Int. J. Open Problems Compt. Math., Vol. 4, No. 1, March 2011
ISSN 1998-6262; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

Note On Two New Wilker-Type Inequalities

Ladislav Matejíčka

Faculty of Industrial Technologies in Púchov, TnUAD, Slovakia
e-mail:matejicka@tnuni.sk

Abstract

In this paper, we give answer to an open problem posed in the paper [Ling Zhu, Some New Wilker-Type inequalities for Circular and Hyperbolic Functions, Abstract and Applied Analysis, Volume 2009, Article ID 485842, 9 pages].

Keywords: *Bernoulli number, power series, Wilker-type inequality in exponential form.*

2000 Mathematics Subject Classification: *26D15.*
[1]

1 Introduction

In the paper [2], Ling Zhu has posted the following open problem: find the respective largest range of α such that the inequalities (1) and (2) hold.

Theorem 1.1 *(see [2]) Let $0 < x < \pi/2$ and $\alpha \geq 1$. Then the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^{\alpha} > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} > 2 \tag{1}$$

holds.

Theorem 1.2 *(see [2]) Let $x > 0$ and $\alpha \geq 1$. Then the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} > 2 \tag{2}$$

holds.

[1]This work was supported by Kega no. 3/7414/09.

There are a lot of papers (see [2]) devoted to Wilker's inequality:

Problem 1.3 *Let* $0 < x < \pi/2$. *Then*

$$\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right) > 2 \tag{3}$$

holds.

Zhu (see [2]) in his paper obtained four new Wilker-type inequalities in exponential form for circular and hyperbolic functions [(1), (2), (4), (5)].

Theorem 1.4 *(see [2]) Let* $0 < x < \pi/2$ *and* $\alpha \in R$, *and* $\alpha \neq 0$. *Then*

(*i*) *when* $\alpha > 0$, *the inequality*

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^{\alpha} > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} \quad \textit{holds.} \tag{4}$$

(*ii*) *when* $\alpha < 0$, *inequality* (4) *is revered.*

Theorem 1.5 *(see [2]) Let* $x > 0$, $\alpha \in R$, *and* $\alpha \neq 0$. *Then*

(*i*) *when* $\alpha > 0$, *the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^{\alpha} > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} \quad \textit{holds.} \tag{5}$$

(*ii*) *when* $\alpha < 0$, *inequality* (5) *is revered.*

2 Main Results

In this paper we prove two lemmas.

Lemma 2.1 *Let* $0 < x < \pi/2$ *and* $\alpha \geq \alpha_0 = \ln 2/\left(2(\ln \pi - \ln 2)\right)$. *Then*

$$\left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^{\alpha} > 2 \tag{6}$$

holds.
Furthermore, $\alpha_0 = \ln 2/\left(2(\ln \pi - \ln 2)\right)$ *is the best constant in* (6).

Lemma 2.2 *Let* $x > 0$ *and* $\alpha \geq \alpha_1 = 0.6$. *Then the inequality*

$$\left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^{\alpha} > 2 \tag{7}$$

holds.
Furthermore, $\alpha_1 = 0.6$ *is the best constant in* (7).

3 Proof of Lemmas

The following lemmas are necessary.

Lemma 3.1 *(see [2]). Let $f, g : [a, b] \to R$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions $(f(x) - f(b))/(g(x) - g(b))$ and $(f(x) - f(a))/(g(x) - g(a))$ are also increasing (or decreasing) on (a, b).*

Lemma 3.2 *(see [2]). Let a_n and b_n $(n = 0, 1, 2, ...)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, ...$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, ...$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Proof of Lemma 2.1. Denote $H(\alpha, x) = (x/\sin x)^{2\alpha} + (x/\tan x)^{\alpha} - 2$ for $\alpha \geq \alpha_0$, $x \in (0, \pi/2)$. By a simple computation we obtain

$$\frac{\partial H(\alpha, x)}{\partial x} = 2\alpha \left(\frac{x}{\sin x}\right)^{2\alpha-1} \frac{(\sin x - x\cos x)}{\sin^2 x} + \alpha \left(\frac{x}{\tan x}\right)^{\alpha-1} \frac{(\sin x \cos x - x)}{\sin^2 x}.$$

Let α is an arbitrary fixed value such that $1 > \alpha \geq \alpha_0$. Denote $H_\alpha(x) = H(\alpha, x)$ for x $\in (0, \pi/2)$. It is evident that $H'_\alpha(x) = \partial H(\alpha, x)/\partial x \geq 0$ on $(0, \pi/2)$ iff

$$\alpha \geq \frac{\ln\left(\frac{x - \sin x \cos x}{2\cos x(\sin x - x\cos x)}\right)}{\ln\left(\frac{2x}{\sin 2x}\right)}, \quad x \in (0, \pi/2).$$

