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Boundedness for Multilinear Commutator of Littlewood-Paley Operator on Hardy and Herz-Hardy Spaces

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Abstract

In this paper,the $(H^p_{\vec{b}}, L^q)$ ***and*** $(H\dot{K}^{\alpha,p}_{q_1,\vec{b}}, \dot{K}^{\alpha,p}_{q_2})$ ***type boundedness for the multilinear commutator associated with the Littlewood-Paley operator are obtained.***

Keywords: *Littlewood-paley operator, Multilinear commutator, BMO, Hardy space, Herz-Hardy space.*

1 Introduction and definition

Let T be the Calderón-Zygmund operator and $b \in BMO(R^n)$. Then we can define the commutator $[b, T]$ generated by b and T as follows,

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [2] Coifman, Rochberg and Weiss prove the boundedness of the commutator $[b, T]$ on $L^p(R^n)$ $(1 < p < \infty)$. However, it is well known that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if a suitable atomic space $H^P_{\vec{b}}(R^n)$ or $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ substituted for $H^p(R^n)$, then $[b, T]$ maps continuously $H^P_{\vec{b}}(R^n)$ into $L^p(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ into $\dot{K}^{\alpha,p}_q$. Moreover, it was observed that $H^p_{\vec{b}}(R^n) \subset H^p(R^n)$, $\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n) \subset H\dot{K}^{\alpha,p}_q(R^n)$. In recent years, the theory of Herz type Hardy spaces have been developed.In this paper, we will establish the continuity of the multilinear commutators related to the Littlewood-Paley operators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces.

At first, let us introduce some definitions (see [1][3-9][11][12]). Suppose a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \backslash \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 1. Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \le 1$. A bounded measurable function a on R^n is said a $(p, \vec{b})$ atom, if

(1) supp $a \subset B = B(x_0, r)$

(2) $||a||_{L^\infty} \le |B|^{-1/p}$

(3) $\int_B a(y)dy = \int_B a(y) \prod_{l \in \sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \le j \le m$.

We say that a temperate distribution f belongs to $H_{\vec{b}}^p(R^n)$,if in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x).$$

where $a_j' s$ are $(p, \vec{b})$ atoms, $\lambda \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H_{\vec{b}}^p(R^n)} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Definition 2. Let $0 < p, q < \infty,\ \alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_0||_{L^q}^p \right]^{1/p}.$$

Definition 3. Let $\alpha \in R,\ 1 < q < \infty,\ \alpha \geq n(1-\frac{1}{q}),\ \ b_i \in BMO(R^n),\ 1 \leq i \leq m$. A function $a(x)$ is called a central $(\alpha, q, \vec{b})$ -atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type), if

(1) supp $a \in B = B(x_0, r)$(or for some $r \geq 1$),

(2) $||a||_{L^q} \leq |B|^{-\alpha/n}$

(3) $\int_B a(x)x^\beta dx = \int_B a(x)x^\beta \prod_{i\in\sigma} b_i(x)dx = 0$ for any $\sigma \in C_j^m$ $, 1 \leq j \leq m$.

We say that a temperate distribution f belongs to $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$(or $HK^{\alpha,p}_{q,\vec{b}}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, \vec{b})$-atom(or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}(\text{ or } ||f||_{HK^{\alpha,p}_{q,\vec{b}}}) = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

where the infimum are taken over all the decompositions of f as above.

Definition 4. Fix $\delta > 0$ and let ψ be a function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,

(3) $|\psi(x + y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g^{\vec{b}}_{\mu,\delta}(f)(x) = \left[\int\int_{R^{n+1}_+} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F^{\vec{b}}_t(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F^{\vec{b}}_t(f)(x, y) = \int_{R^n} \left[\prod_{j=1}^{m}(b_j(x) - b_j(z))\right] \psi_t(y - z)f(z)dz;$$

When $m = 1$, set

$$g^{b}_{\mu,\delta}(f)(x) = \left[\int\int_{R^{n+1}_+} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F^{b}_t(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F^b_t(f)(x, y) = \int_{R^n} (b(x) - b(z))\psi_t(y - z)f(z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define that

$$g_{\mu,\delta}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley function (see [14]).

