Int. J. Open Problems Compt. Math., Vol. 4, No. 2, June 2011
ISSN 1998-6262; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

A New Hermite-Hadamard Type Inequality

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Abstract

In this paper, we firstly establish a new generalization of the classical Hermite-Hadamard inequality for a real-valued convex function. Then the convexity of the matrix function $g(A) = f(\det A)$ ***is proved under certain conditions on the function f and the matrix A. Based on these, finally we derive a new Hermite-Hadamard type inequality for the function*** $g(t) = f(\det A(t))$***.***

Keywords: *convex function; convexity; Hermite-Hadamard inequality; determinant; Minkowski inequality.*

1 Introduction and Main Results

Inequalities with well symmetry are important and interesting in Analysis and PDE, and among the inequality theory, the inequalities relating to convexity of functions are extremely valuable. Moreover in the research of the convexity of functions, a well-known example is the famous Hermite-Hadamard inequality. It was firstly published in [1], which gives us an estimate of the mean value of a convex function.

In this paper we consider the function $g(t) = f(\det A(t))$, and present a new generalization of the classical Hermite-Hadamard inequality. Firstly we recall some basic facts.

Throughout this note, we denote by I the closed interval [a, b]. A real-valued function f is said to be convex on I if

$$f(\mu x + (1-\mu)y) \leq \mu f(x) + (1-\mu) f(y),$$

and concave on I if

$$f(\mu x + (1-\mu)y) \geq \mu f(x) + (1-\mu) f(y)$$

for all $x, y \in I$ and $0 \leq \mu \leq 1$.

A matrix $A \in M_n$ is said to be positive definite if $\mathrm{Re}\left(x^T A x\right) > 0$, and is said to be non-negative definite if $\mathrm{Re}\left(x^T A x\right) \geq 0$ for all nonzero $x \in \mathbb{C}^n$. The convex set of positive definite matrices is denoted by M_n^+, and the convex set of non-negative definite matrices is denoted by SM_n. Together with the definition of real-valued convex functions, we have the definition of convexity of matrix functions as follows:

Definition 1.1. *A real valued function f defined on M_n^+ or SM_n is said to be convex if*

$$f\left(\mu A + (1-\mu) B\right) \leq \mu f(A) + (1-\mu) f(B),$$

and is said to be concave if

$$f\left(\mu A + (1-\mu) B\right) \geq \mu f(A) + (1-\mu) f(B)$$

for all $0 \leq \mu \leq 1$ and all $A, B \in M_n^+$ or SM_n, $A \neq B$.

Recall that it has been proved by Horn and Johnson in [2] that the function $g(A) = \log(\det A)$ is a strictly concave function on the convex set of positive definite Hermitian matrices M_n^+, and by the following famous Minkowski inequality, we obtain that the function $g(A) = (\det A)^{\frac{1}{n}}$ is also concave on the set of positive definite Hermitian matrices.

Theorem 1.2 (Minkowski Inequality). *If $A, B \in M_n^+(R)$, then*

$$(\det(A+B))^{\frac{1}{n}} \geq (\det A)^{\frac{1}{n}} + (\det B)^{\frac{1}{n}}. \tag{1}$$

But in general, the function $g(A) = (\det A)^m$ is not concave for $m \neq \frac{1}{n}$, not to mention a general function $g(A) = f(\det A)$. In this paper, we will derive the convexity of the matrix function $g(A) = f(\det A)$ under certain conditions on the function f and the matrix A as follows:

Theorem 1.3. *Let A be a positive definite matrix with the eigenvalues $\lambda_i(A)$, and B be a symmetric matrix with the eigenvalues $\lambda_i(B)$, where $i = 1, \cdots, n$. Then for arbitrary monotonic increasing and convex function $f(x)$, the inequality*

$$f\left(\det\left(\mu A + (1-\mu) B\right)\right) \leq \mu f(\det A) + (1-\mu) f(\det B) \tag{2}$$

holds true for all $0 \leq \mu \leq 1$ if one of the following conditions is satisfied:
(i) $\lambda_i(B) \geq \max\limits_{1 \leq i \leq n} \lambda_i(A)$ *for all $i = 1, \cdots, n$;*
(ii) $\lambda_i(B) \leq \min\limits_{1 \leq i \leq n} \lambda_i(A)$ *for all $i = 1, \cdots, n$.*

Theorem 1.4. *Let A, B be two non-negative definite matrices such that $AB = BA$, and $\lambda_i(A)$, $\lambda_i(B)$, where $i = 1, \cdots, n$, be the eigenvalues of A and B. Then for arbitrary monotonic increasing and convex function $f(x)$, the inequality*

$$f(\det(\mu A + (1-\mu)B)) \leq \mu f(\det A) + (1-\mu) f(\det B)$$

holds true for all $0 \leq \mu \leq 1$ if one of the following conditions is satisfied:
(i) $\lambda_i(B) \geq \lambda_i(A)$ *for all* $i = 1, \cdots, n$;
(ii) $\lambda_i(B) \leq \lambda_i(A)$ *for all* $i = 1, \cdots, n$.

