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Some Open Problems in Chaos Theory and Dynamics

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Abstract

In this paper, some new open problems are proposed about the following topics: 2-D rational mappings, hyperbolic structure, structural stability, some questions about periodic, homoclinic and heteroclinic orbits and classification of chaos, and finally some questions about the notion of chaotification of dynamical systems. The presentation of each problem is easy with some relevant work related to it.

Keywords: *Chaos, 2-D rational mappings, hyperbolic nature, S-unimodal mappings, structural stability, Chen system, periodic, homoclinic, and heteroclinic orbits, chaotification.*

1 Introduction

In this paper, some new open problems are proposed. These problems are related to questions about advanced results concerning the dynamics of some dynamical systems. For example, 2-D rational mappings, the hyperbolic nature of a modulated logistic map, the joint function between regular and chaotic attractors, some problems related to the composition and sum of S-unimodal mappings, the structural stability of the Chen system and more generally, the structural stability of general 3-D quadratic continuous-time systems, some questions about periodic, homoclinic and heteroclinic orbits and classification of chaos, and finally some questions about the notion of chaotification of dynamical systems. These questions have not been previously addressed in the literature. Solving such a problem opens an interesting field in chaos studies concerned with the classification and determination of the type of chaos observed experimentally, proved analytically, or tested numerically in theory and practice.

2 Advanced results about the dynamics of 2-D rational mappings

It well known that chaos can occur in simple dynamical systems and that the complexity of an equation does not necessarily correlate with the complexity of its dynamics. Rational chaotic systems are rather rare in theory and practice. In [Lu *et al.*, 2004], the following new 1-D discrete iterative system with a rational fraction was discovered in a study of evolutionary algorithms:

$$g(x) = \frac{1}{0.1 + x^2} - ax, \tag{1}$$

where a is a parameter. The map (1) describes different random evolutionary processes, and it is much more complicated than the logistic system (4) below. In [Chang *et al.*, 2005], an extended version (with more complicated dynamical behavior) of this map to two-dimensions is given as follows:

$$h(x, y) = \begin{pmatrix} \frac{1}{0.1+x^2} - ay \\ \frac{1}{0.1+y^2} + bx \end{pmatrix}, \tag{2}$$

where a and b are parameters. In [Zeraoulia & Sprott, 2011], a new and very simple 2-D map, characterized by the existence of only one rational fraction with no vanishing denominator is constructed and given by:

$$f(x, y) = \begin{pmatrix} \frac{-ax}{1+y^2} \\ x + by \end{pmatrix}, \tag{3}$$

where a and b are bifurcation parameters. The map (3) is algebraically simpler than map (2), but it produces several new complicated chaotic attractors obtained *via* the quasi-periodic route to chaos.

More and advanced results about the dynamics of some 2-D rational mappings such as (2) and (3) [Zeraoulia & Sprott, 2011]. The map (3) is algebraically simpler but with more complicated behavior than the map studied in [Lu *et al.*, 2004], and it produces several new chaotic attractors obtained *via* the quasi-periodic route to chaos. The main question concerns the rigorous boundedness and chaoticity of the map (3) for some values of its bifurcation parameters, i.e., Find regions in the $a-b$ plane in which the map (3) is bounded and chaotic in the rigorous mathematical definition of chaos and boundedness of attractors.

3 About the hyperbolic nature of a modulated logistic map

Generally, the dynamics of a system is interesting if it has a closed, bounded, and hyperbolic attractor. In this case, the coexistence of highly complicated long-term behavior, sensitive dependence on initial conditions, and the overall stability of the orbit structure are the main characteristics of the system. In other words, let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^r real function that defines a discrete map also called f and Ω is a manifold. Hence an f-invariant subset Λ of $\mathbb{R}^n$ satisfies $f(\Lambda) \subset \Lambda$. Then one has the following definitions given in [Abraham & Marsden, 1978]:

Definition 1 *If f is a diffeomorphism defined on some compact smooth manifold $\Omega \subset \mathbb{R}^n$, an f-invariant subset Λ of $\mathbb{R}^n$ is said to be hyperbolic if there exists a $0 < \lambda < 1$ and a $c > 0$ such that*

(1) $T_\Lambda\Omega = E^s \oplus E^u$, where $\oplus$ means the algebraic direct sum.

(2) $Df(x) E^s_x = E^s_{f(x)}$, and $Df(x) E^u_x = E^u_{f(x)}$ for each $x \in \Lambda$.

(3) $\left\|Df^k v\right\| \leq c\lambda^k \|v\|$ for each $v \in E^s$ and $k > 0$.

