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Prime Spectrum of a C-algebra

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Abstract

In this paper we defined prime ideal and maximal ideal and proved several properties of these. We have discussed the space of prime ideals of a C-algebra A with respect to the hull-kernel topology, which is called the prime spectrum of A and denoted by Spec A. It is also proved that Spec A is a T_0 space.

Keywords: *C-algebra, Boolean algebra, Prime ideal, Maximal ideal, Stone topology, Spec A.*

1 Introduction

In [3] Fernando Guzman and Craig C.Squier introduced the variety of C-algebras as the variety generated by the three element algebra $C = \{T, F, U\}$ with the operations $\wedge, \vee$ and $'$ of type (2,2,1), which is the algebraic form of the three valued conditional logic. They proved that C and the two element Boolean algebra $B = \{T, F\}$ are the only subdirectly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. In [7] U.M.Swamy et.al., have worked on three valued logic and introduced the concept of the Centre $B(A)$ of a C-algebra A and proved that the centre of a C-algebra is a Boolean algebra. Later in [4] S.Kalesha Vali et.al., introduced the notion of an ideal and principal ideal of a C-algebra and discuss various properties of these including the set of ideals forms an algebraic distributive lattice. In this paper we define prime ideal, maximal ideal and proved several properties of these. We discuss the space of prime ideals of a C-algebra with respect to the hull-kernel topology. If X is the set of all prime ideals of A and, for any $a \in A$, $N_a = \{P \mid a \notin P\}$, then the class $\{N_a \mid a \in A\}$ forms a base for a topology on the set X. Also, the topology generated by $\{N_a \mid a \in A\}$ on the set X of prime ideals of A is the Stone topology. X together with the Stone topology is called Stone space or the prime spectrum and is denoted by Spec A. The Stone topology on X is also called the hull-kernel topology.

2 C-algebra

In this section we recall the definition of a C-algebra and some results from [3, 5, 7, 8]. Let us start with the definition of a C-algebra.

Definition 2.1 ([3])**.** *By a C-algebra we mean an algebra of type* $(2, 2, 1)$ *with binary operations* $\wedge$ *and* $\vee$ *and unary operation* $'$ *satisfying the following identities.*

(1) $x'' = x$ *(2)* $(x \wedge y)' = x' \vee y'$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *(4)* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$ *(6)* $x \vee (x \wedge y) = x$
(7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Example 2.2 ([3])**.** *The three element algebra* $C = \{T, F, U\}$ *with the operations given by the following tables is a C-algebra.*

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

x	x'
T	F
F	T
U	U

Note 2.3 ([3])**.** *The identities 1.1(1), 1.1(2) imply that the variety of C-algebras satisfies all the dual statements of 1.1(2) to 1.1(7).* $\wedge$ *and* $\vee$ *are not commutative in* C*. The ordinary distributive law of* $\wedge$ *over* $\vee$ *fails in* C*. Every Boolean algebra is a C-algebra.*

Now we give some results on C-algebra collected from [3, 5, 7, 8].

Lemma 2.4. *Every C-algebra satisfies the following identities:*

(1) $x \wedge x = x$ *(2)* $x \wedge x' = x' \wedge x$
(3) $x \wedge y \wedge x = x \wedge y$ *(4)* $x \wedge x' \wedge y = x \wedge x'$
(5) $x \wedge y = (x' \vee y) \wedge x$ *(6)* $x \wedge y = x \wedge (y \vee x')$
(7) $x \wedge y = x \wedge (x' \vee y)$ *(8)* $x \wedge y \wedge x' = x \wedge y \wedge y'$
(9) $(x \vee y) \wedge x = x \vee (y \wedge x)$ *(10)* $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x = x$.

The dual statements of the above identities are also valid in a C-algebra.

3 Ideals of a C-algebra

In this section we recall the the definition of an ideal and principal ideal of a C-algebra and some results from [4] which are useful in proving the results in the forthcoming sections. Let us start with the definition of an ideal of a C-algebra.

