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A Modified Wavelet Algorithm to Solve BVPs with An Infinite Number of Boundary Conditions

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Abstract

The main purpose of this research is to solve Boundary Value Problems (BVPs) with an infinite number of boundary conditions. A new algorithm has been designed and developed for solving this type of problems. This algorithm is consisting of three stages: the first stage is dealing with finding the infinite boundary values; the second stage is deeling with finding the value of $y'(a)$ using one of the most famous interpolation methods (namely; the Spline method); and the third stage is dealing with solving the problem by using the wavelet method. We have considered two test examples to prove the robustness and effectiveness of the modified suggested algorithm.

Keywords: Boundary Value Problems, Spline Function, Wavelet Method.**2000**
2010 Mathematics Subject Classification: 30E25, 65D07, 42C40.

1 Introduction

In solving Ordinary Differential Equations (ODEs) by using Haar wavelet related method, Chen and Hsiao [4] had derived an operational matrix of integration based on Haar wavelet. Lepik [19, 20, 21] had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation method. Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities, for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani [3] but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets-the simplicity gets to some extent lost. The other way is to make use of the integral method, which was proposed by Chen and Hsiao [4]. The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [7] who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [5] who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang proposed a key idea to transform the time varying function and its product with states into a Haar product matrix [9].

2 Preliminaries

2.1 Numerical solutions of Boundary Value Problems (BVP):

The well-known numerical method for solving BVPs in ODEs are:
i) Finite different method . ii) Shooting method. iii) Collocation method.

Finite differences method is based on dividing the given interval of the independent variables by nodes and then approximating the differential equation by a given finite difference formulas at each node. This will produce a set of algebraic equations, mostly non-linear, which can be solved by Newton iteration or one of its alternatives (for more details see [12] and [13]). To get accurate results for these methods, we have to increase the number of nodes which will produce a greater number of algebraic equations which increase the complexity of the solution and takes a lot of computer time. Mostly, the iteration processes at the nodes will create noisy data and this noise can be accumulated by iteration processes and render the solution meaningless. Shooting methods are probably the most popular numerical method for solving BVP. It is a successive substitution method based on the idea of guessing the initial condition which associate solution satisfies the desired boundary condition.

However, any finite-difference algorithm can be considered to solve this "new" Initial Value Problem (IVP). For details see [1, 2, 11, 18]. Unfortunately, these methods can be quite inefficient as they may often converge quite slowly, or not at all, and a wrong guess could substantially increases the computer time. Furthermore, the numerical errors can be magnified. The possible difficulties with shooting methods are frequently discussed in the literature, see for example [6, 10, 11, 15]. As an alternative, BVPs can be solved using some projection-based method, such as Collocation Method (CM). In particular, those based on Splines are commonly used, see for example Varga [8, 14, 16, 22]. In this context CM' often have better performance than other numerical methods, but the choice of the collocation points greatly influences the effectiveness of the method. Furthermore, if the solution path exhibit some abrupt changes, the approximation could be inaccurate. In numerical analysis, the discovery of compactly supported wavelets has proven to be a useful tool for the approximation of functions, differential and integral operators. The use of wavelets based algorithms is superficially similar to other projection methods, but these algorithms are more efficient because of the localization of wavelet bases in both space and frequency domain. Therefore, the approximation of a function using wavelets bases may be advantageous.

Batiha, et. al. in [23] compared the variational iteration method (VIM) with the Adomian decomposition method (ADM) for solving nonlinear integro-differential equations. From the computational viewpoint, the VIM is more efficient, convenient and easy to use. El-Hawary, et. al. in [24] describing the solving of the second order neutral delay differential equations (NDDEs) based on seventh C3-spline collocation methods with three parameters c1, c2, c3 2 (0, 1). It is shown that the proposed methods for second order NDDEs possess a convergence rate of order seven if :

$$1 - c1 - c2 - c3 + c1c2 + c1c3 + c2c3 - 2c1c2c3 <= 0.$$

Numerical results illustrating the behavior of the methods when faced with some difficult problems are presented and the numerical results are compared to those obtained by other methods.

3 Haar Wavelets.

The Haar wavelet function was introduced by Alfred Haar in 1910 in the form of a regular pulse pair . After that many other wavelet functions were generated and introduced. Those include the Shannon, Daubechies and Legendry wavelets. Among those forms, Haar wavelets have the simplest orthonormal series with compact support.

