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On Weakly Von Neumann Regular Rings

Mohammed Kabbour and Najib Mahdou

Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202,
University S.M. Ben Abdellah Fez, Morocco.
e-mail: mkabbour@gmail.com
Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202,
University S.M. Ben Abdellah Fez, Morocco.
e-mail: mahdou@hotmail.com

Abstract

In this paper, we define and study a particular case of von Neumann regular notion called a weak von Neumann regular ring. It shown that the polynomial ring $R[x]$ is weak von Neumann regular if and only if R has exactly two idempotent elements. We provide necessary and sufficient conditions for $R = A \propto E$ to be a weak von Neumann ring. It is also shown that I is a primary ideal imply R/I is a weak von Neumann regular ring.

Keywords: *coherent ring, von Neumann regular ring, trivial extension.*

1 Introduction

All rings considered in this paper are assumed to be commutative, and have identity element $\neq 0$; all modules are unital. A ring R is reduced if its nilradical is zero. The following statements on a ring R are equivalent:

1. Every finitely generated ideal of R is principal and is generated by an idempotent.
2. For each x in R, there is some y in R such that $x = x^2y$.
3. R is reduced 0-dimensional ring.

A ring satisfying the equivalent conditions as above is said to be von Neumann regular. See for instance [1, 3, 4, 7].

In this article we study a new concept, close to the notion of von Neumann regular ring. More exactly we modify (every finitely generated ideal $I \subseteq R$) by (every finitely generated ideal $I \subseteq J = Re \subseteq (\neq)R$, where e is an idempotent element of R) in the assertion (1). In the second section we give some results that allow us to study this notion, and it is containing some applications of such a notion.

A ring R is called a coherent ring if every finitely generated ideal of R is finitely presented. We say that R is coherent ring if and only if $(0 : a)$ is finitely generated ideal for every element a of R and the intersection of two finitely generated ideals of R is a finitely generated ideal of R. Hence every von Neuman regular ring is a coherent ring [[3], p.47].

A ring is called a discrete valuation ring if it is a principal ideal domain with only one maximal ideal. A ring R is a Dedekind ring if it is a Noetherian integral domain such that the localization R_p is a discrete valuation ring for every nonzero prime ideal p of R. Recall that a ring R is a Dedekind domain if and only if R is Noetherian, integrally closed domain and each nonzero prime ideal of R is a maximal ideal [[5], Theorem 3.16 p.13].

Let R be a Dedekind ring and let I be a nonzero ideal of R, we say that there exists some prime ideals $p_1, ..., p_n$ (uniquely determined by I) and certain positive integers $k_1, ..., k_n$ (uniquely determined by I) such that $I = p_1^{k_1}...p_n^{k_n}$ [[5], p.12].

If R is a ring and E is an $R-$module, the idealization (also called trivial ring extension of R by E) $R \propto E$, introduced by Nagata in 1956 is the set of pair (r, e) with pairwise addition and multiplication given by $(r, e)(s, f) = (rs, rf + se)$. The trivial ring extension of R by E, $R \propto E$ has the following property that containing R as sub-ring, where the module E can be viewed as an ideal such that its square is zero.

2 Main Results

Definition 2.1 *A ring R is called a weak von Neumann regular ring (WVNR for short) if for every finitely generated ideals I and J of R satisfying $I \subseteq J \subseteq (\neq)R$, when J is generated by an idempotent element of R, then so is I.*

In particular, any von Neumann regular ring is a weak von Neumann regular ring. Now, we give a class of a weak von Neumann regular ring.

Example 2.2 *If R is a ring in which the only idempotent elements are 0 and 1, then R is a WVNR ring. In particular if R be an integral domain or a local ring, then R is a WVNR ring.*

Now we give an example of a non-coherent WVNR ring.

Example 2.3 *Let X be a connected topological space and $T = C(X, R)$ the ring of numerical continuous functions defined in X (where R is the field of reals numbers). Let f be an idempotent element of T, then for each $x \in X$ $f(x) = 0$ or $f(x) = 1$. Hence*

$$\forall x \in X \ \ f(x) = 0 \ \ or \ \ \forall x \in X \ \ f(x) = 1$$

since X is connected. Therefore T has exactly two idempotents and so T is a WVNR ring.