(4) $\left\|Df^{-k} v\right\| \leq c\lambda^k \|v\|$ for each $v \in E^u$ and $k > 0$.

where E^s, E^u are, respectively, the stable and unstable submanifolds of the map f, *i.e.*, the two Df-invariant submanifolds, and E^s_x, E^u_x are the two $Df(x)$-invariant submanifolds. By using elementary real analysis (ideas from complex variable theory), [Glendinning, 2001] gives a new and elementary proof of the classic results due to [Guckenheimer, 1979 and Misiurewicz, 1981] which imply that the invariant set of the logistic map given by:

$$F_\mu(x) = \mu x(1-x) \tag{4}$$

with $\mu \in (4, 2+\sqrt{5}]$ is hyperbolic. The aim of this section is to ask about the hyperbolic nature of a modulated logistic map studied in [Zeraoulia & Sprott, 2008a] and given by:

$$\begin{cases} x_{n+1} = ax_n(1-x_n) \\ y_{n+1} = (b+cx_n)\, y_n(1-y_n) \end{cases} \tag{5}$$

where $0 \leq a \leq 4$.

About the hyperbolic nature of the modulated logistic map (5).

4 Bridging the gap between two 1-D S-unimodal mappings

Robust chaos [Zeraoulia & Sprott, 2008b] is defined as the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space since the presence of such windows implies that small changes of the parameters would destroy the chaotic behavior. This effect implies the fragility of this type of chaos. Unimodality of maps is an important tool for proving the existence of robust chaos in 1-D discrete systems [Zeraoulia & Sprott, 2008b]. First, we recall the usual notations and a standard definition:

Definition 2 *A map $\varphi : I \longrightarrow I$, is S-unimodal on the interval $I = [a, b]$ if (a) The function $\varphi(x)$ is of class C^3. (b) The point a is a fixed point with b its other preimage, i.e., $\varphi(a) = \varphi(b) = a$. (c) There is a unique maximum at $c \in (a, b)$ such that $\varphi(x)$ is strictly increasing on $x \in [a, c)$ and strictly decreasing on $x \in (c, b]$, (d) φ has a negative Schwarzian derivative, i.e., $S(\varphi, x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 < 0$ for all $x \in I - \{y, \varphi'(y) = 0\}$. Since what matters is only its sign, one may as well work with the product: $\hat{S}(\varphi, x) = 2\varphi'(x)\varphi'''(x) - 3(\varphi''(x))^2$, which has the same sign as $S(\varphi, x)$.*

The importance of S-unimodal maps in chaos theory comes from the fact that an S-unimodal map can have at most one periodic attractor which will attract the critical point. This result is used to formulate the following theorem [Andrecut & Ali, 2001]:

Theorem 1 *Let $\varphi_v(x)$: $I = [a; b] \longrightarrow I$ be a parametric S-unimodal map with the unique maximum at $c \in (a; b)$ and $\varphi_v(c) = b$, $\forall\ v \in (v_{\min}, v_{\max})$. Then $\varphi_v(x)$ generates robust chaos for $v \in (v_{\min}, v_{\max})$.*

Here, the symbol v is the bifurcation parameter of the map φ and $v_{\min}, v_{\max}$ are the minimal and the maximal values of v in which φ_v is a parametric S-unimodal map. Theorem 5 gives the general conditions for the occurrence of robust chaos in S-unimodal maps without any procedure for constructing them. Such a procedure can be found in [Andrecu & Ali, 2001].

What is the main behavior of the map bridging the gap between two one-dimensional S-unimodal mappings that generates robust chaos?

What are the main and new behaviors in the dynamics of the sum of n S-unimodal maps?

Is it possible to generate homoclinic chaos *via* the composition of two 2-D continuous mappings?

What is the behavior of maps (or systems) resulting from compositions of several maps (or systems)?

Find a new method for constructing robust chaotic attractors.

5 Structural stability of the Chen system

The concept of structural stability was introduced by Andronov and Pontryagin in 1937, and it has a crucial role in dynamical systems theory. Conditions for structural stability of high-dimensional systems were formulated in [Smale, 1967]. These conditions are the following: A system must satisfy both Axiom A and the strong transversality condition. Mathematically, let $C^r(\mathbb{R}^n, \mathbb{R}^n)$ denote the space of C^r vector fields of $\mathbb{R}^n$ into $\mathbb{R}^n$. Let $Diff^r(\mathbb{R}^n, \mathbb{R}^n)$ be the subset of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ consisting of the C^r diffeomorphisms: (a) Two elements of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ are C^r ε-close ($k \leq r$), or just C^k close, if they, along with their first k derivatives, are within ε as measured in some norm. (b) A dynamical system (vector field or map) is structurally stable if nearby systems have the same qualitative dynamics. Now it is possible to define formally the notion of structural stability as follows:

Definition 3 *(Structural stability) Consider a map $f \in Diff^r(M, M)$ (or a C^r vector field in $C^r(M, M)$). Then f is structurally stable if there exists a neighborhood N of f in the C^k topology such that f is C^0 conjugate (or C^0 equivalent) to every map (or vector field) in N.*

Hence our questions concern the following:

(a) Structural stability of the Chen system (the second simpler system after the one of Lorenz) given by:

$$\begin{cases} x' = a(y - x) \\ y' = (c - a)\,x + cy - xz \\ z' = xy - bz \end{cases} \tag{6}$$

See [Chen & Ueta, 1999].