3.1 Haar Functions.

The basic and simplest form of Haar wavelet is the Haar scaling function that appears in the form of a square wave over the interval t ε [a, b] as expressed in (1) and illustrated in the first subplot of Fig1.

$$\phi_0(t)=\begin{cases}1, & 0\le t<1,\\ 0, & elsewhere.\end{cases} \tag{1}$$

The above expression, called Haar father wavelet, is the zeroth level wavelet which has no displacement and dilation of unit magnitude. Correspondingly, there is a Haar mother wavelet to match the father wavelet which is described as

$$\phi_i(t)=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1,\\ 0, & elsewhere.\end{cases} \tag{2}$$

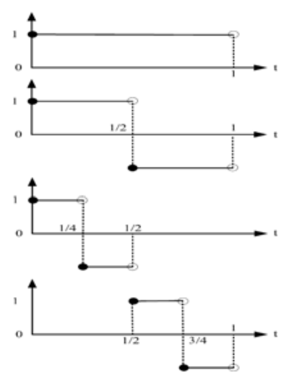


Fig. 1 Graph of Haar function

The Haar mother wavelet is the first level Haar wavelet and its graph is given in the second subplot of Fig.1. This mother wavelet can also be written as the linear combination of the Haar scaling function with translation and compression to half of its original interval

$$\varphi 1(t) = \varphi 0(2t) - \varphi 0(2t - 1) \tag{3}$$

Similarly, the other levels of wavelets can all be generated from $\varphi 1(t)$ by the operations of translation and compression. For example, the third subplot in Fig.1 is formed by compression $\varphi 1(t)$ to left half of its original interval and the forth subplot is the same as the third one plus translating to the right side by 1/2. In general, we can write out the Haar wavelet family as

$$\phi_i(t)=\begin{cases}1, & t\in[\frac{k}{m},\frac{k+0.5}{m}),\\ -1, & t\in[\frac{k+0.5}{m},\frac{k+1}{m}).\\ 0, & \text{elsewhere}.\end{cases} \tag{4}$$

Here m is the level of the wavelet, we assume the maximum level resolution is integer J , then m equals to $2j$ ($j = 0, 1, \ldots, J$); the translation parameter $k = 0, 1, \ldots, m - 1$. The series index number i is defined by m and k and $i = m + k$. For any fixed level m, there are m series of φi to fill the interval $[0, 1)$ corresponding to that level and for a provided J , the index number i can reach the maximum value $M = 2J+1$ when including all levels of wavelets. Each Haar wavelet is composed of a couple of constant steps of opposite sign during its subinterval and is zero else where. Therefore, they have the following relationship:

$$\int_0^1 \phi_i(t)\phi_l(t)dt = \begin{cases} 2^{-j}, & i = l = 2^j + k, \\ 0, & i \neq l. \end{cases} \tag{5}$$

This relationship shows that Haar wavelets are orthogonal to each other and therefore constitute an orthogonal basis. This allows us to transform any function square integral on the interval time $[0, 1)$ into Haar wavelets series.

4 A new algorithm with three different stages

In this part of the research we develop a new algorithm consisting of three stages; the first stage "A" finds the infinite number of boundary condition y(∞), the second stage "B" calculating the value of y'(a) after finding an approximate value to ∞ , and the third stage "C" solves the BVPs using the wavelet method. The details of the new algorithm is a follows:

Stage A :

Let us consider a two point BVP (this will not affect the generality of the method)

$$\frac{d^2y}{dx^2} = f(x, y, y') \quad , \quad x \in (a, b)$$

The boundary conditions

$y(a) = A$

$y(\infty) = B$

Step 1 : we change the second condition when

$$y(\infty) \to B$$
$$b \to \infty$$

Then $y(b^N) = B$

$b^N = a + (N + 1)h$

Let ε be an small arbitrary value.

Step 2 : we use the finite difference method as

Let $f_n = f_n(x) \quad , \quad g_n = g_n(x)$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = f_n y_n + g_n$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f_n y_n + h^2 g_n$$

$$y_{n+1} = \left(2 + h^2 f_n\right) y_n - y_{n-1} + h^2 g_n \tag{6}$$

Step 3 : we substitute $n = 1,2,3,\ldots N + 1$ in equation (6) to get

$$y_2 = \left(2 + h^2 f_1\right) y_1 - y_0 + h^2 g_1$$

and $y_0 = A$, $y_{n+1} = B$

$$B = \left(2 + h^2 f_1\right) y_1 - A + h^2 g_1$$

$$y_1 = \frac{B + A - h^2 g_1}{2 + h^2 f_1}$$

Step 4 if

$\left| y_{(n)}^{N+1} - y_{(n)}^{N} \right| < \varepsilon$, is satisfied stop

Else

Step 5 : Take the n values which we obtained in step 1 and substitute in step 2.