If we show that there is only one $x_\alpha \in (0, \pi/2)$ such that $H'_\alpha(x_\alpha) = 0$ and $H'_\alpha(x) > 0$ for $x \in (0, x_\alpha)$, the proof will be complete. Really, it implies

$$H_\alpha(x) > \min\left\{\lim_{x\to 0^+} H_\alpha(x), \lim_{x\to \frac{\pi}{2}^-} H_\alpha(x)\right\} \quad \textit{for all } x \in (0, \pi/2).$$

$\lim_{x\to\frac{\pi}{2}^-} H_\alpha(x) = (\pi^2/4)^\alpha - 2$ gives $\lim_{x\to\frac{\pi}{2}^-} H_\alpha(x) > 0$ for $1 > \alpha > \alpha_0$. It implies $H_\alpha(x) > 0$ for $x \in (0, \pi/2)$ because of $\lim_{x\to 0^+} H_\alpha(x) = 0$. If $\alpha < \alpha_0$ then $\lim_{x\to\frac{\pi}{2}^-} H_\alpha(x) < 0$. From this and from the continuity of $H_\alpha(x)$ we get that α_0 is the best constant.

To prove the existence of x_α it suffices to show that $f(x)$ is an increasing function for $x \in (0, \pi/2)$, where

$$f(x) = \frac{\ln\left(\frac{x - \sin x \cos x}{2\cos x(\sin x - x\cos x)}\right)}{\ln\left(\frac{2x}{\sin 2x}\right)}.$$

Really, from $f(0.1) = 0.6 < \alpha_0 = 0.767464267$, $\lim_{x\to\pi/2^-} f(x) = 1$ and from the continuity of $f(x)$ on $(0, \pi/2)$ we obtain that for each α such that $\alpha_0 \leq \alpha < 1$ there is only one $x_\alpha \in (0, \pi/2)$ such that $H'_\alpha(x_0) = 0$ and $H'_\alpha(x) > 0$ for $x \in (0, x_\alpha)$.

To prove $f(x)$ is an increasing function on $(0, \pi/2)$ it suffices to show that $p'(x)/q'(x)$ is an increasing function on $(0, \pi/2)$, where

$$p(x) = \ln\left(\frac{x - \sin x\cos x}{2\cos x(\sin x - x\cos x)}\right) \ \textit{and}\ q(x) = \ln\left(\frac{2x}{\sin 2x}\right).$$

It follows from Lemma 3.1 (see [2]), $\lim_{x\to 0^+} 2x/\sin 2x = 1$, and from $\lim_{x\to 0^+}(x - \sin x\cos x)/(2\cos x(\sin x - x\cos x)) = 1$. Direct calculation yields

$$\frac{p'(x)}{q'(x)} = \frac{(\sin^2 x\cos x - 2x^2\cos x + x\sin x)}{(x - \sin x\cos x)(\sin x - x\cos x)} \frac{x\sin^2 x}{(\sin x\cos x - x\cos^2 x + x\sin^2 x)}.$$

Denote

$$G(x) = \frac{(\sin^2 x\cos x - 2x^2\cos x + x\sin x)}{(x - \sin x\cos x)(\sin x - x\cos x)},$$

$$R(x) = \frac{x\sin^2 x}{(\sin x\cos x - x\cos^2 x + x\sin^2 x)}.$$

$G(x)$ is a positive increasing function on $(0, \pi/2)$. (See the proof in the paper [2], page 6.) From

$$R'(x) = \frac{\sin x(\sin^3 x\cos x + x\sin x - 2x^2\cos x)}{(\sin x\cos x - x\cos^2 x + x\sin^2 x)^2} =$$

$$\frac{x^2\sin x\cos x}{(\sin x\cos x - x\cos^2 x + x\sin^2 x)^2}\left(\frac{\sin^2 x}{x^2} + \frac{\tan x}{x} - 2\right) > 0$$

for $x \in (0, \pi/2)$ we have $R(x)$ is a positive increasing function. It implies that $p'(x)/q'(x) = R(x)G(x)$ is an increasing function. The proof of Lemma 2.1 is completed.

