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On Absolutely Almost Convergency of Higher Order of Orthogonal Series

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Abstract

In this paper two methods of summability of infinite numerical series are extended. In addition, certain sufficient conditions expressed in terms of coefficients of an orthogonal series are found which provide that an orthogonal series of functions from L^2 will be summable in the sense of such methods.

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1 Introduction and Preliminaries

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1}$$

where $a_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series, which we shall denote by a, with its partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

When $(p*q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p*q)_n} \sum_{m=0}^{n} p_{n-m} q_m s_m,$$

and we agree with $t_{-1}^{p,q} \equiv 0$.

The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely summable $(N, p, q)_k$ of order k, if for $k \geq 1$ the series

$$\sum_{n=0}^{\infty} n^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|_k.$$

We note that for $k = 1$, $|N, p, q|_k$ summability is the same as $|N, p, q|$ summability introduced by Tanaka [6].

Y. Okuyama [11] has found some sufficient conditions so that the series (1) is summable $|N, p, q|$ almost everywhere. In fact, he generalizes all results published in papers [9], [10]. Let us point out here that the present author [7] also has generalized all Okuyama's results considering $|N, p, q|_k$ summability of order k, $1 \leq k \leq 2$, instead of $|N, p, q|$ summability.

Lorentz [2] has proved that a sequence $v := \{s_n\}$ is almost convergent to a number u if and only if

$$d_{m,n} := d_{m,n}(v) := \frac{1}{m+1} \sum_{i=0}^{m} s_{n+i}$$

tends to the limit u as $m \to \infty$, uniformly in n; m, $n \geq 0$ (see also [5]).

The concept of absolute almost convergence was first introduced by Das, Kuttner, and Nanda [4].

Very recently Das and Ray [3], have extended the definition of $d_{m,n}$ for $m = -1$ setting

$$d_{-1,n} = d_{-1,n}(v) = s_{n-1}.$$

The series a is *absolute almost convergent* if the series

$$\sum_{m=1}^{\infty} |d_{m,n} - d_{m-1,n}|$$

converges uniformly in n.

Note that it was proved in [4] that the convergence of the above series for only one n implies convergence for any other values of n.

We generalize the concept of absolute almost convergence in the following manner:

The series a is *absolute almost convergent of order* k, $k \geq 1$ if the series

$$\sum_{m=1}^{\infty} m^{k-1}|d_{m,n} - d_{m-1,n}|^k$$

converges uniformly in n, and we shall denote in symbols $a \in |\mathcal{AAC}|_k$.

The $(\mathcal{Z}_p)$ method is defined when $d_{m,n}$ tends to a limit as $n \to \infty$ for fixed m. It is showed in [3] that $v \in (\mathcal{Z}_p)$ if and only if

$$\sum_{m=0}^{\infty}(s_m - s_{m+p})$$

converges for fixed $p \geq 1$; and $v \in |\mathcal{Z}_p|$ if and only if

$$\sum_{m=0}^{\infty} |s_m - s_{m+p}|$$

converges for fixed p (see also [1]).

Likewise, we say that $a \in |\mathcal{Z}_p|_\ell$, $\ell \geq 1$, if and only if

$$\sum_{m=0}^{\infty} m^{\ell-1}|s_m - s_{m+p}|^\ell$$

converges for fixed p.

Remark 1.1 *Should be noted that for* $k = 1$ *and* $\ell = 1$ *the concepts absolute almost convergency of order* k *and* $|\mathcal{Z}_p|_\ell$ *–summability reduce to the absolute almost convergency and* $|\mathcal{Z}_1|$ *–summability, respectively. Also it is obvious that inclusions* $|\mathcal{AAC}|_k \subseteq |\mathcal{AAC}|_1$ *and* $|\mathcal{Z}_p|_\ell \subseteq |\mathcal{Z}_p|_1$ *always hold.*

The main purpose of the present paper is to study the absolute almost convergency of order k of the orthogonal series (1), for $1 \leq k \leq 2$. Also, when the orthogonal series (1) is $|\mathcal{Z}_p|_\ell$ –summable, $1 \leq \ell \leq 2$, will be studied.

Throughout this paper K denotes a positive constant and it may be different in different relations.

2 Main Results

We prove the following theorem.

Theorem 2.1 *If for* $1 \leq k \leq 2$ *the series*

$$\sum_{m=1}^{\infty}\left(\frac{1}{m^{\frac{2}{k}}(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}}$$

converges uniformly in n*, then the orthogonal series*

$$\sum_{i=0}^{\infty} a_i\varphi_i(x) \in |\mathcal{AAC}|_k$$

almost everywhere.

