linear system increases. Thus, iterative methods are the natural options for efficient solutions.

Consequently, the concept of the two-stage iterative method has been proposed widely to be one of the efficient methods for solving any linear system. The two-stage iterative method, which is also called as inner/outer iterative scheme was first introduced by Nichols [12]. Actually, there are many two-stage iterative methods can be considered such as Alternating Group Explicit (AGE) [6], Iterative Alternating Decomposition Explicit (IADE) [17], Reduced Iterative Alternating Decomposition Explicit (RIADE) [18], Block Jacobi [2] and Arithmetic Mean (AM) [15] methods. The standard AM method also named as the Full-Sweep Arithmetic Mean (FSAM) method has been modified by combining the half-sweep iteration concept and then called as the Half-Sweep Arithmetic Mean (HSAM) method [19]. The concept of the half-sweep iteration has been introduced by Abdullah [1] via the Explicit Decoupled Group (EDG) method to solve two-dimensional Poisson equations. In [20], another variant of AM method, known as Quarter-Sweep Arithmetic Mean (QSAM) iterative method has been proposed. The QSAM method is derived by combining the standard AM method with quarter-sweep iteration concept [13]. The basic idea of the half- and quarter-sweep iterative methods is to reduce the computational complexity during iteration process. Since the implementation of the half- and quarter-sweep iterations will only consider nearly half and quarter of all interior node points in a solution domain respectively. In this paper, the performance of the FSAM, HSAM and QSAM methods will be investigated in solving dense linear system generated from the discretization of the second kind linear Fredholm integral equations using quadrature method.

The outline of this paper is organized in following way. In Section 2, the formulation of the full-, half- and quarter-sweep quadrature approximation equations will be explained. The latter section of this paper will discuss the formulations of the FSAM, HSAM and QSAM methods, and some numerical results will be shown in fourth section to assert the performance of the proposed methods. Besides that, analysis on computational complexity is mentioned in Section 5. Meanwhile, conclusion and open problem are given in Section 6.

2 Quadrature Approximation Equations

As afore-mentioned, a discretization scheme based on method of quadrature was used to construct approximation equations for problem (1) by replacing the integral to finite sums. In order to facilitate the formulation of the full-, half- and quarter-sweep quadrature approximation equations for problem (1), further discussion will be restricted onto repeated Simpson's $\frac{1}{3}$ (RS1) scheme, which is based on quadratic polynomial interpolation formula with equally spaced data.

The Simpson's $\frac{1}{3}$ scheme for approximating definite integral $\int_a^b y(t)\,dt$ can be defined as follows

$$\int_a^b y(t)dt = \frac{h}{3}\left(y(a) + 4y\left(\bar{t}\right) + y(b)\right) + \varepsilon_n(y) \tag{4}$$

where

$$h = \frac{b-a}{2}, \tag{5}$$

and

$$\bar{t} = \frac{(a+b)}{2} \tag{6}$$

and $\varepsilon_n(y)$ is the truncation error. Meanwhile, RS1 scheme is a repeated application of Eq. (4) for a domain divided into an even number of intervals. Denoting the total number of intervals by n (even), the RS1 scheme is written as

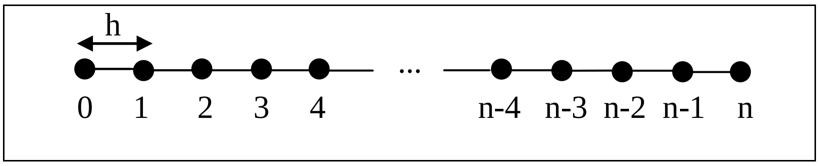
$$\int_a^b y(t)dt = \frac{h}{3}\left(y(a) + 4\sum_{\substack{i=1 \\ odd\ i}}^{n-1} y(a+ih) + 2\sum_{\substack{i=2 \\ even\ i}}^{n-2} y(a+ih) + y(b)\right) + \varepsilon_n(y) \tag{7}$$

where

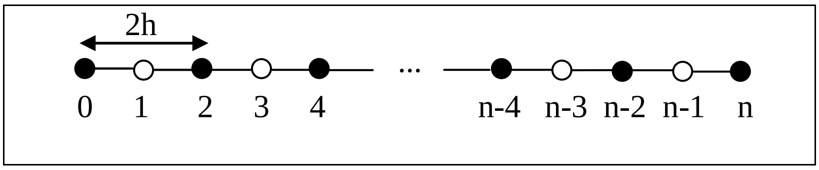
$$h = \frac{b-a}{n}. \tag{8}$$

The first summation is over odd i only and the second summation over even i only.