2 Main results

We begin with two preliminaries lemmas.

Lemma 1. Let $1 < r < \infty$, $b_j \in BMO$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le C \prod_{j=1}^k ||b_j||_{BMO}.$$

Lemma 2. Let $0 < \delta < n$, $1 < s < n/\delta$ and $1/r = 1/s - \delta/n$. Then $g^{\vec{b}}_{\mu,\delta}$ is bounded from $L^s(R^n)$ to $L^r(R^n)$.

Theorem 1. Let $\mu > 3+4/n-2\delta/n$, $n/(n+1-\delta) < q \le 1, 1/q = 1/p-\delta/n, b_i \in BMO$, $1 \le i \le m, \vec{b} = (b_1, \cdots, b_m)$. Then the multilinear commutator $g^{\vec{b}}_{\mu,\delta}$ is bounded from $H^p_{\vec{b}}(R^n)$ to $L^q(R^n)$.

Proof. It suffices to show that there exist a constant $C > 0$, such that for every $(p, \vec{b})$ atom a,

$$||g^{\vec{b}}_{\mu,\delta}(a)||_{L^q} \le C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, 2r)$. When $m = 1$ see [6], and now we assume $m > 1$. Write

$$\int_{R^n} |g^{\vec{b}}_{\mu,\delta}(a)(x)|^q dx = \int_{|x-x_0|\le 2r} |g^{\vec{b}}_{\mu,\delta}(a)(x)|^q dx + \int_{|x-x_0|>2r} |g^{\vec{b}}_{\mu,\delta}(a)(x)|^q dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\delta$ and $1/r = 1/s - \delta/n$, by Hölder's inequality and the (L^s, L^r)-boundedness of $g^{\vec{b}}_{\mu,\delta}$, we get

$$I \le C||g^{\vec{b}}_{\mu,\delta}(a)||^q_{L^r}|B(x_0, 2r)|^{1-q/r} \le C||a||^q_{L^s}|B|^{1-q/r} \le C|B|^{-q/p+q/s+1-q/r} \le C.$$

For II, without loss of generality, we may assume $q \leq 1$. Set $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i = (b_i)_B$, $1 \leq i \leq m$, where $(b_i)_B = \frac{1}{|B(x_0,2r)|}\int_{B(x_0,2r)} b_i(x)dx$, by Hölder's inequality and the vanishing moment of a, we get

$$
\begin{aligned}
& II = \sum_{k=1}^{\infty} \int_{2^{k+1}B \backslash 2^k B} |g_{\mu,\delta}^{\vec{b}}(a)(x)|^q dx \\
\leq \ & C \sum_{k=1}^{\infty} |2^{k+1}B|^{1-q} \left(\int_{2^{k+1}B \backslash 2^k B} |g_{\mu,\delta}^{\vec{b}}(a)(x)| dx \right)^q \\
\leq \ & C \sum_{k=1}^{\infty} |2^{k+1}B|^{1-q} \left[\int_{2^{k+1}B \backslash 2^k B} \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \right. \right. \\
& \left. \left. \times \left(\int_B |\psi_t(y-z) - \psi_t(y)| \prod_{j=1}^m |b_j(x) - b_j(z)||a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dx \right]^q,
\end{aligned}
$$

noting that $z \in B, y \in 2^{k+1}B \backslash 2^k B$, then

$$
\begin{aligned}
& \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \right. \\
& \left. \times \left(\int_B |\psi_t(y-z) - \psi_t(y)| \prod_{j=1}^m |b_j(x) - b_j(z)||a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
\leq \ & C \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \right. \\
& \left. \times \left(\int_B t^{-n+\delta} |a(z)| \prod_{j=1}^m |b_j(x) - b_j(z)| \frac{|z|/t}{(1+|y|/t)^{n+2-\delta}} dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
\leq \ & C \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{t^{1-n} dydt}{(t+|y|)^{2(n+2-\delta)}} \right)^{1/2} \\
& . \times \left(\int_B \prod_{j=1}^m |b_j(x) - b_j(z)||z||a(z)| dz \right),
\end{aligned}
$$