Stage B :

Let us consider a two point BVP (this will not affect the generality of the method)

$$y'' = f(x, y, y'), \quad y(a) = A \ , \ y(b) = B \ , \ x = (a, b) \tag{7}$$

The above problem can be reduced to the following system

$$\left.\begin{array}{l} y_1' = y_2 \\ y_2' = f(x, y_1, y') \\ y_1(a) = A \\ y(b) = B \end{array}\right\} \tag{8}$$

To integrate system (7) in the interval (a, b) we need a value of $y_2(a)$ which is unknown

Step 1: estimate a value S_0 for $y_2(a)$ and integrate the system (7) in the interval (a, b), to get $y_1(b) = m_0$.

Step 2: estimate another different value S_1 for $y_2(a)$ and integrate the system (7) in the interval (a, b), to get $y_1(b) = m_1$ and so on.

Step n: estimate another different value S_n for $y_2(a)$ and integrate the system (7) in the interval (a, b), to get $y_1(b) = m_n$.

Hence we will get the following table of data:

Table (1)

$T = s_0 = y_2(a)$	s_0	s_1	$\dots$	s_n
$P = m_0 = y_1(b)$	m_0	m_1	$\dots$	m_n

Then by using the Spline technique, we can find the value of $y_2(a)$ corresponding to $y_1(b)$ by

$$P(B) = S_0 + (B - m_0)\Delta S_0 + (P - m_0)(P - m_1)\Delta^2 S_0 + \dots + (B - m_0)(B - m_1)\dots(B - m_{n-1})\Delta^2 S_0$$

and the approximate value of $y_1(a)$ corresponding to $y_1(b) = B$ will be $y_2(a) = P(B)$. [12,17].

Stage C: (Wavelet method): we now list outline of stage C as:

Step 1: Let $y^{(n)}(x) = \sum_{i=1}^{m-1} a_i h_i(x)$

where h is Haar matrix and a_i is the wavelet coefficients $m = 2(2^i)$

Step 2: obtain appropriate V order of $y(x)$ by using

$$y^{(V)}(x) = \sum_{i=1}^{m} a_i Z_{n-V,i}(x) + \sum_{\sigma=0}^{n-V-1} \frac{1}{\sigma!}(X - A)^{\sigma} y_0^{(V+\sigma)}$$

Step 3: replace $y^{(n)}(x)$ and all the value of $y^{(V)}(x)$ into the problem.

Step 4: calculate the wavelet coefficients ai.

Step 5: obtain a numerical solution of $y(x)$.

The results may be formulated in a matrix form. Now we discuses Z matrix in the following form:

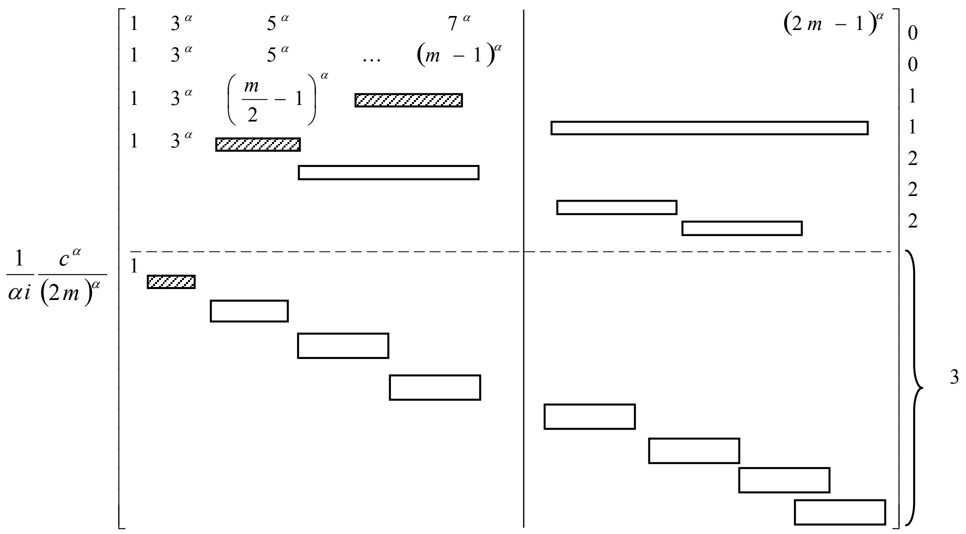


Fig (2): Designation of matrix Z

where $m = 2\left(2^{j}\right)$

represent element which needed to be count

represent element with the same value in same level.