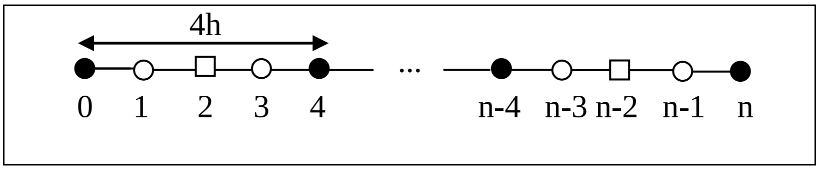
Fig. 1 shows the finite grid networks in order to form the full-, half- and quarter-sweep quadrature approximation equations.



a)



b)



c)

Fig. 1: a), b) and c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively.

Based on Fig. 1, the full-, half- and quarter-sweep iterative methods will compute approximate values onto node points of type ● only until the convergence criterion is reached. Meanwhile, the approximation solutions for the remaining points can be calculated by using direct method [1, 13].

However, in [11], Muthuvalu and Sulaiman carried out a study to investigate the applications of the half-sweep iteration in solving dense linear system generated from the discretization of the second kind Fredholm integral equations using high-order Newton-Cotes schemes. From the results obtained, it has shown that applications of the half-sweep iteration with high-order Newton-Cotes discretization schemes reduce the accuracy of the numerical solutions and it is due to the computational technique for calculating the remaining points by using direct method. Thus, in this paper, we will use second-order Lagrange interpolation method to compute the remaining points for both half- and quarter-sweep iterations in order to overcome the problem mentioned by Muthuvalu and Sulaiman in [11]. Formulations to compute the remaining points using second order Lagrange interpolation method for half- and quarter-sweep iterations are defined in Eqs. (9) and (10) respectively as follows

$$y_i = \begin{cases} \frac{3}{8}y_{i-1} + \frac{3}{4}y_{i+1} - \frac{1}{8}y_{i+3}\ , & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4}y_{i-1} + \frac{3}{8}y_{i+1} - \frac{1}{8}y_{i-3}\ , & i = n-1 \end{cases} \tag{9}$$

$$y_i = \begin{cases} \frac{3}{8}y_{i-2} + \frac{3}{4}y_{i+2} - \frac{1}{8}y_{i+6}\ , & i = 2,6,10,\cdots,n-6 \\ \frac{3}{4}y_{i-2} + \frac{3}{8}y_{i+2} - \frac{1}{8}y_{i-6}\ , & i = n-2 \\ \frac{3}{8}y_{i-1} + \frac{3}{4}y_{i+1} - \frac{1}{8}y_{i+3}\ , & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4}y_{i-1} + \frac{3}{8}y_{i+1} - \frac{1}{8}y_{i-3}\ , & i = n-1 \end{cases} . \tag{10}$$

By applying Eq. (7) into Eq. (1) and neglecting the error, $\varepsilon_n(y)$, a linear system can be formed for approximation values of $y(t)$. Therefore, the full-, half- and quarter-sweep repeated Simpson's $\frac{1}{3}$ approximation equations for Eq. (1) can be generally shown as follows

$$\lambda y_i - A_j\left(K_{i,0}y_0 + \sum_{j=p}^{n-p} K_{i,j}y_j + K_{i,n}y_n\right) = f_i \qquad i = 0,1p,2p,\cdots,n \quad j = 0,1p,2p,\cdots,n \tag{11}$$

where numerical coefficient A_j satisfied following relations

$$A_j = \begin{cases} \frac{1}{3}ph, & j = 0,n \\ \frac{4}{3}ph, & j = 1p,3p,5p,\cdots,n-p \\ \frac{2}{3}ph, & otherwise \end{cases}. \tag{12}$$