On the other hand, recall that the classical Hermite-Hadamard inequality for a real-valued convex function is that:

Theorem 1.5 (Hermite-Hadamard Inequality). *If $f : I \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

An account on the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [4] and [5].

In this paper we obtain a new generalization of the Hermite-Hadamard inequality, and prove that for arbitrary non-negative real-valued integrable function $\Phi : I \to \mathbb{R}$, there exist real numbers l, L such that:

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \leq l \leq \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \leq L \leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{4}$$

In fact, we prove the following theorem:

Theorem 1.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and $\Phi : I \to \mathbb{R}$ be a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also convex. Then for arbitrary $n \in \mathbb{N}$, $\mu_0 = 0, \mu_{n+1} = 1$ and arbitrary $0 \leq \mu_1 \leq \cdots \leq \mu_n \leq 1$, we have*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) &\leq l(\mu_1, \cdots, \mu_n) \leq \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \\ &\leq L(\mu_1, \cdots, \mu_n) \leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \end{aligned}$$

where

$$l(\mu_1, \cdots, \mu_n) = \sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right) f\left(\frac{1}{\left(\mu_{k+1}-\mu_k\right)(b-a)}\int_{(1-\mu_k)a+\mu_k b}^{\left(1-\mu_{k+1}\right)a+\mu_{k+1}b} \Phi(x)\,dx\right)$$

and

$$L(\mu_1, \cdots, \mu_n) = \sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right)\frac{f\circ\Phi((1-\mu_k)a+\mu_k b)+f\circ\Phi\left(\left(1-\mu_{k+1}\right)a+\mu_{k+1}b\right)}{2}.$$

Corollary 1.7. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function, and $\Phi: I\to\mathbb{R}$ be a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also convex, then for arbitrary $n\in\mathbb{N}$, $\mu_0=0,\mu_{n+1}=1$ and arbitrary $0\le\mu_1\le\cdots\le\mu_n\le 1$, we have*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) &\le \sup_{0\le\mu_1\le\cdots\le\mu_n\le 1} l(\mu_1,\cdots,\mu_n) \le \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \\ &\le \sup_{0\le\mu_1\le\cdots\le\mu_n\le 1} L(\mu_1,\cdots,\mu_n) \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \end{aligned}$$

where $l(\mu_1,\cdots,\mu_n)$ and $L(\mu_1,\cdots,\mu_n)$ are defined in Theorem 1.6.

Then by using Theorem 1.3, 1.4 and 1.6, we derive our main result which is a new Hermite-Hadamard type inequality for the function $g(t)=f(\det A(t))$ as follows:

Theorem 1.8. *Let $A(t): I\to M_n^+$ be a family of positive definite real-valued matrices with the eigenvalues $\lambda_i(A(t))$, where $i=1,\cdots,n$, for corresponding $t\in I$. Suppose that for any $t_1\ne t_2\in I$ and $0\le\mu\le 1$, we have*

$$A(\mu t_1+(1-\mu)t_2)\le \mu A(t_1)+(1-\mu)A(t_2),$$

then for $n\in\mathbb{N}$, $\mu_0=0,\mu_{n+1}=1$ and arbitrary $0\le\mu_1\le\cdots\le\mu_n\le 1$, the inequality

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \det A(t)\,dt\right) &\le l_A(\mu_1,\cdots,\mu_n) \le \frac{1}{b-a}\int_a^b f(\det A(t))\,dt \\ &\le L_A(\mu_1,\cdots,\mu_n) \le \frac{f(\det A(a))+f(\det A(b))}{2} \end{aligned}$$

holds true for arbitrary convex function $f:\mathbb{R}\to\mathbb{R}$ if one of the following conditions is satisfied:
(i) $\max\limits_{1\le i\le n}\lambda_i(A(t_1))\le\lambda_i(A(t_2))$ *for all $i=1,\cdots,n$ and $t_1<t_2\in I$;*
(ii) $\min\limits_{1\le i\le n}\lambda_i(A(t_1))\ge\lambda_i(A(t_2))$ *for all $i=1,\cdots,n$ and $t_1<t_2\in I$,*
where

$$l_A(\mu_1,\cdots,\mu_n)=\sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right)f\left(\frac{1}{\left(\mu_{k+1}-\mu_k\right)(b-a)}\int_{(1-\mu_k)a+\mu_k b}^{\left(1-\mu_{k+1}\right)a+\mu_{k+1}b}\det A(t)\,dt\right),$$

$$L_A(\mu_1,\cdots,\mu_n)=\sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right)\frac{f(\det A((1-\mu_k)a+\mu_k b))+f\left(\det A\left(\left(1-\mu_{k+1}\right)a+\mu_{k+1}b\right)\right)}{2}.$$