A nonempty subset I of a C-algebra A is said to be an ideal of A if it satisfies (i) $a, b \in I$ implies that $a \vee b \in I$ and (ii) $a \in I$ implies that $x \wedge a \in I$, for each $x \in A$. The set $\{x \wedge a \mid x \in A\}$ is the smallest ideal containing a and is denoted by $< a >$. An element z of a C-algebra A is called a left zero for $\wedge$ if $z \wedge x = z$ for all $x \in A$. By Lemma 2.4(4), $x \wedge x'$ is a left zero for $\wedge$, for all $x \in A$. In fact, any left zero for $\wedge$ must be of the form $x \wedge x'$ for some $x \in A$ also $x \wedge x' \in I$ for all $x \in A$. Its observed that the set of ideals of a C-algebra A is closed under arbitrary intersections and $I_0 = \{x \wedge x' \mid x \in A\}$ is the smallest ideal of A. If X is a non-empty subset of a C-algebra A then $\{\bigvee_{i=1}^{n}(y_i \wedge x_i) \mid y_i \in A, x_i \in X\}$ is the smallest ideal containing X and is denoted by $< X >$. If $\{I_\alpha\}_{\alpha\in\Delta}$ be a family of ideals of a C-algebra A then $\{\bigvee_{i=1}^{n} a_i \mid a_i \in I_\alpha$, for some $\alpha\}$ is the smallest ideal containing I_α's.

Lemma 3.1. *[4] Let I be an ideal of a C-algebra A and $a, b \in A$. Then*
(1) $y \in < a > \Leftrightarrow y = y \wedge a$.
(2) $< a > = < b > \Leftrightarrow a \wedge b = a$ and $b \wedge a = b$.
(3) $< a \wedge b > = < b \wedge a > = < a > \cap < b >$.
(4) $< a > = < b > \Rightarrow a \vee b = b \vee a$
(5) $< a > = < b >$ it is not necessary that $a = b$.
(6) If $a \wedge b \in I$, then $a \wedge z \wedge b \in I$ for each $z \in A$.
(7) $a \wedge b \in I$ if and only if $b \wedge a \in I$.

Theorem 3.2. *[4] Let A be a C-algebra and $\Im(A)$ the set of all ideals of A. Then $\Im(A)$ is an algebraic distributive lattice with respect to the inclusion ordering.*

4 Prime Ideals

In this section, we first discuss the concept of prime ideal and prove certain important fundamental properties of a prime ideal. In particular, we extend the Stone's theorem on prime ideals of a distributive lattice to C-algebras and prove that every ideal I of a C-algebra A is the intersection of all prime ideals of A containing I.

Definition 4.1. *Let A be a C-algebra. A proper ideal P of A is called a prime ideal if, for any $a, b \in A, a \wedge b \in P$ implies that either $a \in P$ or $b \in P$.*

Theorem 4.2. *The following are equivalent for any proper ideal P of a C-algebra A*
(1) P is a prime ideal
(2) For any ideals I and J of A, $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$
(3) For any ideals I and J of A, $P = I \cap J \Rightarrow P = I$ or $P = J$.

Proof. (1) $\Rightarrow$ (2): Let P be a prime ideal and I and J are ideals of A such that $I \cap J \subseteq P$. Assume that $I \nsubseteq P$. Then there exists $a \in I$ such that $a \notin P$. Let $b \in J$. Then $a \wedge b \in I \cap J \subseteq P$. Since P is Prime, $a \in P$ or $b \in P$. But $a \notin P$. Therefore $b \in P$ (since $a \notin P$). Thus $J \subseteq P$. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1): Assume (3). Let a and $b \in A$ such that $a \wedge b \in P$. Then $< a > \cap < b >=< a \wedge b >\subseteq P$ (by Lemma 3.1) and hence
$(< a > \vee P) \cap (< b > \vee P) = (< a > \cap < b >) \vee P = P$ (since by Theorem 3.2) so that $< a > \vee P = P$ or $< b > \vee P = P$ or equivalently, $< a >\subseteq P$ or $< b >\subseteq P$ and therefore $a \in P$ or $b \in P$. Thus P is a prime ideal. □

Lemma 4.3. *Let P be a prime ideal of a C-algebra A. Then, for any $x \in A$, either $x \in P$ or $x' \in P$.*

In the following, we prove a theorem analogous to that of the Stone's theorem for distributive lattices.