The value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep cases respectively. From Eq. (11), it is obvious that discretization of the Eq. (1) using RS1 scheme leads to the dense linear systems as follows

$$M\underset{\sim}{y} = \underset{\sim}{f} \tag{13}$$

where

$$M = \begin{bmatrix} \lambda - A_0K_{0,0} & -A_pK_{0,p} & -A_{2p}K_{0,2p} & \cdots & -A_nK_{0,n} \\ -A_0K_{p,0} & \lambda - A_pK_{p,p} & -A_{2p}K_{p,2p} & \cdots & -A_nK_{p,n} \\ -A_0K_{2p,0} & -A_pK_{2p,p} & \lambda - A_{2p}K_{2p,2p} & \cdots & -A_nK_{2p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_0K_{n,0} & -A_pK_{n,p} & -A_{2p}K_{n,2p} & \cdots & \lambda - A_nK_{n,n} \end{bmatrix}_{\left(\left(\frac{n}{p}\right)+1\right)\times\left(\left(\frac{n}{p}\right)+1\right)},$$

$$\underset{\sim}{y} = \begin{bmatrix} y_0 & y_p & y_{2p} & \cdots & y_{n-2p} & y_{n-p} & y_n \end{bmatrix}^T$$

and

$$\underset{\sim}{f} = \begin{bmatrix} f_0 & f_p & f_{2p} & \cdots & f_{n-2p} & f_{n-p} & f_n \end{bmatrix}^T.$$

3 Arithmetic Mean Iterative Methods

As stated in previous section, AM methods are one of the two-stage iterative methods and the iterative process involves of solving two independent systems such as $\underset{\sim}{y}^1$ and $\underset{\sim}{y}^2$. To develop the formulation of AM methods, express the coefficient matrix M as the matrix sum

$$M = L + D + U \tag{14}$$

where L, D and U are the strictly lower triangular, diagonal and strictly upper triangular matrices respectively. Thus, by adding positive acceleration parameter, ω the general scheme for FSAM, HSAM and QSAM methods is defined by

$$\left.\begin{aligned} (D+\omega L)\,\underset{\sim}{y}^{1} &= \left((1-\omega)D-\omega U\right)\underset{\sim}{y}^{(k)}+\omega f \\ (D+\omega U)\,\underset{\sim}{y}^{2} &= \left((1-\omega)D-\omega L\right)\underset{\sim}{y}^{(k)}+\omega f \\ \underset{\sim}{y}^{(k+1)} &= \frac{1}{2}\left(\underset{\sim}{y}^{1}+\underset{\sim}{y}^{2}\right) \end{aligned}\right\} \tag{15}$$

where $\underset{\sim}{y}^{(0)}$ is an initial vector approximation to the solution and $0<\omega<2$.

The AM methods require a slight additional computational effort of the sum of two matrices at each iteration k, but its rate of convergence is relatively insensitive to the exact choice of the parameter ω [15]. Practically, the value of ω will be determined by implementing some computer programs and then choose one value of ω, where its number of iterations is the smallest. By determining values of matrices L, D and U as stated in Eq. (14), the general algorithm for FSAM, HSAM and QSAM iterative methods to solve problem (1) would be generally described in Algorithm 1. The FSAM, HSAM and QSAM algorithms are explicitly performed by using all equations at level (1) and (2) alternatively until the specified convergence criterion is satisfied. Generally, the basic idea for the convergence analysis of the AM methods has been proven in [15].

4 Numerical Tests

In order to compare the performances of the iterative methods described in the previous section, several experiments were carried out on the following Fredholm integral equations problems.