Theorem 1.9. *Let $B(t): I\to SM_n$ be a family of non-negative definite real-valued matrices with the eigenvalues $\lambda_i(B(t))$, where $i=1,\cdots,n$, for* *corresponding $t \in I$. Suppose that for any $t_1 \neq t_2 \in I$ and $0 \leq \mu \leq 1$, we have $B(t_1) B(t_2) = B(t_2) B(t_1)$ and*

$$B(\mu t_1 + (1-\mu) t_2) \leq \mu B(t_1) + (1-\mu) B(t_2),$$

then for $n \in \mathbb{N}$, $\mu_0 = 0, \mu_{n+1} = 1$ and arbitrary $0 \leq \mu_1 \leq \cdots \leq \mu_n \leq 1$, the inequality

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \det B(t)\,dt\right) &\leq l_B(\mu_1, \cdots, \mu_n) \leq \frac{1}{b-a}\int_a^b f(\det B(t))\,dt \\ &\leq L_B(\mu_1, \cdots, \mu_n) \leq \frac{f(\det B(a)) + f(\det B(b))}{2} \end{aligned}$$

holds true for arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ if one of the following conditions is satisfied:
(i) $\lambda_i(B(t_1)) \leq \lambda_i(B(t_2))$ *for all $i = 1, \cdots, n$ and $t_1 < t_2 \in I$;*
(ii) $\lambda_i(B(t_1)) \geq \lambda_i(B(t_2))$ *for all $i = 1, \cdots, n$ and $t_1 < t_2 \in I$,*
where

$$l_B(\mu_1, \cdots, \mu_n) = \sum_{k=0}^{n} \left(\mu_{k+1} - \mu_k\right) f\left(\frac{1}{\left(\mu_{k+1}-\mu_k\right)(b-a)} \int_{(1-\mu_k)a+\mu_k b}^{\left(1-\mu_{k+1}\right)a+\mu_{k+1} b} \det B(t)\,dt\right),$$

$$L_B(\mu_1, \cdots, \mu_n) = \sum_{k=0}^{n} \left(\mu_{k+1} - \mu_k\right) \frac{f(\det B((1-\mu_k)a+\mu_k b)) + f\left(\det B\left(\left(1-\mu_{k+1}\right)a+\mu_{k+1} b\right)\right)}{2}.$$

Remark 1. It is obvious that there exist many matrix families satisfying the conditions in Theorem 1.8 and 1.9 such that $\det A(t)$ and $f(\det A(t))$ are both integrable on the interval I.

Moreover by using Theorem 1.8 and 1.9, we can prove that there exist real numbers l_A, L_A and l_B, L_B such that:

Corollary 1.10. *Let $A(t) : I \to M_n^+$ be a family of positive definite real-valued matrices with the eigenvalues $\lambda_i(A(t))$, where $i = 1, \cdots, n$, for corresponding $t \in I$. Suppose that for any $t_1 < t_2 \in I$ and $0 \leq \mu \leq 1$, we have*

$$A(\mu t_1 + (1-\mu) t_2) \leq \mu A(t_1) + (1-\mu) A(t_2),$$

then for $n \in \mathbb{N}$, $\mu_0 = 0, \mu_{n+1} = 1$ and arbitrary $0 \leq \mu_1 \leq \cdots \leq \mu_n \leq 1$, the inequality

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \det A(t)\,dt\right) &\leq \sup_{0\leq\mu_1\leq\cdots\leq\mu_n\leq 1} l_A(\mu_1, \cdots, \mu_n) \\ &\leq \frac{1}{b-a}\int_a^b f(\det A(t))\,dt \\ &\leq \sup_{0\leq\mu_1\leq\cdots\leq\mu_n\leq 1} L_A(\mu_1, \cdots, \mu_n) \\ &\leq \frac{f(\det A(a)) + f(\det A(b))}{2} \end{aligned}$$

holds true for arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ if one of the following conditions is satisfied:
(i) $\max\limits_{1\le i\le n} \lambda_i (A(t_1)) \le \lambda_i (A(t_2))$ *for all* $i = 1, \cdots, n$ *and* $t_1 < t_2 \in I$;
(ii) $\min\limits_{1\le i\le n} \lambda_i (A(t_1)) \ge \lambda_i (A(t_2))$ *for all* $i = 1, \cdots, n$ *and* $t_1 < t_2 \in I$,
where $l_A(\mu_1, \cdots, \mu_n)$ *and* $L_A(\mu_1, \cdots, \mu_n)$ *are defined in Theorem 1.8.*