Now we suppose that $X = [0, 2]$. Let $f_0 \in T$ such that $f_0(x) = 0$ if $0 \leq x \leq 1$ and $f(x) \neq 0$ if $1 < x \leq 2$. Assume that $(0 : f_0)$ is a finitely generated ideal of $T, (0 : f_0) = (f_1, ..., f_n)$. It is easy to see that

$$(0 : f_0) = \{\varphi \in T \ ; \ \forall x \in [1, 2] \ \varphi(x) = 0\}.$$

Let $f \in T$ defined by $f(x) = \sqrt{|f_1(x)| + ... + |f_n(x)|}$. Clearly we have $f \in (0 : f_0)$, then there exists $(g_1, ..., g_n) \in T^n$ such that $f = f_1g_1 + ... + f_ng_n$. We claim that

$$\forall r \in]0, 1[\ \exists x \in [r, 1] \ : \ f(x) \neq 0.$$

Deny. There is some $r \in]0, 1[$ such that $f(x) = 0$ for each $x \in [r, 2]$. Thus for any pair $(i, x) \in \{1, ..., n\} \times [r, 1[$, we have $f_i(x) = 0$. Therefore

$$(0 : f_0) = \{\varphi \in T \ ; \ \forall x \in [r, 2] \ \varphi(x) = 0\}.$$

Which is absurd. We conclude that for each nonnegative integer p there exists $1 - (1/p) \leq x_p \leq 1$ such that $f(x_p) \neq 0$. On the other hand every g_i is bounded mapping. There is some $c > 0$ such that

$$\forall i \in \{1, ..., n\} \ \forall x \in [0, 1] \ |g_i(x)| \leq c.$$

Thus

$$f(x_p) \leq c(|f_1(x_p)| + ... + |f_n(x_p)|).$$

It follows that $1 \leq cf(x_p)$ so that in limit we get $1 \leq c \lim_{p\to\infty} f(x_p)$. But $\lim_{p\to\infty} f(x_p) = 0$, we have the desired contradiction. Consequently T is a non-coherent WVNR ring.

Now we give characterization of weak von Neumann regular rings.

Theorem 2.4 *The following conditions on a ring R are equivalent:*

1. R is a WVNR ring.

2. *For each $a \in Re$ where e is a nonunit idempotent element of R, then $a \in Ra^2$.*

3. *For each $a \in Re$ where e is a nonunit idempotent element of R, then Ra is a direct summand of R.*

Proof. (1) $\Rightarrow$ (3): Let $a \in R$ and $1 \neq e$ an idempotent element of R such that $a \in Re$. Since e is nonunit we have the containments $Ra \subseteq Re \subseteq (\neq)R$. From the definition of a WVNR ring, we can write $Ra = Rf$ for some idempotent $f \in R$. It follows that $Ra \oplus R(1-f) = R$.

(3) $\Rightarrow$ (2): Let $a \in Re$ where e is a nonunit idempotent element of R and let I be an ideal of R such that $I \oplus Ra = R$. We can write $1 = u+v$ for some $u \in I$ and $v \in Ra$. Multiplying the above equality by u (resp., v) we get that $u^2 = u$ (resp., $v^2 = v$). Thus $I = Ru$ and $Ra = Rv$, therefore $a = au + av = av = a^2x$ for some $x \in R$.

(2) $\Rightarrow$ (1): Let J be a principal ideal generated by a nonunit idempotent element e of R, and let I be a finitely generated ideal of R contained in J. It suffices to prove that if $I = (a, b)$, then there exists an idempotent f in R such that $I = Rf$. Since $a \in J = Re$ then $a \in Ra^2$, also $b \in Rb^2$. Let $u = ax$ and $v = by$, where $a^2x = a$ and $b^2y = b$. Hence u and v are idempotent elements of R. The element $f = u + v - uv$ has the required property.

The following Corollary is an immediate consequence of Theorem2.4.

Corollary 2.5 *Let R be a ring. Then the following statements are equivalent:*

1. *R is a von Neumann regular ring.*

2. *R is a WVNR ring and for every nonunit element a of R there exists an idempotent $e \neq 1$ of R such that $a \in Re$.*

Now we give a necessary and sufficient condition for a direct product of rings to be a WVNR ring.

Theorem 2.6 *Let $(R_i)_{1 \leq i \leq n}$ be a family of rings, with $n \geq 2$. Then the following statements are equivalent:*

1. *$\prod_{i=1}^{n} R_i$ is a von Neumann regular ring.*

2. $\prod_{i=1}^{n} R_i$ *is a weak von Neumann regular ring.*

3. *For each* $i \in \{1, ..., n\}$ R_i *is a von Neumann regular ring.*

Proof. Straightforward.

Now we give an example of a non-WVNR Noetherian ring.