(b) Structural stability of the general 3-D quadratic continuous-time system given by

$$\begin{cases} x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \\ y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8xz + b_9yz \\ z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \end{cases} \tag{7}$$

where $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ are the bifurcation parameters.

6 Periodic, homoclinic, and heteroclinic orbits and classification of chaos

Homoclinic and heteroclinic orbits arise in the study of bifurcations and chaos [Aulbach & Flockerzi, 1989], as well as in their applications to fields such as mechanics, biomathematics, and chemistry [Balmforth, 1995; Feng, 1998]. In some cases, it is necessary to determine the nature or the type of chaotic behavior arising in a dynamical system. One of the commonly agreed-upon analytic criteria for proving chaos in autonomous systems is [Shi'lnikov, 1965, 1970] and its subsequent embellishments and slight extension [Kennedy *et al.*, 2001; Silva, 2003]. The resulting chaos is called *horseshoe type* or *Shi'lnikov chaos.* By applying the undetermined coefficient method, homoclinic and heteroclinic orbits in some quadratic three-dimensional autonomous systems are found [Zhou, *et al.*, 2004; Tucker, 1999; Lu *et al.*, 2002; Zhou *et al.*, 2005; Sparrow, 1982; Celikovsk & Vanecek, 1994]. These systems have a *Shi'lnikov type* of chaos, and it is conjectured in [Zhou *et al.*, 2004] that the two *Shi'lnikov* theorems can be used to classify chaos in polynomial ODE systems. For such systems, only three kinds of chaos exist: homoclinic chaos, heteroclinic chaos or a combination of homoclinic and heteroclinic chaos, and chaos without homoclinic and heteroclinic orbits.

(a) Find sufficient conditions for the non-existence of periodic orbits in dynamical systems. The importance of this idea is to distinguish between the type of chaos obtained *via* the usual routes to chaos and chaos coming from nothing (without a transient from periodic orbits).

(b) Find sufficient conditions for the non-existence of homoclinic and heteroclinic orbits in dynamical systems. The importance of this idea is to distinguish between Shi'linikov chaos and chaos without homoclinic and heteroclinic orbits.

7 Chaotification of dynamical systems

Chaotification, or *anticontrol of chaos*, is the reverse of suppressing chaos in a dynamical system. The aim of this process is to create or enhance the system complexity for some special novel, time- or energy-critical interdisciplinary applications such as high-performance circuits and devices, liquid mixing, chemical reactions, biological systems, crisis management, secure information processing, and critical decision-making in politics, economics, military applications, *etc.* Some of the methods used for this purpose (for discrete mappings) can be found in [Li, 2004; Chen & Lai, 1997; Lin *et al.*, 2002; Wang

& Chen, 1999; Zhou et al., 2004 and references therein] using Lyapunov exponents, or by the use of several modified versions of the Marotto theorem [Marotto, 1978, 2005; Chen *et al.*, 1998; Lin *et al.*, 2002] or by the use of the Li-Yorke definition of chaos [Li & Yorke, 1975]. In this section, we state two open problems related to this terminology. These two problems are related to the S-unimodality defined above and the *Collet-Eckmann criterion* defined by: Let B^k be the closed unitary ball in $\mathbb{R}^k$.

Definition 4 *A k-parameter family of unimodal maps is a map $\Gamma : B^k \times I \to I$ such that for $p \in B^k, \gamma_p(x) = \Gamma(p, x)$ is a unimodal map. Such a family is said to be C^n or analytic, according to Γ being C^n or analytic.*

The natural topology can be introduced in spaces of smooth families (C^n with $n = 2, ..., \infty$), but it is not necessary to introduce any topology in the space of analytic families.

Definition 5 *(a) A unimodal map f is called Collet-Eckmann (CE) if there exists constants $C > 0, \lambda > 1$ such that for every $n > 0$,*

$$|Df^n(f(0))| > C\lambda^n \tag{8}$$

(b) A unimodal map f is called backwards Collet-Eckmann (BCE) if there exists $C > 0, \lambda > 1$ such that for any $n > 0$ and any x with $f^n(x) = 0$, we have

$$|Df^n(x)| > C\lambda^n \tag{9}$$

Thus, the unimodal Collet-Eckmann and the backwards Collet-Eckmann maps are strongly hyperbolic along the critical orbit. Finally, the propsed problems are given by:

(a) Chaotifying 1-D discete mappings using S-unimodality criteria

(b) Chaotifying 1-D discrete mappings using the Collet-Eckmann criterion.

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