Corollary 1.11. *Let* $B(t) : I \to SM_n$ *be a family of non-negative definite real-valued matrices with the eigenvalues* $\lambda_i(B(t))$, *where* $i = 1, \cdots, n$, *for corresponding* $t \in I$. *Suppose that for any* $t_1 \neq t_2 \in I$ *and* $0 \le \mu \le 1$, *we have* $B(t_1) B(t_2) = B(t_2) B(t_1)$ *and*

$$B(\mu t_1 + (1-\mu) t_2) \le \mu B(t_1) + (1-\mu) B(t_2),$$

then for $n \in \mathbb{N}$, $\mu_0 = 0, \mu_{n+1} = 1$ *and arbitrary* $0 \le \mu_1 \le \cdots \le \mu_n \le 1$, *the inequality*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \det B(t)\,dt\right) &\le \sup_{0\le\mu_1\le\cdots\le\mu_n\le 1} l_B(\mu_1, \cdots, \mu_n) \\ &\le \frac{1}{b-a}\int_a^b f(\det B(t))\,dt \\ &\le \sup_{0\le\mu_1\le\cdots\le\mu_n\le 1} L_B(\mu_1, \cdots, \mu_n) \\ &\le \frac{f(\det B(a)) + f(\det B(b))}{2} \end{aligned}$$

holds true for arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ if one of the following conditions is satisfied:
(i) $\lambda_i (B(t_1)) \le \lambda_i (B(t_2))$ *for all* $i = 1, \cdots, n$ *and* $t_1 < t_2 \in I$;
(ii) $\lambda_i (B(t_1)) \ge \lambda_i (B(t_2))$ *for all* $i = 1, \cdots, n$ *and* $t_1 < t_2 \in I$,
where $l_B(\mu_1, \cdots, \mu_n)$ *and* $L_B(\mu_1, \cdots, \mu_n)$ *are defined in Theorem 1.9.*

The paper is organized as follows. In section 2, we firstly derive a useful lemma, by which we prove Theorem 1.3 and 1.4. In section 3, we prove Theorem 1.6 by using the famous Jensen's Inequality and directly calculating. In section 4, based on Theorem 1.3, 1.4 and 1.6, we present the proof of our main results. In section 5, we give two interesting open problems related to our paper.

2 Lemmas and Proof of Theorem 1.3 and 1.4

In order to prove Theorem 1.3 and 1.4, we shall need the following lemma:

Lemma 2.1. *If* $0 \leq \alpha, \beta \leq 1$ *satisfying* $\alpha + \beta = 1$, *and* $\mu_i \geq \nu_i$ *for arbitrary* $1 \leq i \leq n$ *or* $\mu_i \leq \nu_i$ *for arbitrary* $1 \leq i \leq n$, *then*

$$\prod_{i=1}^{n} (\mu_i \alpha + \nu_i \beta) \leq \alpha \prod_{i=1}^{n} \mu_i + \beta \prod_{i=1}^{n} \nu_i. \tag{5}$$

Proof. The approach we use is mathematical induction. Firstly we consider $n = 2$, since $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$, we have

$$\begin{aligned} &(\mu_1 \alpha + \nu_1 \beta)(\mu_2 \alpha + \nu_2 \beta) \\ &= ((\mu_1 - \nu_1)\alpha + \nu_1)(\mu_2 - (\mu_2 - \nu_2)\beta) \\ &= \mu_2 (\mu_1 - \nu_1)\alpha - \nu_1 (\mu_2 - \nu_2)\beta - (\mu_1 - \nu_1)(\mu_2 - \nu_2)\alpha\beta + \mu_2 \nu_1 \\ &= \mu_1 \mu_2 \alpha + \nu_1 \nu_2 \beta - (\mu_1 - \nu_1)(\mu_2 - \nu_2)\alpha\beta. \end{aligned} \tag{6}$$

It follows from the hypotheses that

$$(\mu_1 \alpha + \nu_1 \beta)(\mu_2 \alpha + \nu_2 \beta) \leq \mu_1 \mu_2 \alpha + \nu_1 \nu_2 \beta.$$

Assume that (5) is true for $n = k$, we prove that it is also true for $n = k + 1$. Since (5) holds for $n = k$, we have

$$\begin{aligned} \prod_{i=1}^{k+1} (\mu_i \alpha + \nu_i \beta) &= \left(\mu_{k+1}\alpha + \nu_{k+1}\beta\right) \prod_{i=1}^{k} (\mu_i \alpha + \nu_i \beta) \\ &\leq \left(\mu_{k+1}\alpha + \nu_{k+1}\beta\right) \left(\alpha \prod_{i=1}^{k} \mu_i + \beta \prod_{i=1}^{k} \nu_i\right) \end{aligned}$$

As the proof of $n = 2$ we obtain that

$$\begin{aligned} &\left(\mu_{k+1}\alpha + \nu_{k+1}\beta\right) \left(\alpha \prod_{i=1}^{k} \mu_i + \beta \prod_{i=1}^{k} \nu_i\right) \\ &= \left(\alpha \mu_{k+1} \prod_{i=1}^{k} \mu_i + \beta \nu_{k+1} \prod_{i=1}^{k} \nu_i\right) - \alpha\beta \left(\mu_{k+1} - \nu_{k+1}\right) \left(\prod_{i=1}^{k} \mu_i - \prod_{i=1}^{k} \nu_i\right) \end{aligned} \tag{7}$$

Since the second term in (7) is non-positive for $\mu_i \geq \nu_i$ or for $\mu_i \leq \nu_i$, it follows that

$$\prod_{i=1}^{k+1} (\mu_i \alpha + \nu_i \beta) \leq \alpha \prod_{i=1}^{k+1} \mu_i + \beta \prod_{i=1}^{k+1} \nu_i$$

and consequently that inequality (5) holds for $n = k + 1$. □

With the help of Lemma 2.1, we now turn to prove Theorem 1.3 and 1.4.