Example 2.7 *Let* n *be a positive integer such that* $n \geq 2$*, and let* Z *be the ring of integers. Then* Z^n *is not a WVNR ring since* Z *is not a von Neumann regular ring. Consequently,* Z^n *is non-WVNR Noetherian ring.*

For an ideal I of a WVNR ring R, R/I is not necessarily a WVNR ring. For this, we claim that $Z/12Z$ is not a WVNR ring, where Z is the ring of integers. Indeed, 9 is an idempotent and $9.2 = 6$ but $6^2 = 0$. Thus $6 \neq 6^2 x$ for each $x \in Z/12Z$. By applying condition (2) of Theorem 2.4, we get the result.

In the next theorem we give a sufficient condition for R/I to be a WVNR ring.

Theorem 2.8 *Let* R *be a ring and let* I *be a primary ideal. Then* R/I *is a WVNR ring.*

Proof. We denote $\overline{a} = a + I$ for every $a \in R$. To prove Theorem 2.8, it is enough to show that R/I has exactly two idempotent elements which are $\overline{0}$ and $\overline{1}$. Let $a \in R$ such that $\overline{a}$ a nonzero idempotent element of R/I. We have $a^2 - a \in I$. Since I is a primary ideal of R and $a \notin I$, there exists a nonnegative integer n such that $(a-1)^n \in I$. By the binomial theorem (which is valid in any commutative ring),

$$(a-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} a^k \in I.$$

We put $a^2 = a + x$. By induction we claim that for each $k \geq 2$, $a^k = a + x\left(1 + a + ... + a^{k-2}\right)$. Indeed, it is certainly true for $k = 2$. Suppose the statement is true for k, then we get the following equalities

$$a^{k+1} = a^2 + x\left(a + a^2 + ... + a^{k-1}\right) = a + x\left(1 + a + ... + a^{k-1}\right)$$

We conclude that for each nonnegative integers n, there is some $x_k \in I$ such that $a^k = a + x_k$. We can also deduce that

$$(-1)^n 1 + \sum_{k=1}^{n} (-1)^{n-k} \left(\begin{array}{c} n \\ k \end{array} \right) (a + x_k) \in I.$$

But

$$(-1)^n 1 + \sum_{k=1}^{n} (-1)^{n-k} \left(\begin{array}{c} n \\ k \end{array} \right) a = (-1)^n (1 - a),$$

hence $1 - a \in I$ and so $\overline{a} = \overline{1}$. By applying Example 2.2 we get that R is a WVNR ring. This completes the proof.

Remark 2.9 *The converse of Theorem 2.8 is not true in general. For example $Z/6Z$ (where Z is the ring of integers) is a WVNR ring because*

$$\forall x \in Z/6Z \quad x^3 - x = x(x-1)(x+1) = 0.$$

But $6Z$ is not a primary ideal of Z.

Corollary 2.10 *Let R be a Dedekind ring and let $I = p_1^{k_1} ... p_n^{k_n}$ be a nonzero ideal of R, where $p_1, ..., p_n$ are the prime ideals containing I. Then R/I is a WVNR ring if and only if $n = 1$ or $k_1 = ... = k_n = 1$.*

Proof. We shall need to use the following property:
if p and q are distinct maximal ideals of any ring A then $p^k + q^l = A$ for every positive integers k and l.

Assume that $n \geq 2$. Thus $p_i^{k_i} + p_j^{k_j} = R$ if $i \neq j$. By using the Chinese remainder theorem we deduce that

$$R/I \cong R/p_1^{k_1} \times ... \times R/p_n^{k_n}.$$

We can now apply Theorem 2.6 to obtain that R/I is a WVNR ring if and only if for each $i \in \{1, ..., n\}, R/p_i^{k_i}$ is a von Neumann regular ring. On the other hand every power of a maximal ideal m of R is a primary ideal. By applying Theorem 2.8 we deduce that $R/p_i^{k_i}$ has exactly two idempotent elements which are 0 and 1. But a WVNR ring A is a von Neumann regular ring if and only if for every nonunit element a of A there exists a nonunit idempotent e of A such that $a \in Ae$. It follows that R/I is a WVNR ring if and only for each $i \in \{1, ..., n\},\ R/p_i^{k_i}$ is a field (i.e $p_i^{k_i} = p_i$).

Finally, if $n = 1$ then R/I is a WVNR ring by Theorem 2.8. We conclude that R/I is a WVNR ring if and only if $n = 1$ or $k_1 = ... = k_n = 1$.

Example 2.11 *Let n be a positive integer and let Z be the ring of integers. Then Z/nZ is a WVNR ring if and only if n is a power of a prime integer or $v_p(n) \in \{0, 1\}$ for every prime integer p ($v_p(n)$ is the $p-$valuation of n).*

Example 2.12 *Let K be a field, and let f be a nonconstant polynomial in $K[x]$. Thus $K[x]/(f)$ is a WVNR ring if and only if f is a power of a irreducible polynomial or $v_p(f) \in \{0, 1\}$ for every irreducible polynomial p.*

Now, we give a characterization that a polynomial ring is a WVNR ring.