So we use the following algorithm for counting the elements which are required

$$\frac{1}{\alpha i}\frac{1}{(2m)^{\alpha}}\left[\left(c\left(\frac{m}{2^{L}}+2k-1\right)\right)^{\alpha}-2\left(c\,(2k-1)^{\alpha}\right)\right]$$

where $L = 0, 1, \ldots, j$ (Level of Haar wavelet) [3]

$\alpha = n - V$, $c = B - A$

$k = 1, 2, \ldots, \frac{m}{2\left(2^{L}\right)}$, .

5 Numerical applications

Problem (1):

We have the following BVP $y'' = 2y - e^{-x}$ with boundary conditions $y(0) = 1$, $y(\infty) = 0$. The exact solution for this problem is $y(x) = e^{-x}$.

Solution:

1. **For stage A:**

We take $h = 0.5$ and $\varepsilon = 10^{-18}$

$$\frac{-y_{n+1}+2y_n-y_{n-1}}{h^2}+2y_n = e^{-x_n}$$

$$-y_{n+1}+2y_n-y_{n-1}+0.5y_n = 0.25\,e^{\frac{-n}{2}}$$

$$y_{n+1} = 2.5y_1 - y_{n-1} - 0.25\,e^{\frac{-n}{2}} \qquad (9)$$

Substituted $n = 1, 2, 3, \ldots$ in equation (9) to obtain

$$y_2 = 2.5y_1 - y_0 - 0.25\,e^{\frac{-1}{2}}$$

$$y_2 = 2.5y_1 - 1 - 0.25\,e^{\frac{-1}{2}}$$

by substituted $y_2 = 0$ we obtain

$$y_1 = 0.46065$$

$$y_3 = 2.5y_2 - y_1 - 0.25\,e^{-1}$$

substituted $y_3 = 0$ to obtain

$y_1 = 0.56591$

But $|0.5659 - 0.4606| \not< 10^{-8}$

So we continue to obtain

$$y_{20} = \frac{3.6650387e+011}{524288} y_1 \frac{9.125968e+010}{262144} - \frac{9.1625968e+010}{262144} e^{-0.5}$$

$$-\frac{2.2906492e+010}{65536} e^{-1} - \frac{5726623061}{262144} e^{-1.5} - \frac{1431655765}{131072} e^{-2}$$

$$-\frac{357913941}{65536} e^{-2.5} - \frac{89478485}{32768} e^{-3} - \frac{22369621}{16384} e^{-3.5} - \frac{5592405}{8192} e^{-4}$$

$$-\frac{1398101}{4096} e^{-4.5} - \frac{349525}{2048} e^{-5} - \frac{87381}{1024} e^{-5.5} - \frac{21845}{512} e^{-6} - \frac{5461}{256} e^{-6.5}$$

$$-\frac{1365}{128} e^{-7} - \frac{341}{64} e^{-7.5} - \frac{85}{32} e^{-8} - \frac{21}{16} e^{-8.5} - \frac{5}{8} e^{-9} - \frac{1}{4} e^{-9.5}.$$

We continue to $n = 19$ the second condition became as follows

$b^{n+1} = a + (n+1)h$

$b^{20} = 0 + (19+1)\frac{1}{2}$

$\therefore b^{(20)} = 10$

The boundary condition became as follows $y(0) = 1$, $y(10) \cong 0$.

2. **Using stage B** (Spline method) transform the problem to an (IVP)

Table (2)
The initial condition became as follows

X	3	2	1	0	-1
Y	1.9607	1.4706	0.9804	0.4902	0.0000

$y(0) = 1$, $y'(0) = -1$

So by using stage B (spline methods) the problem became as follows

$y'' = 2y - e^{-x}$

$y(0) = 1$, $y'(0) = -1$

3. By using stage C (wavelet) we solve the above problem as:

$u = 2$, $V = 0$, $\alpha = 2$

Step 1 : $y'' = \sum_{i=0}^{m-1} a_i h_i(x)$

Step 2 : $y(x) = \sum_{i=0}^{m-1} a_i P_{2,i}(x) + \sum_{\sigma=0}^{2-0-1} \frac{1}{\sigma!}(x-0)^{\sigma} y_0^{\sigma}$

$$= \sum_{i=0}^{m-1} a_i P_{2,i}(x) + 1 - x$$

Step 3 : $y''(x) - 2y(x) = -e^{-x}$

$$\sum_{i=0}^{m-1} a_i h_i(x) - 2\left(\sum_{i=0}^{m-1} a_i P_{2,i}(x) + 1 - x \right) = -e^{-x}$$