Proof of Theorem 1.3. Since $f(x)$ is a convex function, we have

$$f(\mu x + (1-\mu) y) \leq \mu f(x) + (1-\mu) f(y)$$

for arbitrary $x, y \in \mathbb{R}$ and $0 \leq \mu \leq 1$. Putting $x = \det A$ and $y = \det B$ it follows that

$$f(\mu \det A + (1-\mu)\det B) \leq \mu f(\det A) + (1-\mu) f(\det B). \tag{8}$$

On the other hand, it is known from [2] that for the positive definite matrix A and symmetric matrix B there exists a nonsingular matrix C such that $A = C^T C$ and $B = C^T \Lambda C$, where $\Lambda = diag\{\lambda_1, \cdots, \lambda_n\}$. The inequality

$$\det(\mu A + (1-\mu) B) \leq \mu \det A + (1-\mu)\det B$$

is then equivalent to

$$\det(\mu I + (1-\mu)\Lambda) \leq \mu + (1-\mu)\det \Lambda,$$

where I is the identity matrix. That is

$$\prod_{i=1}^{n} (\mu + \lambda_i (1-\mu)) \leq \mu + (1-\mu) \prod_{i=1}^{n} \lambda_i.$$

It follows from Ostrowski Theorem (see [2]) that for each $1 \leq i \leq n$, there exists $\theta_i > 0$ such that

$$\min_{1 \leq i \leq n} \lambda_i(A) \leq \theta_i \leq \max_{1 \leq i \leq n} \lambda_i(A)$$

and

$$\lambda_i(B) = \theta_i \lambda_i.$$

Thus we conclude that

$$\frac{\lambda_i(B)}{\max\limits_{1 \leq i \leq n} \lambda_i(A)} \leq \lambda_i \leq \frac{\lambda_i(B)}{\min\limits_{1 \leq i \leq n} \lambda_i(A)}. \tag{9}$$

Therefore, if the condition (i) satisfies, we have $\lambda_i \geq 1$ for arbitrary $1 \leq i \leq n$, and if the condition (ii) is satisfied, we have $\lambda_i \leq 1$ for arbitrary $1 \leq i \leq n$. Then it follows from Lemma 2.1 that

$$\prod_{i=1}^{n} (\mu + \lambda_i (1-\mu)) \leq \mu + (1-\mu) \prod_{i=1}^{n} \lambda_i,$$

which is equivalent to

$$\det(\mu A + (1-\mu) B) \leq \mu \det A + (1-\mu)\det B.$$

Furthermore, since $f(x)$ is a monotonic increasing function, together with (8) we have

$$f(\det(\mu A + (1-\mu) B)) \leq \mu f(\det A) + (1-\mu) f(\det B).$$

□

Proof of Theorem 1.4. As the proof of Theorem 1.3, we only need to prove

$$\det(\mu A + (1-\mu) B) \leq \mu \det A + (1-\mu) \det B \tag{10}$$

is also satisfied under the hypotheses of Theorem 1.4. Indeed, since $AB = BA$, it is known from [2] that there exists a orthogonal matrix C such that $A = C^T \Lambda_A C$, $B = C^T \Lambda_B C$, where $\Lambda_A = diag\{\lambda_1(A), \cdots, \lambda_n(A)\}$ and $\Lambda_B = diag\{\lambda_1(B), \cdots, \lambda_n(B)\}$. Thus (10) is equivalent to

$$\det(\mu \Lambda_A + (1-\mu)\Lambda_B) \leq \mu \det \Lambda_A + (1-\mu) \det \Lambda_B,$$

that is

$$\prod_{i=1}^{n} (\mu \lambda_i(A) + (1-\mu)\lambda_i(B)) \leq \mu \prod_{i=1}^{n} \lambda_i(A) + (1-\mu) \prod_{i=1}^{n} \lambda_i(B).$$

Therefore, if the condition (i) satisfies, we have $\lambda_i(A) \leq \lambda_i(B)$ for arbitrary $1 \leq i \leq n$, and if the condition (ii) is satisfied, we have $\lambda_i(A) \geq \lambda_i(B)$ for arbitrary $1 \leq i \leq n$. Then it follows from Lemma 2.1 that

$$\prod_{i=1}^{n} (\mu \lambda_i(A) + (1-\mu)\lambda_i(B)) \leq \mu \prod_{i=1}^{n} \lambda_i(A) + (1-\mu) \prod_{i=1}^{n} \lambda_i(B),$$

which is equivalent to

$$\det(\mu A + (1-\mu) B) \leq \mu \det A + (1-\mu) \det B.$$

□

3 Proof of Theorem 1.6

In order to prove Theorem 1.6, we also need some lemmas as follows:

