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A Positive Solution For a Nonlocal Boundary Value Problem

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Abstract

In this work, we apply Guo-Krasnosel'skii fixed point theorem and use the appropriate Green's function in the study of existence of positive solution for the three point boundary value problem:

$$u'' + g(t)f(u) = 0, \quad 0 < t < 1$$

$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta)$$

where $\eta \in (0,1)$, $\alpha,\ \beta \in \mathbb{R}_+,\ g \in C[0,1], f \in C[0,\infty[$.

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1 Introduction

Three point boundary value problems (BVP) for second order differential equations, are models for many problems in physics, biology, chemistry....For example, second order three point BVP are used as models for the membrane response of a spherical cap, in nonlinear diffusion generated by nonlinear sources and in chemical reactor theory. Our attention will be focused in particular on a BVP that requires nonlocal boundary conditions, this is because such conditions allows more precise measurements needed in some cases.

We study the existence of nontrivial positive solution for the following second order three point boundary value problem:

$$u'' + g(t)f(u(t)) = 0, \quad 0 < t < 1 \tag{1}$$

$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta) \tag{2}$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}_+, g \in C[0,1]$, $f \in C([0,\infty[)$.

This study is motived by Il'in and Moiseev's results [2], on similar boundary value problems for certain linear ordinary differential equations, by Gupta's results in [7] for nonlinear ordinary differential equations and by the fact that existence of positive solutions of nonlinear multi-point boundary value problems has recently attracted particular attention from many authors and various methods including, among others, coincidence degree theory, Leray-Schauder fixed point alternative theorem are used. Also, inspired by [1,3,4,5,8-10], and under certain conditions on the nonlinearity of f and by using the well known Guo's fixed point theorem and the Green's function for this problem, we study the existence of positive solutions to problem (1)-(2).

This paper is organized as follows. In section 2 we present some preliminaries, give the Green's function for the problem (1)-(2), state some definitions then expose the Guo-Krasnosel'skii Theorem. In section 3 we give the main results and proofs.

2 Description of the problem and preliminaries Lemmas

Let $E = C[0,1]$, with supremum norm $||y|| = \max_{t\in[0,1]} |y(t)|$, $\forall\ y \in E$. Denote E^+ the set $\{x \in E, x(t) \geq 0, t \in [0,1]\}$.

Now we state two preliminary results.

Lemma 2.1 *Let $y \in E^+$. If $\beta \neq \alpha + 1$, then the three point BVP*

$$\begin{cases} u'' + g(t)f(y(t)) = 0, \quad 0 < t < 1 \\ u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta) \end{cases} \tag{3}$$

has a unique solution

$$\begin{aligned} u(t) \;&=\; -\int_0^t (t-s)\, g(s)f(y(s))ds + \frac{t+\alpha}{1+\alpha-\beta}\int_0^1 (1-s)\, g(s)f(y(s))ds - \\ &\beta\frac{t+\alpha}{1+\alpha-\beta}\int_0^\eta g(s)f(y(s))ds. \end{aligned}$$

Proof. The proof is easy, then we omit it.

Lemma 2.2 *Under the assumptions of Lemma 2.1, the solution of the problem (3) can be written as*

$$u(t) = \int_0^1 G(t,s)g(s)f(y(s)))ds$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{(1-\beta-t)(s+\alpha)}{1+\alpha-\beta}, & 0 \le s \le t < 1; 0 \le s \le \eta < 1 \\ \dfrac{(s+\alpha)(1-t)+\beta(t-s)}{1+\alpha-\beta}, & 0 < \eta \le s \le t \le 1 \\ \dfrac{(t+\alpha)(1-\beta-s)}{1+\alpha-\beta}, & 0 \le t \le s \le \eta < 1 \\ \dfrac{(t+\alpha)(1-s)}{1+\alpha-\beta}, & 0 \le t \le s < 1, 0 < \eta \le s < 1 \end{cases} \tag{4}$$

Proof. Let $t < \eta$, using Lemma 2.1, we get

$$\begin{aligned} u(t) &= \int_0^t \frac{(1-\beta-t)(s+\alpha)}{1+\alpha-\beta} g(s)f(y(s))ds \\ &+ \int_t^\eta \frac{(t+\alpha)(1-\beta-s)}{1+\alpha-\beta} g(s)f(y(s))ds \\ &+ \frac{t+\alpha}{1+\alpha-\beta}\int_\eta^1 (1-s)g(s)f(y(s))ds. \end{aligned} \tag{5}$$

Now if $t > \eta$, then

$$\begin{aligned} u(t) &= \int_0^\eta \left[(-t+s) + \frac{t+\alpha}{1+\alpha-\beta}(1-s) - \beta\frac{t+\alpha}{1+\alpha-\beta}\right] g(s)f(y(s))ds \\ &+ \int_\eta^t \left[(-t+s) + \frac{t+\alpha}{1+\alpha-\beta}(1-s)\right] g(s)f(y(s))ds \\ &+ \frac{t+\alpha}{1+\alpha-\beta}\int_t^1 (1-s)g(s)f(y(s))ds \end{aligned}$$

$$\begin{aligned} &= \int_0^\eta \frac{(1-\beta-t)(s+\alpha)}{1+\alpha-\beta} g(s)f(y(s))ds \\ &+ \int_\eta^t \frac{(s+\alpha)(1-t)+\beta(t-s)}{1+\alpha-\beta} g(s)f(y(s))ds \\ &+ \frac{t+\alpha}{1+\alpha-\beta}\int_t^1 (1-s)g(s)f(y(s))ds. \end{aligned} \tag{6}$$

From (5) and (6) we obtain $u(t) = \int_0^1 G(t,s)g(s)f(y(s)))ds$, where $G(t,s)$ is defined by (4).