Proof of Lemma 2.2 Denote $J(\alpha, x) = (x/\sinh x)^{2\alpha} + (x/\tanh x)^\alpha - 2$ for $\alpha \geq \alpha_1$, $x > 0$. Direct computation results

$$\frac{\partial J(\alpha, x)}{\partial x} =$$

$$2\alpha\left(\frac{x}{\sinh x}\right)^{2\alpha-1}\left(\frac{\sinh x - x\cosh x}{\sinh^2 x}\right) + \alpha\left(\frac{x}{\tanh x}\right)^{\alpha-1}\left(\frac{\sinh x\cosh x - x}{\sinh^2 x}\right).$$

Let α is an arbitrary fixed value such that $1 > \alpha \geq \alpha_1$. Denote $J_\alpha(x) = J(\alpha, x)$ for $x > 0$. If we show that $J'_\alpha(x) = \partial J(\alpha, x)/\partial x > 0$, the proof will be complete because of $\lim_{x\to 0^+} J_\alpha(x) = 0$. Elementary calculation gives that $J'_\alpha(x) > 0$ on $(0,\infty)$ iff

$$\left(\frac{x}{\tanh x}\right)^{\alpha-1} (\sinh x\cosh x - x) > 2\left(\frac{x}{\sinh x}\right)^{2\alpha-1} (x\cosh x - \sinh x)\,. \qquad (8)$$

From $\sinh x\cosh x - x > 0$ and $x\cosh x - \sinh x > 0$, we obtain that (8) is equivalent to

$$\alpha > \frac{\ln\left(\frac{\sinh x\cosh x - x}{2\cosh x(x\cosh x - \sinh x)}\right)}{\ln\left(\frac{2x}{\sinh 2x}\right)}. \qquad (9)$$

Denote

$$g(x) = \frac{\ln\left(\frac{\sinh x\cosh x - x}{2\cosh x(x\cosh x - \sinh x)}\right)}{\ln\left(\frac{2x}{\sinh 2x}\right)}, \qquad x > 0.$$

We show that $g(x)$ is a decreasing function and $\lim_{x\to 0^+} g(x) = 0.6$. Cusa-Huygeus' inequality $(\sinh x)/x < 2/3 + (\cosh x)/3$ implies

$(\sinh x\cosh x - x)/(2\cosh x(x\cosh x - \sinh x)) < 1$. From this and from $2x < \sinh(2x)$ we get that $g(x)$ is a positive function. Using

$$\lim_{x\to 0^+} \frac{\sinh x\cosh x - x}{2\cosh x(x\cosh x - \sinh x)} = 1, \qquad \lim_{x\to 0^+} \frac{2x}{\sinh 2x} = 1$$

and Lemma 3.1 [2] we obtain that if

$$gg(x) = \frac{\left(\ln\left(\frac{\sinh x\cosh x - x}{2\cosh x(x\cosh x - \sinh x)}\right)\right)'}{\left(\ln\left(\frac{2x}{\sinh 2x}\right)\right)'}$$

is a decreasing function, then $g(x)$ is a decreasing function. Straightforward computation leads to

$$\left(\ln\left(\frac{\sinh x\cosh x - x}{2\cosh x(x\cosh x - \sinh x)}\right)\right)' =$$

$$\frac{2\sinh(2x) + 4x - \sinh(4x) + 8x^2\sinh(2x) - 4x\cosh(2x)}{(\sinh(2x) - 2x)(2x + 2x\cosh(2x) - 2\sinh(2x))}$$

and to

$$\left(\ln\left(\frac{2x}{\sinh 2x}\right)\right)' = \frac{\sinh(2x) - 2x\cosh(2x)}{x\sinh(2x)}.$$

From this $gg(x) = r(x)s(x)$, where

$$r(x) = \frac{x\sinh^2 x}{2x\cosh^2 x - x\sinh x\cosh x},$$

$$s(x) = \frac{x \sinh x + \sinh^2 x \cosh x - 2x^2 \cosh x}{(\sinh x \cosh x - x)(x \cosh - \sinh x)}.$$

It is evident that $2x \cosh^2 x - x \sinh x \cosh x > 0$ and $x \sinh x + \sinh^2 x \cosh x - 2x^2 \cosh x > 0$ for $x > 0$. We show that $r(x)$, $s(x)$ are decreasing functions. It implies that $gg(x) = r(x)s(x)$ is a decreasing function. By a simple computation we obtain

$$r'(x) = \frac{rr(x)}{(2x \cosh^2 x - x \sinh x \cosh x)^2},$$

where

$$rr(x) = x^2 \sinh x \cosh x \left(2 - \frac{\tanh x}{x} - \frac{\sinh^2 x}{x^2}\right) < 0.$$