Proof. Denote

$$\phi_{m,n}(x) := d_{m,n}(x) - d_{m-1,n}(x).$$

Then we have that

$$\begin{aligned}
\phi_{m,n}(x) &= \frac{1}{m+1}\sum_{i=0}^{m} s_{n+i}(x) - \frac{1}{m}\sum_{i=0}^{m-1} s_{n+i}(x) \\
&= \frac{ms_{n+m}(x) - (s_n(x) + \cdots + s_{n+m-1}(x))}{m(m+1)} \\
&= \frac{1}{m(m+1)}\sum_{\nu=1}^{m}\nu a_{n+\nu}\varphi_{n+\nu}(x)
\end{aligned}$$

where $s_r(x) = \sum_{j=0}^{r} a_j\varphi_j(x)$ are partial sums of order r of the series (1).

Using the Hölder's inequality and orthogonality to the latter equality, we have that

$$\begin{aligned}
\int_a^b |\phi_{m,n}(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}}\left(\int_a^b |d_{m,n}(x) - d_{m-1,n}(x)|^2 dx\right)^{\frac{k}{2}} \\
&= (b-a)^{1-\frac{k}{2}}\left(\frac{1}{m^2(m+1)^2}\int_a^b\left|\sum_{\nu=1}^{m}\nu a_{n+\nu}\varphi_{n+\nu}(x)\right|^2 dx\right)^{\frac{k}{2}} \\
&= (b-a)^{1-\frac{k}{2}}\left(\frac{1}{m^2(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}}.
\end{aligned}$$

Hence, the series

$$\sum_{m=1}^{\infty} m^{k-1}\int_a^b |\phi_{m,n}(x)|^k dx \leq (b-a)^{1-\frac{k}{2}}\sum_{m=1}^{\infty}\left(\frac{1}{m^{\frac{2}{k}}(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}} \tag{2}$$

converges uniformly in n by the assumption. From this fact, and since the functions $|\phi_{m,n}(x)|$ are non-negative then by the Lemma of Beppo-Lévi the series

$$\sum_{m=1}^{\infty} m^{k-1}|\phi_{m,n}(x)|^k$$

converges uniformly in n almost everywhere. The proof of theorem 2.1 is completed.

Corollary 2.2 *If the series*

$$\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\sqrt{\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2}$$

converges uniformly in n, then the orthogonal series

$$\sum_{i=0}^{\infty} a_i\varphi_i(x) \in |\mathcal{AAC}| \equiv |\mathcal{AAC}|_1$$

almost everywhere.

Now we shall prove a theorem which is more general than Theorem 2.1 under some additional conditions. Before doing this we put

$$\Lambda^{(k)}(\nu) := \frac{1}{\nu^{\frac{2}{k}-1}}\sum_{m=\nu}^{\infty}\left[\frac{m^{\frac{1}{k}}}{m(m+1)}\right]^2. \tag{3}$$

Theorem 2.3 *Let $1 \leq k \leq 2$ and $\{\Omega(m)\}$ be a positive sequence such that $\{\Omega(m)/m\}$ is a non-increasing sequence and the series $\sum_{m=1}^{\infty}\frac{1}{m\Omega(m)}$ converges. If the series*

$$\sum_{\nu=1}^{\infty}|a_{n+\nu}|^2\Omega^{\frac{2}{k}-1}(\nu)\Lambda^{(k)}(\nu)$$

converges, then the orthogonal series $\sum_{i=0}^{\infty} a_i\varphi_i(x) \in |\mathcal{AAC}|_k$ almost everywhere, where $\Lambda^{(k)}(\nu)$ is defined by (3).