Lemma 2.3 *For all* $t, s \in [0,1]$ *and if* $\beta \neq \alpha + 1$ *and* $0 < \beta \leq 1$, *then*

$$\begin{aligned} 0 &< k(\tau)G(s,s) \leq G(t,s) \leq \frac{2+\alpha}{\alpha}G(s,s) \\ & \textit{where} \\ k(\tau) &= \min\left\{\frac{\alpha}{(1+\alpha)}, 1-\beta-\tau\right\} \\ 0 &< \tau < 1-\beta. \end{aligned} \tag{7}$$

Proof. Let $t, s \in [0,1]$ such that $G(s,s) \neq 0$ then by using (4) it yields

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(1-\beta-t)(s+\alpha)}{(1-\beta-s)(s+\alpha)} \leq \frac{(1-\beta-t)}{(1-\beta-s)} \leq 1 \\ 0 &\leq s \leq t < 1; 0 \leq s \leq \eta < 1. \end{aligned}$$

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(s+\alpha)(1-t)+\beta(t-s)}{(s+\alpha)(1-s)} \leq \frac{2+\alpha}{\alpha} \\ 0 &< \eta \leq s \leq t \leq 1. \end{aligned}$$

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(t+\alpha)(1-\beta-s)}{(s+\alpha)(1-\beta-s)} \leq 1 \\ 0 &\leq t \leq s \leq \eta < 1. \end{aligned}$$

$$\frac{G(t,s)}{G(s,s)} = \frac{(t+\alpha)}{(s+\alpha)} \leq 1, \quad 0 \leq t \leq s < 1, 0 < \eta \leq s < 1$$

so, we get

$$\frac{G(t,s)}{G(s,s)} \leq \frac{2+\alpha}{\alpha}, \ \forall s, t \in [0,1].$$

Now we look for lower bounds of $G(t,s)$ for $0 \leq t \leq \tau < 1-\beta$.

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{1-\beta-t}{1-\beta-s} \geq \frac{1-\beta-\tau}{1-\beta} \geq (1-\beta-\tau) \\ 0 &\leq s \leq t \leq 1; \ 0 \leq s \leq \eta < 1. \end{aligned}$$

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(s+\alpha)(1-t)+\beta(t-s)}{(s+\alpha)(1-s)} \\ &\geq \frac{(1-t)}{(1-s)} \geq \frac{1-\tau}{1-\eta} \geq (1-\tau), \\ 0 &< \eta \leq s \leq t < 1. \end{aligned}$$

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(t+\alpha)}{(s+\alpha)} \geq \frac{\alpha}{(1+\alpha)} \\ 0 &\leq t \leq s \leq \eta < 1. \end{aligned}$$

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \frac{(t+\alpha)}{(s+\alpha)} \geq \frac{\alpha}{(1+\alpha)}\\ 0 &\leq t \leq s < 1,\ 0 < \eta \leq s < 1.\end{aligned}$$

Finally, if we put $k(\tau) = \min\left\{\frac{\alpha}{(1+\alpha)}, 1-\beta-\tau\right\}$, we get $\frac{G(t,s)}{G(s,s)} \geq k(\tau)$.

Lemma 2.4 *Under the assumptions of Lemma 2.3 and if $y \in E^+$, then the solution of problem (3) is nonnegative and satisfies*

$$\min_{t\in[0,\tau]} u(t) \geq \lambda \|u\|$$

where $\lambda = k(\tau)\frac{\alpha}{2+\alpha}$.

Proof. Applying the right hand side of inequality (7) we get

$$u(t) \leq \frac{2+\alpha}{\alpha}\int_0^1 G(s,s)g(s)f(y(s))ds,$$

then $\|u\| \leq \frac{2+\alpha}{\alpha}\int_0^1 G(s,s)g(s)f(y(s))ds$. Consequently

$$\int_0^1 G(s,s)g(s)f(y(s))ds \geq \frac{\alpha}{2+\alpha}\|u\|.$$

Taking into account the left hand side of inequality (7) we obtain for all $t \in [0,\tau]$

$$u(t) \geq k(\tau)\int_0^1 G(s,s)g(s)f(y(s))ds \geq k(\tau)\frac{\alpha}{2+\alpha}\|u\|.$$

Thus, $\min_{t\in[0,\tau]} u(t) \geq k(\tau)\frac{\alpha}{2+\alpha}\|u\| = \lambda\|u\|$.

Now we provide some background definitions.