Example 1 [21]

$$y(x)-\int_0^1\left(4xt-x^2\right)y(t)dt=x \quad 0\le x\le 1 \tag{16}$$

and the exact solution is given by

$$y(x)=24x-9x^2.$$

Example 2 [14]

$$y(x)-\int_0^1(x^2+t^2)y(t)dt=x^6-5x^3+x+10 \quad 0\le x\le 1 \tag{17}$$

with the exact solution

$$y(x)=x^6-5x^3+\frac{1045}{28}x^2+x+\frac{2141}{84}.$$

Algorithm 1 FSAM, HSAM and QSAM algorithms

i) Level (1)
For $i = 0, p, 2p, \cdots, n-2p, n-p, n$ and $j = 0, p, 2p, \cdots, n-2p, n-p, n$
Calculate

$$y_i^1 \leftarrow \begin{cases} (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=p}^{n} A_j K_{i,j} y_j^{(k)} + \omega f_i \right) \Big/ (\lambda - A_i K_{i,i}) & ,i = 0 \\ (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=0}^{n-p} A_j K_{i,j} y_j^1 + \omega f_i \right) \Big/ (\lambda - A_i K_{i,i}) & ,i = n \\ (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=0}^{i-p} A_j K_{i,j} y_j^1 + \omega \sum_{j=i+p}^{n} A_j K_{i,j} y_j^{(k)} + \omega f_i \right) \Big/ (\lambda - A_i K_{i,i}) & ,others \end{cases}$$

ii) Level (2)
For $i = n, n-p, n-2p \cdots, 2p, p, 0$ and $j = 0, p, 2p, \cdots, n-2p, n-p, n$
Calculate

$$y_i^2 \leftarrow \begin{cases} (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=p}^{n} A_j K_{i,j} y_j^2 + \omega f_i \right) \Big/ (\lambda - A_i K_{i,i}) & ,i = 0 \\ (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=0}^{n-p} A_j K_{i,j} y_j^{(k)} + \omega f_i \right) \Big/ (\lambda - A_i K_{i,i}) & ,i = n \\ (1-\omega)y_i^{(k)} + \left(\omega \sum_{j=0}^{i-p} A_j K_{i,j} y_j^{(k)} + \omega \sum_{j=i+p}^{n} A_j K_{i,j} y_j^2 + \omega f_i \right) \Big/ (\lambda - A_i K_{ii}) & ,others \end{cases}$$

iii) For $i = 0, p, 2p, \cdots, n-2p, n-p, n$
Calculate

$$y_i^{(k+1)} \leftarrow \frac{1}{2}\left(y_i^1 + y_i^2\right)$$

In comparison, the Gauss-Seidel (GS) method acts as the comparison control of numerical results. There are three parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. Throughout the simulations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$ and carried out on several different values of n. All the simulations were implemented by a computer with processor Intel(R) Core(TM) 2 CPU 1.66GHz and computer codes were written in C programming language.

Results of numerical simulations, which were obtained from implementations of the GS, FSAM, HSAM and QSAM iterative methods for Examples 1 and 2, have been recorded in Tables 1 and 2 respectively. Meanwhile, reduction percentages of the number of iterations and execution time for the FSAM, HSAM and QSAM methods compared with GS method have been summarized in Table 3.

Table 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 1)

Methods	**Number of iterations**				
	n				
	480	**960**	**1920**	**3840**	**7680**
GS	194	194	195	195	195
FSAM	84	84	84	84	84
HSAM	84	84	84	84	84
QSAM	83	84	84	84	84
	Execution time (seconds)				
Methods	***n***				
	480	**960**	**1920**	**3840**	**7680**
GS	3.11	10.08	42.32	134.77	566.50
FSAM	1.95	7.76	26.96	89.43	445.76
HSAM	1.02	2.04	8.38	30.11	130.87
QSAM	0.56	1.09	3.22	16.85	68.44
	Maximum absolute error				
Methods	***n***				
	480	**960**	**1920**	**3840**	**7680**
GS	7.156 E-10	7.552 E-10	6.868 E-10	6.961 E-10	7.008 E-10
FSAM	1.480 E-10	1.502 E-10	1.513 E-10	1.519 E-10	1.521 E-10
HSAM	1.465 E-10	1.496 E-10	1.510 E-10	1.518 E-10	1.521 E-10
QSAM	2.366 E-10	1.467 E-10	1.496 E-10	1.510 E-10	1.518 E-10