Proof. Applying Hölder's inequality to the inequality (2) we get that

$$
\begin{aligned}
&\sum_{m=1}^{\infty} m^{k-1}\int_a^b |\phi_{m,n}(x)|^k dx \leq \\
&\qquad \leq K\sum_{m=1}^{\infty}\left(\frac{1}{m^{\frac{2}{k}}(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}} \\
&\qquad = K\sum_{m=1}^{\infty}\frac{1}{(m\Omega(m))^{\frac{2-k}{2}}}\left(\frac{\Omega^{\frac{2}{k}-1}(m)}{m(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}} \\
&\qquad \leq K\left(\sum_{m=1}^{\infty}\frac{1}{(m\Omega(m))}\right)^{\frac{2-k}{2}}\left(\sum_{m=1}^{\infty}\frac{\Omega^{\frac{2}{k}-1}(m)}{m(m+1)^2}\sum_{\nu=1}^{m}|\nu a_{n+\nu}|^2\right)^{\frac{k}{2}} \\
&\qquad \leq K\left(\sum_{\nu=1}^{\infty}|\nu a_{n+\nu}|^2\sum_{m=\nu}^{\infty}\frac{\Omega^{\frac{2}{k}-1}(m)}{m(m+1)^2}\right)^{\frac{k}{2}} \\
&\qquad \leq K\left\{\sum_{\nu=1}^{\infty}|\nu a_{n+\nu}|^2\left(\frac{\Omega(\nu)}{\nu}\right)^{\frac{2}{k}-1}\sum_{m=\nu}^{\infty}\left[\frac{m^{\frac{1}{k}}}{m(m+1)}\right]^2\right\}^{\frac{k}{2}} \\
&\qquad = K\left\{\sum_{\nu=1}^{\infty}|a_{n+\nu}|^2\Omega^{\frac{2}{k}-1}(\nu)\Lambda^{(k)}(\nu)\right\}^{\frac{k}{2}},
\end{aligned}
$$

which is finite by assumption, and this completes the proof.

The following theorems concerning $|\mathcal{Z}_p|_\ell$ –summability of orthogonal series (1) can be proved in a similar manner as we did above. This is why we shall omit the proofs.

Theorem 2.4 *If for $1 \leq \ell \leq 2$ the series*

$$\sum_{m=1}^{\infty}\left(m^{2-\frac{2}{\ell}}\sum_{i=1}^{p}|a_{m+i}|^2\right)^{\frac{\ell}{2}}$$

converges for fixed p, then the orthogonal series

$$\sum_{i=0}^{\infty} a_i\varphi_i(x) \in |\mathcal{Z}_p|_\ell$$

almost everywhere.

Theorem 2.5 *Let $1 \leq \ell \leq 2$, $\{\Omega(m)\}$ be a positive sequence, and the series $\sum_{m=1}^{\infty}\frac{1}{m\Omega(m)}$ converges. If the series*

$$\sum_{m=1}^{\infty} m\Omega^{\frac{2}{k}-1}(m)\sum_{i=1}^{p}|a_{m+i}|^2$$

converges for fixed p, then the orthogonal series $\sum_{i=0}^{\infty} a_i\varphi_i(x) \in |\mathcal{Z}_p|_\ell$ *almost everywhere.*

From theorem 2.4 as a special case for $p = 1$ and $\ell = 1$ we obtain:

Corollary 2.6 *If the series*

$$\sum_{m=2}^{\infty} |a_m|$$

converges, then the orthogonal series

$$\sum_{m=0}^{\infty} a_m\varphi_m(x) \in |\mathcal{Z}_1|$$

almost everywhere.

Remark 2.7 *It is showed in [3] that* $|\mathcal{AAC}| \subset |C,1|$ *and* $|\mathcal{AAC}| \not\subseteq |N|$*, but* $|\mathcal{AAC}| \not\subseteq |C,\alpha|$*, for* $0 < \alpha < 1$*, where* $|C,1|$ *and* $|N|$ *denote the absolute Cesàro and harmonic methods of summability respectively. Besides, the* $|\mathcal{Z}_p|$*,* $p \geq 2$*, method is not comparable with methods* $|C,\alpha|$*, for* $0 < \alpha \leq 1$*,* $|N|$*, and* $|\mathcal{AAC}|$*. These facts provide that results obtained in this paper are new.*

3 Open Problem

The definition of $d_{m,n}$ is generalized in the following way [8]:

A sequence $v := \{s_n\}$ is said to be almost matrix summable to ξ provided

$$D_{m,n} := t_{m,n}(v) := \sum_{i=0}^{m} a_{m,m-i}s_{m-i,n}$$

tends to the limit ξ uniformly in n, where

$$s_{m-i,n} = \frac{1}{m-i+1}\sum_{j=n}^{m-i+n} s_j$$

and $(a_{n,k})$ is an infinite triangular matrix that satisfies the well-known Silverman-Teoplitz condition of regularity.

We say that the series a is *absolute almost matrix summable of order* k*,* $k \geq 1$ if the series

$$\sum_{m=1}^{\infty} m^{k-1}|D_{m,n} - D_{m-1,n}|^k$$

converges uniformly in n.

We finalize the paper with the following open question: Under what condition the series (1) is absolute almost matrix summable of order k, $1 \leq k \leq 2$.

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