Theorem 4.4. *Let I be an ideal of a C-algebra and $a \in A \setminus I$. Then there exists a prime ideal P containing I and not containing a.*

Proof. Let $a \notin I$. Let $\wp = \{J \mid J$ is an ideal of A, $I \subseteq J$ and $a \notin J\}$. Clearly $I \in \wp$. Therefore $\wp$ is nonempty and is a partially ordered set under the inclusion. It is easily verify that the union of all ideals in chain of ideals is again an ideal. Therefore by Zorn's lemma there exists a maximal member M in $\wp$. Then, clearly M is a proper ideal of A not containing a and $I \subseteq M$. We shall prove that M is a prime ideal of A. let $x, y \in A$ such that $x \notin M$ and $y \notin M$. Then M is properly contained in $M \vee < x >$ and $M \vee < y >$ and hence by the maximality of M, we have
$a \in (M \vee < x >) \cap (M \vee < y >) = M \vee (< x > \wedge < y >) = M \vee (< x \wedge y >)$
Therefore $x \wedge y \notin M$ (for, if $x \wedge y \in M$ then $M \vee (< x > \wedge < y >) = M$, which is a contradiction, since $a \notin M$). Therefore M is prime containing I and not containing a. □

Corollary 4.5. *For any ideal I of a C-algebra A,*
$I = \cap\{P \mid P$ *is a prime ideal of A and* $I \subseteq P\}$.

Proof. If $a \notin I$, then by above theorem, there exists a prime ideal P containing I and not containing a and therefore $a \notin \bigcap_{I \subseteq P, P prime} P$.Therefore $\bigcap_{I \subseteq P, P prime} P \subseteq I$. Thus $I = \cap\{P \mid P$ is a prime ideal of A and $I \subseteq P\}$. □

Corollary 4.6. *The intersection of all prime ideals coincides with the set of all left zeros for $\wedge$ in A.*

Theorem 4.4 is strengthened in the following.

Theorem 4.7. *Let I be an ideal of a C- algebra A and S a nonempty subset of A which is closed under the operation $\wedge$ and is disjoint with I. Then there exists a prime ideal P of A containing I and disjoint with S.*

5 Maximal Ideals

In this section, we discuss maximal ideals of a C-algebra A. As usual, a proper ideal of A is called maximal, if it is not contained in any proper ideal except itself. We prove here that every maximal ideal is prime and that the converse is true in certain special cases.

Definition 5.1. *A proper ideal M of a C-algebra A is said to be a Maximal ideal of A if M is maximal among all the proper ideals of A.*

Lemma 5.2. *Let A be a C-algebra. Every maximal ideal in A is a prime ideal.*

Proof. Let M be a maximal ideal of A. Then clearly M is a proper ideal of A. Suppose $a, b \in A$ such that $a \notin M$ and $b \notin M$.
Then $M \subsetneqq < M \cup \{a\} >= A$; $M \subsetneqq < M \cup \{b\} >= A$ (since M is maximal).
Then $b \in < M \cup \{a\} >= \{\bigvee_{i=1}^{n} (y_i \wedge x_i) \mid y_i \in A, x_i \in M \cup \{a\}\}$.
That is, $b = \bigvee_{i=1}^{n} (y_i \wedge x_i)$, for some $y_i \in A, x_i \in M \cup \{a\}$.
Now, $b = b \wedge b = b \wedge \bigvee_{i=1}^{n} (y_i \wedge x_i) = \bigvee_{i=1}^{n} (b \wedge y_i \wedge x_i)$.
If $x_i \in M$, then clearly $b \wedge y_i \wedge x_i \in M$. Since $b \notin M$ and $b = \bigvee_{i=1}^{n} (b \wedge y_i \wedge x_i)$, it follows that $b \wedge y_i \wedge a \notin M$ for some y_i (since $x_i \in M$ or $x_i = a$). By Lemma 3.1, we get that $b \wedge a \notin M$, and hence $a \wedge b \notin M$. Thus M is a prime ideal of A. □

The validity of the converse of the above theorem is not known. In Boolean algebras every prime ideal is maximal but in C-algebras, we do not know that every prime ideal is maximal, it is under investigation.

In the following, we discuss another class of C-algebras where every prime ideal is maximal. First let us recall the following.

Definition 5.3. *[7] Let A be a C-algebra with T (that is, T is the identity element for $\wedge$ in A). Then the Boolean centre of A is defined as the set $\mathbb{B}(A) = \{a \in A \mid a \vee a' = T\}$. $\mathbb{B}(A)$ is known to be a Boolean algebra under the operations induced by those on A.*

Theorem 5.4. *[7] Let A be a C-algebra with T. Suppose that, for any $x \in A$, there exists a smallest $x_0 \in \mathbb{B}(A)$ such that $x \wedge x_0 = x$ and $x_0' \vee x = T$. Then every prime ideal of A is a maximal ideal.*