Table 3: Reduction percentages of the number of iterations and execution time for the FSAM, HSAM and QSAM methods compared with GS method

Methods	**Number of iterations**	
	Example 1	**Example 2**
FSAM	56.70 – 56.93%	42.85 – 42.86%
HSAM	56.70 – 56.93%	42.85 – 42.86%
QSAM	56.70 – 57.22%	42.85 – 42.86%
Methods	**Execution time**	
	Example 1	**Example 2**
FSAM	21.31 – 37.30%	22.58 – 49.03%
HSAM	67.20 – 80.20%	60.48 – 87.34%
QSAM	81.99 – 92.40%	83.06 – 94.04%

Table 2: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 2)

	Number of iterations				
Methods	***n***				
	480	**960**	**1920**	**3840**	**7680**
GS	56	56	56	56	56
FSAM	32	32	32	32	32
HSAM	32	32	32	32	32
QSAM	32	32	32	32	32
	Execution time (seconds)				
Methods	***n***				
	480	**960**	**1920**	**3840**	**7680**
GS	1.24	3.53	18.40	60.34	220.45
FSAM	0.96	2.11	9.38	35.85	126.69
HSAM	0.49	1.04	2.33	10.86	40.34
QSAM	0.21	0.54	1.49	3.60	16.94
	Maximum absolute error				
Methods	***n***				
	480	**960**	**1920**	**3840**	**7680**
GS	5.8823 E-10	8.3052 E-10	1.2601 E-10	1.3006 E-10	1.3088 E-10
FSAM	6.1321 E-10	6.1889 E-10	1.0376 E-10	1.0660 E-10	1.0661 E-10
HSAM	4.1221 E-10	5.1096 E-10	6.3888 E-10	8.9604 E-10	2.0551 E-10
QSAM	1.7833 E-9	3.5589E-10	8.2415 E-10	9.6604 E-10	4.0062 E-10

5 Computational Complexity Analysis

In order to measure the computational complexity of the FSAM, HSAM and QSAM methods, an estimation amount of the computational work required for iterative methods has been conducted. The computational work is estimated by considering the arithmetic operations performed per iteration. Based on Algorithm 1, it can be observed that there are $\frac{2n}{p}+7$ additions/subtractions (ADD/SUB) and $\frac{4n}{p}+9$ multiplications/divisions (MUL/DIV) operations in computing a value for each node point in the solution domain for second kind linear Fredholm integral equations. From the order of the coefficient matrix, M, the total number of arithmetic operations per iteration for the FSAM, HSAM and QSAM iterative methods for solving Eq. (1) has been summarized in Table 4.

Table 4: Total number of arithmetic operations per iteration for FSAM, HSAM and QSAM methods

Methods	**Arithmetic Operation**	
	ADD/SUB	**MUL/DIV**
FSAM	$2n^2+9n+7$	$4n^2+13n+9$
HSAM	$\frac{n^2}{2}+\frac{9n}{2}+7$	$n^2+\frac{13n}{2}+9$
QSAM	$\frac{n^2}{8}+\frac{9n}{4}+7$	$\frac{n^2}{4}+\frac{13n}{4}+9$

6 Conclusion and Open Problem

In this paper, we present an application of the QSAM iterative method for solving dense linear systems arising from the discretization of the second kind linear Fredholm integral equations by using repeated Simpson's $\frac{1}{3}$ scheme. Through numerical results obtained for Examples 1 and 2 (Tables 1 and 2), it clearly shows that by applying the AM methods can reduce number of iterations and execution time compared to the GS method. At the same time, it has been shown that, applying the half- and quarter-sweep iterations reduces the computational time in the implementation of the iterative method, see Table 3. Overall, the numerical results show that the QSAM method is a better method compared to the GS, FSAM and HSAM methods in the sense of number of iterations and execution time. For the future works, applications of the quarter-sweep iteration concept with other iterative method to solve linear system can be investigate.

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