So, $r(x)$ is a decreasing function. We note that $\lim_{x \to 0^+} r(x) = 3/4$. Now we prove that $s(x)$ is a decreasing function and $\lim_{x \to 0^+} s(x) = 8/10$. After rewriting $s(x)$ and using elementary formulas we obtain

$$s(x) = \frac{4x \sinh x + \cosh(3x) - (1 + 8x^2) \cosh x}{x \sinh(3x) - \cosh(3x) + 5x \sinh x + (1 - 4x^2) \cosh x}.$$

From the following power series expansions $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, $\sinh x = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$, we get

$$s(x) = \frac{4x \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} + \sum\limits_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} - (1 + 8x^2) \sum\limits_{n=0}^{\infty} \frac{(x)^{2n}}{(2n)!}}{x \sum\limits_{n=0}^{\infty} \frac{(3x)^{2n+1}}{(2n+1)!} - \sum\limits_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} + 5x \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} + (1 - 4x^2) \sum\limits_{n=0}^{\infty} \frac{(x)^{2n}}{(2n)!}}.$$

Direct computation leads to $s(x) = \sum_{n=1}^{\infty} a_n x^{2n} / \sum_{n=1}^{\infty} b_n x^{2n}$, where

$$a_n = \frac{3^{2n} + 24n - 32n^2 - 1}{(2n)!} \qquad b_n = \frac{3^{2n-1}(2n - 3) + 18n - 16n^2 + 1}{(2n)!}.$$

It implies that $\lim_{x \to 0^+} s(x) = a_3/b_3 = 512/640 = 8/10$ and so we have $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} gg(x) = \lim_{x \to 0^+} s(x) \lim_{x \to 0^+} r(x) = 6/10$. We show that $\{a_n/b_n\}_{n=3}^{\infty}$ is a decreasing sequence. Then $s(x)$ will be a decreasing function for $x > 0$ (see Lemma 3.2). To show this, it suffices to prove that the function

$$\frac{p(x)}{q(x)} = \frac{9^x + 24x - 32x^2 - 1}{9^x(\frac{2}{3}x - 1) + 18x - 16x^2 + 1}$$

is decreasing on $(2, \infty)$. Because of $p(2) = q(2) = 0$, it suffices to prove that

$$t(x) = \frac{p'(x)}{q'(x)} = \frac{9^x \ln 9 + 24 - 62x}{9^x \left(\frac{2 \ln 9}{3} x + \frac{2}{3} - \ln 9\right) + 18 - 32x}$$

is decreasing on $(2, \infty)$ (see Lemma 3.1). By a simple computation we obtain that

$$t'(x) = \frac{tt(x)}{\left(9^x \left(\frac{2\ln 9}{3}x + \frac{2}{3} - \ln 9\right) + 18 - 32x\right)^2},$$

where

$$tt(x) = -\frac{2}{3}(\ln^2 9)9^{2x} + 9^x \left(\frac{128}{3}(\ln^2 9)x^2 - x\left(112\ln^2 9 - \frac{128}{3}\ln 9\right) + \right.$$
$$\left. 42\ln^2 9 + 64\ln 9 - \frac{128}{3}\right) - 384.$$

Denote $pp(x) = (3)9^x - 206x^2 + 446.9x + 301$. From $pp'(x) = 3(\ln 9)9^x - 412x + 446.9$, $pp''(x) = 3(\ln^2 9)9^x - 412$, $pp''(2) = 761.1544$, $pp'(2) = 156.8256$, $pp(2) = 613.8$ we get $pp(x) > 0$ for $x \in < 2, \infty)$. Because $-2\ln^2 9/3 < -3.2$, $128\ln^2 9/3 < 206$, $112\ln^2 9 - 128\ln 9/3 > 446.9$, $42\ln^2 9 + 64\ln 9 - 128/3 < 301$ we have $tt(x) < -3.2(9^{2x}) + 9^x(206x^2 - 446.9x - 301) - 384 < -0.2(9^{2x}) - 384 < 0$ for $x \geq 2$. We note that, if $\alpha < \alpha_1 = 0.6$ then there is $\varepsilon_{\alpha_1} > 0$ such that $J'_\alpha(x) < 0$ for $x \in (0, \varepsilon_{\alpha_1})$. Because $\lim_{x\to 0^+} J_\alpha(x) = 0$, we have that $J_\alpha(x) < 0$ on $(0, \varepsilon_{\alpha_1})$ and the inequality (3) does not hold on $(0, \varepsilon_{\alpha_1})$. The proof is complete.

4 Open Problem

It has already been proved a lot of other inequalities for circular and hyperbolic functions in exponential form. Some of them are valid for larger sets of their exponents as they were defined. So, there is a question of finding the largest ranges of their exponents.

ACKNOWLEDGEMENTS. The author thanks the referee and editor for useful comments.

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