Proof. Let P be a prime ideal of A and Q be any ideal such that $P \subsetneqq Q$. Since, $P \subsetneqq Q$, there exists $x \in Q$ such that $x \notin P$. Then there exists smallest $x_0 \in \mathbb{B}(A)$ such that $x \wedge x_0 = x$ and $x_0' \vee x = T$. Therefore $x_0 \notin P$ (for, if $x_0 \in P$ then $x \in P$, a contradiction). Since P is prime, $x_0' \in P \subseteq Q$. Therefore both x and x_0' belong to Q and hence $T = x_0' \vee x \in Q$ (since, Q is an ideal). Then $a = T \wedge a \in Q$ for all $a \in A$. Therefore $Q = A$. Thus P is a maximal ideal of A. $\square$

Theorem 5.5. *Let X be a nonempty set and $C = \{T, F, U\}$ be the three-element C-algebra. Let C^X be the set of all mappings of X into C. Then C^X is a C-algebra under the pointwise operations. For any $Y \subseteq X$, let $f_Y \in C^X$ be defined by $f_Y(x) = \begin{cases} T, & \text{if } x \in Y; \\ F, & \text{if } x \notin Y. \end{cases}$ and, for any $x \in X$, let $f_x = f_{\{x\}}$. Also, for any $f \in C^X$, let $|f| = \{x \in X \mid f(x) = T\}, |f|$ is called the support of X. Then every prime ideal of C^X is a maximal ideal.*

Proof. It can be easily verified that C^X is a C-algebra under the pointwise operations. Before proving every prime ideal of C^X is maximal first we prove that for any $g \in C^X$ and $Y \subseteq X$ (1)$f_Y' = f_{X \smallsetminus Y}$, (2) $f_Y \wedge f_Y' = \bar{F}$, the constant map, (3) $f_Y \in \mathbb{B}(C^X)$ and (4) $g \wedge f_{|g|} = g$. Define $\bar{F} : X \to C$ by $\bar{F}(x) = F$, for all $x \in X$. (1) This follows from the facts that $T' = F$ and $f'(x) = f(x)'$ for all $x \in X$ and for all $f \in C^X$. (2) If $z \in Y, (f_Y \wedge f_Y')(z) = T \wedge F = F$. If $z \notin Y, (f_Y \wedge f_Y')(z) = F \wedge T = F$. Therefore $f_Y \wedge f_Y' = \bar{F}$. (3) Since $f_Y \wedge f_Y' = \bar{F}$, $(f_Y \wedge f_Y')' = (\bar{F})'$ and hence $f_Y' \vee f_Y = \bar{T}$, so that $f_Y \vee f_Y' = \bar{T}$. Therefore $f_Y \in \mathbb{B}(C^X)$. (4) For $x \in |g|$, $f_{|g|}(x) = T$ and $g(x) \wedge f_{|g|}(x) = g(x) \wedge T = T \wedge T = T = g(x)$. For $x \notin |g|, f_{|g|}(x) = F$ $g(x) \wedge f_{|g|}(x) = g(x) \wedge F = g(x)$ (since $g(x) = U$ or $F, g(x) \wedge F = g(x)$). Therefore $g \wedge f_{|g|} = g$.
Let P be a prime ideal of C^X. Let Q be any ideal of C^X such that $P \subsetneqq Q$. Then there exists $g \in Q$ such that $g \notin P$. Since $g \in C^X, g \wedge f_{|g|} = g$. Then $f_{|g|} \notin P$ (since $g \notin P$). Since $f_{|g|} \wedge f_{|g|}' = \bar{F} \in P$ and P is a prime ideal, $f_{|g|}' \in P \subseteq Q$. We shall prove that $f_{|g|}' \vee g = \bar{T}$. If $x \in |g|$ then $g(x) = T$ and $f_{|g|}'(x) = F$ and hence $(f_{|g|}' \vee g)(x) = F \vee T = T = g(x)$. If $x \notin |g|$ then $g(x) = U$ or F and $f_{|g|}'(x) = T$ and hence $(f_{|g|}' \vee g)(x) = T \vee U$ or $T \vee F = T)$. Therefore $f_{|g|}' \vee g = \bar{T}$. Since $f_{|g|}' \in Q$ and $g \in Q, f_{|g|}' \vee g \in Q$ (since Q is an

ideal). Therefore $\bar{T} \in Q$ which implies that $Q = C^X$. Therefore P is maximal ideal of C^X. □

6 The Prime Spectrum

Stone's celebrated theorem [6]on the topological representation of Boolean algebras was extended to several algebraic structures like Post algebras, distributive lattices, Stone lattices etc. In each of these, the set of prime ideals together with the Stone topology (or the Hull- Kernel topology) plays the key role. In this section we discuss the space of prime ideals of a C-algebra with respect to the hull- kernel topology. We begin with the following.