set $B' = B'(x,t)$, we have

$$\begin{aligned}
& t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+2-\delta)}} \\
\leq\ & t^{-n}\left(\int_{B'}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+2-\delta)}}.\right. \\
& \left.+\sum_{k=1}^{\infty}\int_{2^kB'\backslash 2^{k-1}B'}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y|)^{2(n+2-\delta)}}\right) \\
\leq\ & Ct^{-n}\left(t^n+\sum_{k=1}^{\infty}2^{-kn\mu}2^{2k(n+2-\delta)}(2^kt)^n\right)\frac{1}{(t+|x|)^{2(n+2-\delta)}} \\
\leq\ & C\left(1+\sum_{k=1}^{\infty}2^{k(-n\mu+3n+4-2\delta)}\right)\frac{1}{(t+|x|)^{2(n+2-\delta)}} \\
\leq\ & C\frac{1}{(t+|x|)^{2(n+2-\delta)}},
\end{aligned}$$

it is easy to calculate that

$$\int_0^{\infty}\frac{tdt}{(t+|x|)^{2(n+2-\delta)}} = C|x|^{-2(n+1-\delta)}.$$

So

$$\begin{aligned}
II \ \leq\ & C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-q} \\
& \times\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{-(n+1-\delta)}\left(\int_B\prod_{j=1}^m|b_j(x)-b_j(z)||z||a(z)|dz\right)dx\right]^q \\
\leq\ & C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\left(\int_B|(\vec{b}(z)-\lambda)_{\sigma^c}||z||a(z)|dz\right)^q \\
& \times\sum_{k=1}^{\infty}|2^{k+1}B|^{1-q}\left[\int_{2^{k+1}B\backslash 2^kB}|x|^{-(n+1-\delta)}|(\vec{b}(x)-\lambda)_\sigma|dx\right]^q \\
\leq\ & C||\vec{b}||_{BMO}^q\sum_{k=1}^{\infty}k^q\cdot 2^{-knq(1+1/n-1/p)} \\
\leq\ & C||\vec{b}||_{BMO}^q.
\end{aligned}$$

This completes the proof of Theorem 1.

Theorem 2. Let $0 < \delta < n$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$,$1/q_1 - 1/q_2 = \delta/n$, $n(1-1/q_1) \le \alpha < n(1-1/q_1)+1$ and $b_i \in BMO(R^n), 1 \le i \le m$, $\vec{b} = (b_1, \cdots, b_m)$. Then $g^{\vec{b}}_{\mu,\delta}$ is bounded from $H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(R^n)$ to $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Proof. Let $f \in H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3, we write

$$
\begin{aligned}
&||g^{\vec{b}}_{\mu,\delta}(f)(x)||_{\dot{K}^{\alpha,p}_{q_2}} \\
\le\ & C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}}\Big)^p\right]^{1/p} \\
\le\ & C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j|||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}}\Big)^p\right]^{1/p} \\
& + C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-2}^{\infty} |\lambda_j|||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}}\Big)^p\right]^{1/p} \\
=\ & I + II.
\end{aligned}
$$