Lemma 3.1. *If $f : \mathbb{R} \to \mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be integrable functions, then we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f \circ \Phi(x)\,dx &= \int_0^1 f \circ \Phi(\mu a + (1-\mu) b)\,d\mu \\ &= \int_0^1 f \circ \Phi(\mu b + (1-\mu) a)\,d\mu.\end{aligned}$$

Proof. We could use the change of variables $x = \mu a + (1-\mu) b$ and $x = \mu b + (1-\mu) a$ to complete the proof. □

Lemma 3.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function, and $\Phi: I\to\mathbb{R}$ be a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also convex, then we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right)\le \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{11}$$

Proof. Observing that the first inequality is actually the famous Jensen's inequality (see [6]), thus we only need to prove the second one. Since $f\circ\Phi(x)$ is a convex function, we have for arbitrary $\mu\in[0,1]$

$$\frac{f\circ\Phi(\mu a+(1-\mu)b)+f\circ\Phi((1-\mu)a+\mu b)}{2}\le\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{12}$$

Integrating (12) over [0, 1] and using Lemma 3.1 we have

$$\frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx\le\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}.$$

□

Remark 2. If $f:\mathbb{R}\to\mathbb{R}$ is a concave function, and $\Phi: I\to\mathbb{R}$ is a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also concave, then as the proof of Lemma 3.2 we have

$$\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}\le\frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx\le f\left(\frac{1}{b-a}\int_a^b\Phi(x)\,dx\right).$$

With the help of Lemma 3.1 and 3.2, we now turn to prove Theorem 1.6.

Proof of Theorem 1.6. It follows from the hypothesis that $f(x)$ and $f\circ\Phi(x)$ are both convex functions, therefore by applying Lemma 3.2 we have

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right)\le \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \le \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \tag{13}$$

By assumption $\lambda_0=0$, so

$$[a,(1-\lambda_1)a+\lambda_1 b]=[(1-\lambda_0)a+\lambda_0 b,(1-\lambda_1)a+\lambda_1 b].$$

Then applying (13) to

$$[(1-\lambda_k)a+\lambda_k b,(1-\lambda_{k+1})a+\lambda_{k+1}b],$$

for $k=0,1,\cdots,n$ we have

$$\begin{aligned}
&f\left(\tfrac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)\\
&\le\tfrac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)\,dx \qquad (14)\\
&\le\tfrac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}.
\end{aligned}$$

Multiplying each term in (14) by corresponding $(\lambda_{k+1}-\lambda_k)$, and adding the resulting inequalities, we get

$$\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\, f\left(\tfrac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)$$

$$\leq \sum_{k=0}^{n}\tfrac{1}{(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f\circ\Phi(x)\,dx$$

$$\leq \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\,\tfrac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2},$$

that is

$$l(\lambda_1,\cdots,\lambda_n)\leq \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \leq L(\lambda_1,\cdots,\lambda_n),$$

where $l(\lambda_1,\cdots,\lambda_n)$ and $L(\lambda_1,\cdots,\lambda_n)$ are defined in Theorem 1.6.

To prove the remaining two inequalities:

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) &\leq l(\lambda_1,\cdots,\lambda_n)\\ &\leq L(\lambda_1,\cdots,\lambda_n)\\ &\leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \end{aligned}$$

we use the fact $f:\mathbb{R}\to\mathbb{R}$, $f\circ\Phi(x)$ are both convex functions and observe that $\sum\limits_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$

$$\begin{aligned} & f\left(\tfrac{1}{b-a}\int_a^b \Phi(x)\,dx\right)\\ &= f\left(\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\tfrac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)\\ &\leq \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\, f\left(\tfrac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)\,dx\right)\\ &\leq \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\,\tfrac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}\\ &\leq \tfrac{1}{2}\sum_{k=0}^{n}\left(((1-\lambda_k)-(1-\lambda_{k+1}))\,((1-\lambda_k)+(1-\lambda_{k+1}))\, f\circ\Phi(a)\right)\\ &\quad+\tfrac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)(\lambda_{k+1}+\lambda_k)\, f\circ\Phi(b)\\ &= \tfrac{1}{2}\sum_{k=0}^{n}\left(\left((1-\lambda_k)^2-(1-\lambda_{k+1})^2\right) f\circ\Phi(a)+\left(\lambda_{k+1}^2-\lambda_k^2\right) f\circ\Phi(b)\right)\\ &= \tfrac{f\circ\Phi(a)+f\circ\Phi(b)}{2}. \end{aligned}$$