Definition 6.1. *Let A be a C-algebra and X the set of all prime ideals of A. For any $a \in A$, let $N_a = \{P \mid a \notin P\}$.*

Theorem 6.2. *The class $\{N_a \mid a \in A\}$ forms a base for a topology on the set X of all prime ideals of a C-algebra A. Also, for any $a, b \in A$, the following hold.*
(1) $N_a \cap N_b = N_{a\wedge b}$ *(2) $N_{a\vee b} \subseteq N_a \cup N_b$*
(3) $N_{a'} \subseteq X \setminus N_a$ *(4) $N_{a\wedge a'} = \phi$.*

Proof. Let P be any prime ideal of A. Then
(1) $P \in N_a \cap N_b \Leftrightarrow a \notin P$ and $b \notin P \Leftrightarrow a \wedge b \notin P \Leftrightarrow P \in N_a \cap N_b$.
Therefore $N_a \cap N_b = N_{a\wedge b}$
(2) $P \in N_{a\vee b} \Rightarrow a \vee b \notin P \Rightarrow a \notin P$ or $b \notin P \Rightarrow P \in N_a$ or $P \in N_b \Rightarrow P \in N_a \cup N_b$. Therefore $N_{a\vee b} \subseteq N_a \cup N_b$.
(3) First note that $a \wedge a'$ is a left zero for $\wedge$ and hence $a \wedge a' \in P$.
$P \in N_{a'} \Rightarrow a' \notin P \Rightarrow a \in P \Rightarrow P \notin N_a$. Therefore $N_{a'} \subseteq X \setminus N_a$.
(4) By (3), $N_a \wedge N_{a'} = \phi$, and hence, by (1), $N_{a\wedge a'} = \phi$. □

Thus $\{N_a \mid a \in A\}$ is a class of subsets of X which is closed under finite intersections. Also, since every prime ideal is a proper ideal, we get that $\bigcup_{a\in A} N_a = X$. Thus $\{N_a \mid a \in A\}$ is a base for a topology on X.

Note that $N_{a\vee b}$ may not be equal to $N_a \cup N_b$ and $N_{a'}$ may not be equal to $X \setminus N_a$. For, consider the following

Example 6.3. *Consider the three element C-algebra $C = \{T, F, U\}$. Here there is only one prime ideal, namely $I_0 = \{F, U\}$ and hence $X = \{I_0\}$. Note that $N_U = \phi, N_T = X, N_U \cup N_T = X$ and $N_{U\vee T} = N_U = \phi$. Also, $N_{U'} = N_U = \phi$.*

It is well known that, for any prime ideal P of a Boolean algebra B and for any $a \in B$, either $a \in P$ or $a' \in P$ but not both; that is, exactly one of a and a' belongs to P. But this is not true for prime ideals of a C-algebra. In fact, we have the following.

Theorem 6.4. *The following are equivalent for any C-algebra A.*
(1) A is a Boolean algebra
(2) $X \setminus N_a = N_{a'}$, for any $a \in A$.
(3) For any prime ideal P of A and $a \in A$, exactly one of a and a' belongs to P.
(4) $N_a \cup N_b = N_{a \vee b}$ for any $a, b \in A$.