Theorem 2.13 *Let R be a ring. Then the polynomial ring $R[x]$ is a WVNR ring if and only if R has exactly two idempotent elements.*

Proof. Assume that R has exactly two idempotent elements. Since the set of all idempotent elements of $R[x]$ is $\{a \in R\ ;\ a^2 = a\}$, then $R[x]$ is a WVNR ring (Example 2.2). Conversely, suppose that $R[x]$ is a WVNR ring and let e be a nonunit idempotent element of R. We have $ex \in eR[x]$. By using condition (2) of Theorem2.4, we get that $ex \in (ex)^2R[x]$. There is some $f \in R[x]$ such that $ex = ex^2f(x)$. Thus $e = 0$, completing the proof of Theorem2.13.

Corollary 2.14 *Let R be a ring. Then the polynomial ring $R[x_1, ..., x_n]$ in several indeterminates is a WVNR ring if and only if R has exactly two idempotent elements which are 0 and 1.*

Proof. By induction on n from Theorem 2.13.

Example 2.15 *Let R be a von Neumann regular ring which is not a field. Then $R[x_1, ..., x_n]$ is not a WVNR ring. For instance if R is a Boolean ring such that $R \neq \{0, 1\}$ then $R[x_1, ..., x_n]$ is not a WVNR ring.*

Remark 2.16 *Let R be a ring and let $R[[x]]$ be the ring of formal power series in x with coefficients in R. With a similar proof as in Theorem 2.13, we get that $R[[x]]$ is a WVNR ring if and only if R has exactly two idempotent elements.*

We end this paper by studying the transfer of a WVNR property to trivial ring extensions.

Theorem 2.17 *Let A be a ring, E an $A-$module and let $R = A \propto E$ be the trivial ring extension of A by E. Then R is a WVNR ring if and only if the following statements are true:*

1. *A is a WVNR ring.*

2. *$aE = 0$ for every idempotent element $1 \neq a \in A$.*

Proof. It is easy to see that an element (a, x) of R is idempotent if and only so is a and $x = 0$.

Assume that R is a WVNR ring. Let $a \in Ae$ for some nonunit idempotent e of A, then $(a, 0) \in R(e, 0)$. The element $(e, 0)$ is a nonunit idempotent of R, by Theorem 2.4 we get that $(a, 0) \in R(a, 0)^2$. Hence there exists $(b, x) \in R$ such that $(a, 0) = (a, 0)^2(b, x)$. Therefore $a \in Aa^2$. We deduce that A is a WVNR ring. Now we consider a nonunit idempotent element a of A and $x \in E$. We have $(0, ax) = (a, 0)(0, x)$, then $(0, ax) \in R(a, 0)$. Since $(a, 0)$ is a nonunit idempotent element of R, then $(0, ax) \in R(0, ax)^2$ and so $ax = 0$. It follows that $aE = 0$.

Conversely, suppose that A is a WVNR ring and $bE = 0$ for each nonunit idempotent element b of A. Let $(a, x) \in R(b, 0)$ for some nonunit idempotent element b of A, there is some $(c, y) \in R$ such $(a, x) = (b, 0)(c, y)$. Hence $a \in Ab$ and $x = by$ therefore $x = 0$ and $a \in Aa^2$. It follows that $(a, x) \in R(a, x)^2$. This completes the proof of Theorem 2.17.

Example 2.18 *Let A be a ring and let E be an $A-$module. Suppose that A has exactly two idempotent elements. Then $A \propto E$ is a WVNR ring.*
For instance let G be a commutative group, then $Z \propto G$ is a WVNR ring, where Z is the ring of integers. This is an other WVNR ring which is neither local nor integral domain. Finally by Theorem 2.13 the polynomial ring $(Z \propto G))[x]$ is also a WVNR ring.

Corollary 2.19 *Let A be a ring and $Q(A)$ its full ring of quotient. Then the following statements are equivalent:*

1. *A has exactly two idempotent elements.*

2. *$A \propto A$ is a WVNR ring.*

3. *$A \propto Q(A)$ is a WVNR ring.*

Proof. (1) $\Rightarrow$ (2): The ring $A \propto A$ has exactly two idempotent elements which are 0 and 1.
(2) $\Rightarrow$ (1): Let a be a nonunit idempotent element of A. By Theorem 2.17 $aA = 0$, then $a = 0$.
(1) $\Leftrightarrow$ (3): By the same way we get this equivalence.

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