For II, by the (L^{q_1}, L^{q_2})-boundedness of $g^{\vec{b}}_{\mu,\delta}$ and the Hölder's inequality, we have

$$
\begin{aligned}
II \le\ & C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|||g^{\vec{b}}_{\mu,\delta}(a_j)||_{L^{q_2}}\right)^p\right]^{1/p} \\
\le\ & C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|||a_j||_{L^{q_1}}\right)^p\right]^{1/p} \\
\le\ & C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| \cdot 2^{-j\alpha}\right)^p\right]^{1/p} \\
\le\ & C\begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p}, & 0 < p \le 1 \\ \Big[\sum_{j=-\infty}^{\infty} |\lambda_j|^p (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}) & \\ \quad \times (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2})^{p/p'}\Big]^{1/p}, & 1 < p < \infty \end{cases} \\
\le\ & C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \\
\le\ & C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.
\end{aligned}
$$

For I, when m=1, let $C_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{C_k}$, $b^i_j = |B_j|^{-1}\int_{B_j} b_i(x)dx$,

$1 \le i \le m, \vec{b'} = (b_j^1, \cdots, b_j^m)$. Similar to the proof of II in Theorem 1, we have

$$\begin{aligned} & g_{\mu,\delta}^{b_1}(a_j)(x) \\ \le\ & C\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\right. \\ & .\times\left.\left(\int_{B_j} t^{-n+\delta}|a_j(z)||b_1(x)-b_1(z)|\frac{|z|/t}{(1+|y|/t)^{n+2-\delta}}dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ \le\ & C\left(\int_0^\infty\frac{tdt}{(t+|x|)^{2(n+2-\delta)}}\right)^{1/2}\left(\int_{B_j}|b_1(x)-b_1(z)||z||a_j(z)|dz\right) \\ \le\ & C|x|^{-(n+1-\delta)}\int_{B_j}|z||a_j(z)||b_1(x)-b_1(z)|dz \\ \le\ & C|x|^{-(n+1-\delta)}\left(\int_{B_j}|z||a_j(z)||b_1(x)-b_j^1|dz+\int_{B_j}|z||a_j(z)||b_1(z)-b_j^1|dz\right) \\ \le\ & C|x|^{-(n+1-\delta)}\left(|b_1(x)-b_j^1|2^{j(1+n(1-1/q_1)-\alpha)}+2^{j(1+n(1-1/q_1)-\alpha)}||b_1||_{BMO}\right). \end{aligned}$$

Then

$$\begin{aligned} & ||g_{\mu,\delta}^{b_1}(a_j)\chi_k||_{L_{q_2}} \\ \le\ & C2^{j(1+n(1-1/q_1)-\alpha)}\left[\left(\int_{B_k}|x|^{-q_2(n+1-\delta)}|b_1(x)-b_j^1|^{q_2}dx\right)^{1/q_2}\right. \\ & \left.+\left(\int_{B_k}|x|^{-q_2(n+1-\delta)}dx\right)^{1/q_2}||b_1||_{BMO}\right] \\ \le\ & C||b_1||_{BMO}2^{j(1+n(1-1/q_1)-\alpha)}\cdot 2^{-k(1+n(1-1/q_1))}, \end{aligned}$$

thus

$$\begin{aligned} I\ =\ & C\left[\sum_{j=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||g_{\mu,\delta}^{b_1}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p} \\ \le\ & C||b_1||_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))}\right)^p\right]^{1/p} \end{aligned}$$

$$\leq C||b_1||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{p(j-k)(1+n(1-1/q_1)-\alpha)}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q_1)-\alpha)p/2}\right)\right.\\ \left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q_1)-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||b_1||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

When $m>1$, similar to the proof of $g^{b_1}_{\mu,\delta}(a_j)(x)$, we have

$$\begin{aligned} g^{\vec{b}}_{\mu,\delta}(a_j)(x) &\leq C|x|^{-(n+1-\delta)}\int_{B_j}|z||a_j(z)|\prod_{i=1}^{m}|b_i(x)-b_i(z)|dz\\ &\leq C|x|^{-(n+1-\delta)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|\int_{B_j}|z||a_j(z)||(\vec{b}(x)-\vec{b}')_{\sigma^c}|dz\\ &\leq C|x|^{-(n+1-\delta)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|2^j\cdot 2^{-j\alpha}\cdot 2^{jn(1-1/q_1)}||\vec{b}_{\sigma^c}||_{BMO}\\ &\leq C|x|^{-(n+1-\delta)}\cdot 2^{j(1+n(1-1/q_1)-\alpha)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|||\vec{b}_{\sigma^c}||_{BMO}. \end{aligned}$$