Proof. (1) $\Rightarrow$ (2): Let A be a Boolean algebra and $P \in X \setminus N_a$. Then $P \notin N_a$ and hence $a \in P$ so that $a' \notin P$ (since A is a Boolean algebra and $a \vee a' = 1 \notin P$). Therefore $P \in N_{a'}$. Thus, $X \setminus N_a \subseteq N_{a'}$. By Theorem 6.2(3), $N_{a'} \subseteq X \setminus N_a$. Therefore $X \setminus N_a = N_{a'}$, for any $a \in A$.
(2) $\Rightarrow$ (3): Suppose that $X \setminus N_a = N_{a'}$, for any $a \in A$. Let P be a prime ideal of A and $a \in A$. Since $a \wedge a' \in P$, we get that $a \in P$ or $a' \in P$. Suppose that both $a, a' \in P$. Then $P \notin N_a$ and $P \notin N_{a'}$, which is a contradiction to (2). Therefore exactly one of a and a' belongs to P.
(3) $\Rightarrow$ (4): Suppose exactly one of a and $a' \in P$, for any prime ideal P of A and $a \in A$. Then $N_{a \vee b} \subseteq N_a \cup N_b$ (by Theorem 6.2(2)).
Now $P \in N_a \cup N_b \Rightarrow P \in N_a$ or $P \in N_b \Rightarrow a \notin P$ or $b \notin P \Rightarrow a' \in P$ or $b' \in P \Rightarrow a' \wedge b' \in P \Rightarrow (a' \wedge b')' \notin P$ (since by assumption)$\Rightarrow a \vee b \notin P \Rightarrow P \in N_{a \vee b}$. Therefore $N_a \cup N_b \subseteq N_{a \vee b}$. Thus $N_a \cup N_b = N_{a \vee b}$ for any $a, b \in A$.
(4) $\Rightarrow$ (1): Suppose $N_a \cup N_b = N_{a \vee b}$ for any $a, b \in A$. Then $N_{a \vee b} = N_{b \vee a}$ (Since $N_a \cup N_b = N_b \cup N_a$). We know that $< a \vee b >=< b \vee a >$ and hence $a \vee b = (a \vee b) \wedge (b \vee a)$ (by Lemma 3.1) $= (b \vee a) \wedge (a \vee b) = b \vee a$. Therefore $a \vee b = b \vee a$ for all $a, b \in A$. In [7] it is proved that a C-algebra is a Boolean algebra if and only if $a \vee b = b \vee a$ for all $a, b \in A$. Thus A is a Boolean algebra. □

We have proved in Theorem 6.2 that the class $\{N_a \mid a \in A\}$ forms a base for a topology on X.

Definition 6.5. *The topology generated by $\{N_a \mid a \in A\}$ on the set X of prime ideals of A is called the Stone topology. X together with the Stone topology is called Stone space or the prime spectrum and denote it by Spec A.*

The Stone topology on X is also called the hull-kernel topology, the reason being the following.

Theorem 6.6. *Let X be the prime spectrum of a C-algebra A. For any $S \subseteq A$ and $Y \subseteq X$, we define the hull of S and the kernel of Y respectively by $h(S) = \{P \in X \mid S \subseteq P\}$ and $K(Y) = \bigcap_{P \in Y} P$.*
Then, for any $Y \subseteq X$, the closure $\bar{Y}$ of Y is equal to the hull of the kernel of Y; that is $\bar{Y} = h(K(Y))$.

Proof. First we prove that $h(S)$ is closed in X, for any $S \subseteq A$.
We have $h(S) = \{P \in X \mid S \subseteq P\} = X \setminus \{P \in X \mid S \nsubseteq P\} = X \setminus \bigcup_{a \in S} N_a$.
Since each N_a is open, $\bigcup_{a \in S} N_a$ is also open. Therefore $X \setminus \bigcup_{a \in S} N_a$ is closed.
That is $h(S)$ is closed. Thus $h(K(Y))$ is closed.
Let $Y \subseteq X$ and $Q \in Y$. Then $\bigcap_{P \in Y} P \subseteq Q$. Therefore $Q \in h(\bigcap_{P \in Y} P)$. Hence $h(\bigcap_{P \in Y} P)$ is a closed set containing Y.
Let C be any closed set in X containing Y. Then $X \setminus C$ is an open set in X and hence $X \setminus C = \bigcup_{a \in I} N_a$ for some $I \subseteq A$, which implies that $C = X \setminus \bigcup_{a \in I} N_a = h(I)$. Therefore $Y \subseteq h(I)$ (since C is closed such that $Y \subseteq C$).
Now let $P \in Y$. Then $P \in h(I)$ and hence $I \subseteq P$ (since by definition of $h(S)$).
Therefore $I \subseteq P$, for every $P \in Y$. That is $I \subseteq \bigcap_{P \in Y} P$. Therefore
$h(\bigcap_{P \in Y} P) \subseteq h(I) = C$. Thus $h(\bigcap_{P \in Y} P)$ is the smallest closed set containing Y.
Hence $h(K(Y))$ is the closure of Y in X. □

Definition 6.7. *For any $S \subseteq A$, we define*
$$N(S) = X \setminus h(S) = \{P \in X \mid S \nsubseteq P\}.$$

Since $N(S) = \bigcup_{a \in S} N_a$, it follows that the open subsets of the prime spectrum X are precisely of the form $N(S), S \subseteq A$, and the closed subsets of X are of the form $h(S), S \subseteq A$. It can be easily verified that $N(S) = N(< S >)$ for any $S \subseteq A$.