□

Remark 3. In fact the key point of our proof is Lemma 3.2, thus by the following inequality in Remark 2:

$$\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}\leq\frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx\leq f\left(\frac{1}{b-a}\int_a^b\Phi(x)\,dx\right),$$

we have

Theorem 3.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a concave function, and $\Phi:I\to\mathbb{R}$ be a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also concave. Then for arbitrary $n\in\mathbb{N}$, $\mu_0=0,\mu_{n+1}=1$ and arbitrary $0\leq\mu_1\leq\cdots\leq\mu_n\leq 1$, we have*

$$\begin{aligned}\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}&\leq l(\mu_1,\cdots,\mu_n)\leq\frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx\\&\leq L(\mu_1,\cdots,\mu_n)\leq f\left(\frac{1}{b-a}\int_a^b\Phi(x)\,dx\right),\end{aligned}$$

where

$$l(\mu_1,\cdots,\mu_n)=\sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right)\frac{f\circ\Phi((1-\mu_k)a+\mu_k b)+f\circ\Phi\left(\left(1-\mu_{k+1}\right)a+\mu_{k+1}b\right)}{2},$$

$$L(\mu_1,\cdots,\mu_n)=\sum_{k=0}^{n}\left(\mu_{k+1}-\mu_k\right)f\left(\frac{1}{\left(\mu_{k+1}-\mu_k\right)(b-a)}\int_{(1-\mu_k)a+\mu_k b}^{\left(1-\mu_{k+1}\right)a+\mu_{k+1}b}\Phi(x)\,dx\right).$$

Corollary 3.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a concave function, and $\Phi:I\to\mathbb{R}$ be a non-negative real-valued integrable function such that $f\circ\Phi(x)$ is also concave, then for arbitrary $n\in\mathbb{N}$, $\mu_0=0,\mu_{n+1}=1$ and arbitrary $0\leq\mu_1\leq\cdots\leq\mu_n\leq 1$, we have*

$$\begin{aligned}\frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}&\leq\sup_{0\leq\mu_1\leq\cdots\leq\mu_n\leq 1}l(\mu_1,\cdots,\mu_n)\leq\frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx\\&\leq\sup_{0\leq\mu_1\leq\cdots\leq\mu_n\leq 1}L(\mu_1,\cdots,\mu_n)\leq f\left(\frac{1}{b-a}\int_a^b\Phi(x)\,dx\right),\end{aligned}$$

where $l(\mu_1,\cdots,\mu_n)$ and $L(\mu_1,\cdots,\mu_n)$ are defined in Theorem 3.3.

4 Proof of the Main Results

In this section, with the help of Theorem 1.3, 1.4 and 1.6, we prove our main results. We only prove Theorem 1.8, and the proof of Theorem 1.9 is similar.

Proof of Theorem 1.8. If one of the following conditions is satisfied:
(i) $\max_{1\leq i\leq n} \lambda_i (A(t_1)) \leq \lambda_i (A(t_2))$ for all $i = 1, \cdots, n$ and $t_1 < t_2 \in I$;
(ii) $\min_{1\leq i\leq n} \lambda_i (A(t_1)) \geq \lambda_i (A(t_2))$ for all $i = 1, \cdots, n$ and $t_1 < t_2 \in I$,
then by Theorem 1.3 we have for arbitrary monotonic increasing and convex function $f(x)$, the inequality

$$f(\det(\mu A(t_1) + (1-\mu) A(t_2))) \leq \mu f(\det A(t_1)) + (1-\mu) f(\det A(t_2)) \quad (15)$$

holds for any $t_1 < t_2 \in I$ and $0 \leq \mu \leq 1$. Moreover since for any $t_1 \neq t_2 \in I$ and $0 \leq \mu \leq 1$ we have

$$A(\mu t_1 + (1-\mu) t_2) \leq \mu A(t_1) + (1-\mu) A(t_2).$$

Since $A(t) : I \to M_n^+$ be a family of positive definite real-valued matrices, it follows that $A(\mu t_1 + (1-\mu) t_2)$ and $\mu A(t_1) + (1-\mu) A(t_2)$ are both positive definite real-valued matrices. By [2] we have that there exists a orthogonal matrix C such that

$$A(\mu t_1 + (1-\mu) t_2) = C^T \Lambda_{A(\mu t_1+(1-\mu)t_2)} C$$

and

$$\mu A(t_1) + (1-\mu) A(t_2) = C^T \Lambda_{\mu A(t_1)+(1-\mu)A(t_2)} C,$$

where

$$\Lambda_{A(\mu t_1+(1-\mu)t_2)} = diag\{\lambda_1 (A(\mu t_1 + (1-\mu) t_2)), \cdots, \lambda_n (A(\mu t_1 + (1-\mu) t_2))\}$$

and

$$\Lambda_{\mu A(t_1)+(1-\mu)A(t_2)} = diag\{\lambda_1 (\mu A(t_1) + (1-\mu) A(t_2)), \cdots, \lambda_n (\mu A(t_1) + (1-\mu) A(t_2))\}.$$