So

$$\begin{aligned} &||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}}\\ \leq\ & C2^{j(1+n(1-1/q_1)-\alpha)}||\vec{b}_{\sigma^c}||_{BMO}\\ &.\times\left[\int_{B_k}\left(|x|^{-(n+1-\delta)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|\right)^{q_2}dx\right]^{1/q_2}\\ \leq\ & C||\vec{b}_{\sigma^c}||_{BMO}2^{j(1+n(1-1/q_1)-\alpha)}\cdot 2^{-k(n+1-\delta)+kn/q_2}||\vec{b}_\sigma||_{BMO}\\ \leq\ & C||\vec{b}||_{BMO}2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))}. \end{aligned}$$

then

$$I = C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||g^{\vec{b}}_{\mu,\delta}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p}$$

$$\leq C||\vec{b}||_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))}\right)^p\right]^{1/p}$$

$$\leq C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q_1)-\alpha)p}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q_1)-\alpha)p/2}\right)\right. & \\ \quad\left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)(1+n(1-1/q_1)-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||\vec{b}||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

Remark. Theorem 2 also hold for nonhomogeneous Herz-type spaces, we omit the details.

3 Open Problem

In this paper, the continuity of the multilinear commutators related to the Littlewood-Paley operators and BMO functions on certain Hardy and Herz-Hardy spaces are obtained.

The open problem is to study the boundedness of the multilinear operators generated by the Littlewood-Paley operators and others locally integrable functions on others spaces.

References

[1] J. Alvarez, Continuity properties for linear commutators of Calderón-Zygmund operators, Collect. Math., 49(1998), 17-31.

[2] R.Coifman, R.Rochberg and G.Weiss, Factorization theorem for Hardy space in several variables, Ann. of Math., 103(1976), 611-635.

[3] J.Garcia-Cuerva and M.L.Herrero, A Theory of Hardy spaces associated to Herz Spaces, Proc. London Math. Soc., 69(1994), 605-628.

[4] L.Z.Liu, Weighted weak type (H^1, L^1) estimates for commutators of Littlewood-Paley operators, Indian J. of Math., 45(1)(2003), 71-78.

[5] L.Z.Liu, Weighted weak type estimates for commutators of Littlewood-Paley operator, Japanese J. of Math., 29(1)(2003), 1-13.

[6] L.Z.Liu, S.Z.Lu and J.S.Xu, Boundedness for commutators of Littlewood-Paley operators, Adv. in Math.(China), 32(2003), 473-480.

[7] S.Z.Lu and D.C.Yang, The local versions of $H^p(R^n)$ spaces at the origin, Studia. Math., 116(1995), 147-158.

[8] S.Z.Lu and D.C.Yang, The decomposition of the weighted Herz spaces and its applications, Sci. in China(ser.A), 38(1995), 147-158.

[9] S.Z.Lu and D.C.Yang, The weighted Herz type Hardy spaces and its applications, Sci. in China(ser.A), 38(1995), 662-673.

[10] S.Z.Lu and D.C.Yang, The continuity of commutators on Herz-type space, Michigan Math. J., 44(1997), 255-281.

[11] C.Perez, Endpoint estimate for commutators of singular integral operators, J. Func. Anal., 128(1995), 163-185.

[12] C.Perez and R.Trujillo-Gonzalez , Sharp weighted estimates for multilinear commutators, J. London Math. Soc., 65(2002), 672-692.

[13] E.M.Stein, Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton NJ., 1993.

[14] A.Torchinsky, The real variable methods in harmonic analysis, Pure and Applied Math., 123, Academic Press, New York, 1986.