Theorem 6.8. *For any ideal I of A, consider the open set $N(I) = \{P \in X \mid I \nsubseteq P\}$. Then the map $I \mapsto N(I)$ is an isomorphism of the lattice $\Im(A)$ of ideals of A onto the lattice of open subsets of X.*

Proof. Let $I, J \in \Im(A)$. Then
$$\begin{aligned} N(I \cap J) &= \{P \in X \mid I \cap J \nsubseteq P\} \\ &= X \setminus \{P \in X \mid I \cap J \subseteq P\} \\ &= X \setminus (\{P \in X \mid I \subseteq P\} \cup \{P \in X \mid J \subseteq P\}) \\ &= (X \setminus \{P \in X \mid I \subseteq P\}) \cap (X \setminus \{P \in X \mid J \subseteq P\}) \\ &= N(I) \cap N(J) \\ N(I \cup J) &= \{P \in X \mid I \cup J \nsubseteq P\} \\ &= \{P \in X \mid I \nsubseteq p\} \cup \{P \in X \mid J \nsubseteq P\} \\ &= N(I) \cup N(J) \end{aligned}$$
Therefore the map $I \mapsto N(I)$ is a homomorphism of the lattice $\Im(A)$ into the lattice of open subsets of X.
Suppose $I \neq J$. Then $I \nsubseteq J$ or $J \nsubseteq I$. $I \nsubseteq J$ then there exists $a \in I$ such that $a \notin J$. Since $a \notin J$, there exists a prime ideal P such that $J \subseteq P$ and $a \notin P$. Therefore $J \subseteq P$ and $I \nsubseteq P$ (since $a \in I$ and $a \notin P$). Therefore $P \notin N(J)$

and $P \in N(I)$. Thus $N(I) \nsubseteq N(J)$. Therefore $N(I) \neq N(J)$.
Therefore the map $I \mapsto N(I)$ is an injective map.
Further, if G is any subset of X, then $G = \bigcup\limits_{a \in S} N_a$ for some subset S of A and hence $G = N(S) = N(< S >)$. Therefore the map $I \mapsto N(I)$ is a surjective map. Thus the map is an isomorphism of lattices. □

In the following we prove certain topological properties of the prime spectrum of a C-algebra.

Theorem 6.9. *Let X be the prime spectrum of a C-algebra A. Then N_a is compact for any $a \in A$.*

Proof. To prove the theorem it is enough if we prove that every basic open cover of N_a contains a finite sub cover. Let $B \subseteq A$ such that $N_a = \bigcup\limits_{a \in B} N_b$.
Let $I =< B >$, the ideal generated by B in A. If $a \notin I$ then there exists a prime ideal P such that $I \subseteq P$ and $a \notin P$, which imply that $P \in N_a$ and $P \notin N_b$ for all $b \in B$, a contradiction to the assumption that $N_a \subseteq \bigcup\limits_{b \in B} N_b$.
Thus $a \in I =< B >$ and hence $a = \bigvee\limits_{i=1}^{n} (y_i \wedge x_i)$ where $y_i \in A, x_i \in B$.

$$\begin{aligned} \text{Now } N_a = N_{\bigvee\limits_{i=1}^{n}(y_i \wedge x_i)} &\subseteq \bigcup_{i=1}^{n} N_{y_i \wedge x_i} \quad \text{(by Theorem 6.2(2))} \\ &\subseteq \bigcup_{i=1}^{n} N_{x_i} \quad (\text{since } N_{y \wedge x} \subseteq N_y \cap N_x \subseteq N_x) \\ &= \bigcup_{b \in S} N_b \text{ where } S = \{x_1, x_2, ..., x_n\}, \text{ which is} \end{aligned}$$

a finite subset of B. Thus N_a is compact. □

Corollary 6.10. *Let Y be an open subset of the prime spectrum X of a C-algebra A. Then Y is compact if and only if $Y = N(I)$ for some finitely generated ideal I of A.*

Proof. Suppose that Y is compact. Since Y is an open subset of X, we have that $Y = \bigcup\limits_{a \in S} N_a$ for some $S \subseteq A$. From the compactness of Y, it follows that $Y = \bigcup\limits_{i=1}^{n} N_{s_i} = N(< \{s_1, s_2, ..., s_n\} >)$. The converse follows from the facts that any finite union of compact sets is compact and that $N(I) = N(< F >) = \bigcup\limits_{a \in F} N_a$ where F is a finite set generating I. □

Corollary 6.11. *If A is a C-algebra with T, then Spec A is compact.*

Proof. Let A be a C-algebra with T; that is T is the identity for $\wedge$ in A. Then $< T >= A$ and hence Spec $A = N_T$, which is compact. □

In general the prime spectrum of a C-algebra may not be compact. For consider the following example, in which the C-algebra is not possessing identity for $\wedge$.