Definition 2.5 *Let E be a Banach space. A nonempty closed convex subset $K \subset E$ is called a cone if it satisfies the following two conditions:*
(i) $x \in K$ and $\lambda \geq 0$ implies $\lambda x \in K$.
(ii) $x \in K$ and $-x \in K$ implies $x = 0$.

Definition 2.6 *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Now we state the well known Guo-Krasnosel'skii fixed point Theorem [6].

Theorem 2.7 *Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$$

be a completely continuous operator such that
(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$.
Then $\mathcal{A}$ has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

Definition 2.8 *A function $u(t)$ is called positive solution of problem (1)-(2) if* $u(t) \geq 0$, $\forall t \in [0, 1]$.

Define the operator T by

$$Tu(t) = \int_0^1 G(t, s)g(s)f(u(s))ds, \ \forall u \in C\,[0, 1] \, .$$

Lemma 2.9 *A function $u(t) \in E$ is a solution of the BVP (1)-(2) if and only if $Tu\,(t) = u(t)$.*

3 Main Results

In this section, we present and prove our main results, before that, we make the following assumptions
($\mathbf{I}_1$)$g \in C([0; 1]; [0; \infty))$ and there exists $x_0 \in [0, 1]$ such that $g(x_0) > 0$.
($\mathbf{I}_2$)$f \in C([0; \infty); [0; \infty))$ and there exists nonnegative constants a and A, such that

$$a = \lim_{u \to 0^+} \frac{f\,(u)}{u}, \qquad A = \lim_{u \to \infty} \frac{f\,(u)}{u}.$$

Remark. The case $a = 0$ and $A = \infty$ is called superlinear case and the case $a = \infty$ and $A = 0$ is called sublinear case.
The main result of this work is the following

Theorem 3.1 *Under the assumptions I_1 and I_2, the problem* (1)-(2) *has at least one positive solution in the both cases superlinear as well as sublinear.*

Proof. To prove this theorem we apply Guo-Krasnosel'skii fixed point Theorem (see Theorem 2.7). Denote

$$K = \left\{ y/y \in C\,[0, 1]\,,\ y \geq 0,\ \min_{t \in [0, \tau]} y\,(t) \geq \lambda\,||y|| \right\}$$

where λ is defined in Lemma 2.4. It is easy to check that K is a nonempty closed and convex subset of E and satisfies the two statements in Definition

2.5, so it is a cone. Using Lemma 2.4 we see that $TK \subset K$. Applying Ascoli Arzela Theorem we prove that T is completely continuous operator in E.

The superlinear case: Since $a = \lim\limits_{u \to 0^+} \dfrac{f(u)}{u} = 0$, then for any $\varepsilon > 0$, $\exists \delta_1 > 0$, such that $0 < y \leq \delta_1$ implies $f(y) \leq \varepsilon y$. Let Ω_1 be an open set in E defined by $||Tu|| \leq ||u||, \ \forall u \in K \cap \partial\Omega_1$. Then, for $u \in K \cap \partial\Omega_1$, it yields

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds \leq \frac{2+\alpha}{\alpha}\varepsilon ||u|| \int_0^1 G(s,s)g(s)ds. \tag{8}$$

Taking into account that $g(x_0) > 0$, we can choose ε such

$$\varepsilon \leq \frac{\alpha}{(2+\alpha)\int_0^1 G(s,s)g(s)ds} \tag{9}$$

The inequalities (8) and (9) imply that $||Tu|| \leq ||u||, \ \forall u \in K \cap \partial\Omega_1$. Now from $A = \lim\limits_{u \to \infty} \dfrac{f(u)}{u} = \infty$, we have $\forall M > 0$, $\exists H > 0$, such that $f(y) \geq My$ for $y \geq H$. Let $H_1 = \max\left\{2\delta_1, \frac{H}{\lambda}\right\}$ and denote by Ω_2 the open set $\{y \in E / \, ||y|| < H_1\}$. If $u \in K \cap \partial\Omega_2$ then

$$\min_{t \in [0,\tau]} u(t) \geq \lambda ||u|| = \lambda H_1 \geq H$$

and so

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)g(s)f(u(s))ds \geq k(\tau)\int_0^1 G(s,s)g(s)f(u(s))ds \\ &\geq k(\tau)M\,||u|| \int_0^1 G(s,s)g(s)ds. \end{aligned}$$

Let us choose M such that $M \geq \dfrac{1}{k(\tau)\int_0^1 G(s,s)g(s)ds}$, then we get $Tu(t) \geq ||u||$. Hence,

$$||Tu|| \geq ||u||, \ \forall u \in K \cap \partial\Omega_2.$$

By the first part of Theorem 2.7, T has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$ such that $H \leq ||u|| \leq H_1$. This completes the superlinear case of Theorem 3.1.

The sublinear case: Reasoning as in superlinear case, we obtain that the problem (1)-(2) has at least one positive solution. This achieves the proof of Theorem 3.1.

4 Open problem

In this work, we have assumed that f and g are continuous functions, it will be interesting to consider the same problem but with singularities.

As the conditions (2) are imposed for the first time, they can be introduced to study higher order multipoint boundary value problems.

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