Then it follows from

$$A(\mu t_1 + (1-\mu) t_2) \leq \mu A(t_1) + (1-\mu) A(t_2)$$

that

$$C^T \Lambda_{A(\mu t_1+(1-\mu)t_2)} C \leq C^T \Lambda_{\mu A(t_1)+(1-\mu)A(t_2)} C,$$

which is equivalent to

$$\lambda_i (A(\mu t_1 + (1-\mu) t_2)) \leq \lambda_i (\mu A(t_1) + (1-\mu) A(t_2))$$

for any $1 \leq ß \leq n$. Thus

$$\begin{aligned}\det A\left(\mu t_1 + (1-\mu) t_2\right) &= \prod_{i=1}^{n} \lambda_i \left(A\left(\mu t_1 + (1-\mu) t_2\right)\right) \\ &\leq \prod_{i=1}^{n} \lambda_i \left(\mu A\left(t_1\right) + (1-\mu) A\left(t_2\right)\right) \\ &= \det\left(\mu A\left(t_1\right) + (1-\mu) A\left(t_2\right)\right).\end{aligned}$$

Since $f(x)$ is monotonic increasing, it follows that

$$f\left(\det A\left(\mu t_1 + (1-\mu) t_2\right)\right) \leq f\left(\det\left(\mu A\left(t_1\right) + (1-\mu) A\left(t_2\right)\right)\right).$$

Then by using (15) we have

$$f\left(\det A(\mu t_1 + (1-\mu) t_2)\right) \leq \mu f\left(\det A(t_1)\right) + (1-\mu) f\left(\det A(t_2)\right)$$

for any $t_1 \neq t_2 \in I$ and $0 \leq \mu \leq 1$, which implies that the function $f(\det A(t))$ is a convex function of t. Then by using Theorem 1.6 we have

$$\begin{aligned}f\left(\frac{1}{b-a}\int_a^b \det A(t)\, dt\right) \leq l_A\left(\mu_1, \cdots, \mu_n\right) &\leq \frac{1}{b-a}\int_a^b f\left(\det A(t)\right) dt \\ \leq L_A\left(\mu_1, \cdots, \mu_n\right) &\leq \frac{f\left(\det A(a)\right) + f\left(\det A(b)\right)}{2},\end{aligned}$$

where $l_A\left(\mu_1, \cdots, \mu_n\right)$ and $L_A\left(\mu_1, \cdots, \mu_n\right)$ are defined in Theorem 1.8. □

Example 4.1. *Let $a_0 = a_{n+1} = 0, a_1, \cdots a_n \geq 0$ and suppose that there is at least one a_i such that $a_i \neq 0$. For $\mu_k = \frac{\sum_{i=0}^{k} a_i}{\sum_{i=0}^{n} a_i}$ from Theorem 1.8 we get*

$$l_A\left(\mu_1, \cdots, \mu_n\right) = \frac{1}{\sum\limits_{k=0}^{n} a_k} \sum_{k=0}^{n} a_{k+1} f\left(\frac{\sum\limits_{k=0}^{n} a_k}{a_{k+1}(b-a)} \int_{c_k}^{c_{k+1}} \det A(t)\right)$$

and

$$L_A\left(\mu_1, \cdots, \mu_n\right) = \frac{1}{\sum\limits_{k=0}^{n} a_k} \sum_{k=0}^{n} a_{k+1} \frac{f \circ \det A\left(c_k\right) + f \circ \det A\left(c_{k+1}\right)}{2},$$

where $c_k = \left(1-\mu_k\right) a + \mu_k b$.

5 Open Problem

In this section, we present two interesting open problems related to our paper, one open problem is:

Problem 5.1. *Is the inequality*

$$f\left(\frac{1}{b-a}\int_a^b \det A(t)\,dt\right) \leq l_A(\mu_1,\cdots,\mu_n) \leq \frac{1}{b-a}\int_a^b f(\det A(t))\,dt$$
$$\leq L_A(\mu_1,\cdots,\mu_n) \leq \frac{f(\det A(a)) + f(\det A(b))}{2}$$

holds true for some more general matrix families, such as the matrix families without the hypothesis

$$A(\mu t_1 + (1-\mu)t_2) \leq \mu A(t_1) + (1-\mu)A(t_2),$$

where $l_A(\mu_1,\cdots,\mu_n)$ and $L_A(\mu_1,\cdots,\mu_n)$ are defined in Theorem 1.8?

Another open problem is related to Hermite-Hadamard type inequality for the matrices:

Problem 5.2. *Is the inequality*

$$f\left(\frac{1}{Vol(\Omega)}\int_\Omega \det A dV_\Omega\right) \leq \frac{1}{Vol(\Omega)}\int_\Omega f(\det A)dV_\Omega \leq \frac{1}{Vol(\partial\Omega)}\int_{\partial\Omega} f(\det A)dV_{\partial\Omega}$$

holds true for the subset Ω of the convex set of positive definite matrices M_n^+ or non-negative definite matrices SM_n, where dV_Ω and $dV_{\partial\Omega}$ respectively denote the volume elements of Ω and its boundary $\partial\Omega$?

ACKNOWLEDGEMENTS. I would especially like to express my appreciation to my advisor professor Yu Zheng for longtime encouragement and meaningful discussion. I would also especially like to thank the referee for meaningful suggestions that led to improvement of the article.

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