Example 6.12. *Let X be an infinite set and $C = \{T, F, U\}$ be the three element C-algebra. Let $A = \{f : X \to C \mid f(x) = U$ for all but finite number of $x \in X\}$ and $|f| = \{x \in X \mid f(x) \neq U\}$, for any $f \in C^X$. Therefore $A = \{f : X \to C \mid |f|$ is finite $\}$.*
First we show that A is a subalgebra of the C-algebra C^X (refer [7]). Since the variety of C-algebras satisfies the identities $x'' = x$ and $(x \wedge y)' = x' \vee y'$, it is enough if we show that A is closed under $\wedge$ and $'$ (or $\vee$ and $'$). Note that $|f| = |f'|$ for any $f \in C^X$. Let $f \in A$. Then $|f|$ is finite and so is $|f'|$. Therefore $f' \in A$. Let $f, g \in A$. Suppose that $x \notin |f|$. Then $f(x) = U$. Now, $f(x) \vee g(x) = U$ which implies that $(f \vee g)(x) = U$ that is, $x \notin |f \vee g|$. Therefore, $x \in |f \vee g|$ implies that $x \in |f|$. That is, $|f \vee g| \subseteq |f|$. On the other hand, if $|f|$ is finite then $|f \vee g|$ is finite and therefore $f \vee g \in A$. Thus A is a C-algebra with respect to the point wise operations. Suppose if possible that there exist $h \in A$ such that $h \wedge f = f = f \wedge h$, for all $f \in A$. Then $h(x) \wedge f(x) = f(x)$, for all $f \in A$, which is possible if and only if $h(x) = T$, for all $x \in X$. Therefore $h \notin A$. Thus A has no identity for $\wedge$.
For each $x \in X$, let $P_x = \{f \in A \mid f(x) = F$ or $U\}$. Then it can be easily verified that P_x is an ideal of A. Now, let $f \wedge g \in P_x$. Then $f(x) \wedge g(x) = F$ or U that is, $f(x) \wedge g(x) \neq T$. If $f(x) = T$ and $g(x) = T$ then $f(x) \wedge g(x) = T$. Therefore $f(x) \neq T$ or $g(x) \neq T$ that is, $f \in P_x$ or $g \in P_x$. Thus P_x is a prime ideal of A.
Let X be the prime spectrum of A. We have that $X = \bigcup_{a \in A} N_a$. Suppose that X is compact. Then $X = \bigcup_{i=1}^{n} N_{f_i}$, for some $f_1, f_2, ..., f_n \in A$. Now, let $Y = \bigcup_{i=1}^{n} |f_i|$. Then Y is a finite subset of X (since $f_i \in A$, $|f_i|$ is finite). Since X is infinite, we can choose $x_0 \in X \setminus Y$. Now, $P_{x_0} \in X = \bigcup N_{f_i}$ and hence $P_{x_0} \in N_{f_i}$ and therefore $f_i \notin P_{x_0}$ which implies that $f_i(x_0) = T$, which is a contradiction to the fact that $x_0 \notin |f_i|$. Thus X is not compact.

In the following we discuss the separation property of the topological space Spec A, for a general C-algebra A.

Theorem 6.13. *Spec A is a T_0 space, for any C-algebra A.*

Proof. Let $P \neq Q$ be prime ideals. Then $P \nsubseteq Q$ or $Q \nsubseteq P$. Suppose $P \nsubseteq Q$. Choose $a \in P$ such that $a \notin Q$. Then $Q \in N_a, P \notin N_a$. Thus N_a is an open set such that N_a containing Q but not containing P. Thus Spec A is a T_0 space. $\square$

7 Open Problems

1. It is known that in a Boolean algebra, every prime ideal is maximal. It is under investigation that whether every prime ideal is maximal or not in a C-algebra.

2. If A_1, A_2 are two C-algebras in which prime ideals are all maximal. Does the direct product $A_1 \times A_2$ have the same property.

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