$$\sum_{i=0}^{m-1} a_i \left| h_i(x) - 2P_{2,i}(x) \right| = -e^{-x} + 2 - 2x$$

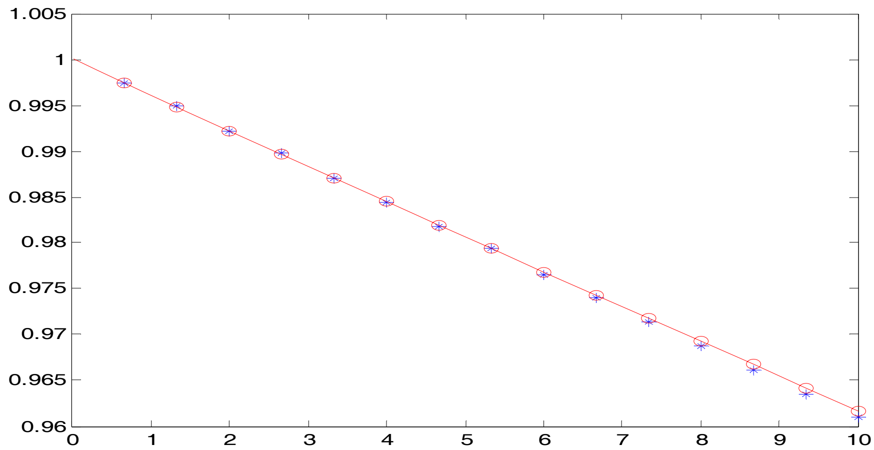
Step 4 : we solve this linear system to obtain wavelet coefficients ai.

Step 5 : after calculating ai we subtitled in step 2 to obtain the numerical solution for $y(x)$.

Table (3)

Numerical solution for problem (1)

X	y-Numerical	y-Exact	Error
0.0000	1.00038	1.00000	0.00038
0.6667	0.99740	0.99740	0.00000
1.3333	0.99495	0.99481	0.00015
2.0000	0.99219	0.99222	0.00003
2.6667	0.98984	0.98964	0.00020
3.3333	0.98698	0.98706	0.00008
4.0000	0.98438	0.98450	0.00012
4.6667	0.98177	0.98194	0.00017
5.3333	0.97947	0.97938	0.00008
6.0000	0.97656	0.97684	0.00027
6.6667	0.97396	0.97429	0.00034
7.3333	0.97135	0.97176	0.00041
8.0000	0.96875	0.96923	0.00048
8.6667	0.96615	0.96671	0.00057
9.3333	0.96354	0.96420	0.00066
10.0000	0.96094	0.96169	0.00075



Fig(3):Comparison between numerical solution and exact solution of problem(1).

Problem (2)

Now, have the following BVP $y'' = 4y - 8$ with boundary conditions $y(0) = 4$, $y(\infty) = 2$. The exact solution for this problem is:
$y(x) = 2e^{-2x} + 2$

Solution :

By using stage A
We take $h = 0.5$ and $\varepsilon = 10^{-8}$ by solving it in the same way on problem (1) we obtain $n = 21$ the second condition will be as follows

$$b^{(n+1)} = a + (n+1)h$$

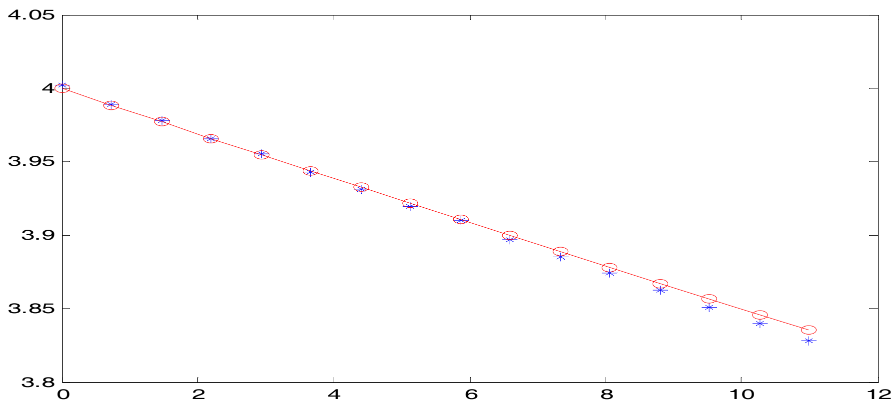
$$b^{(22)} = 0 + (21+1) * \frac{1}{2}$$

$$\therefore b^{(22)} = 11$$

So the boundary conditions became
$y(0) = 4$, $y(11) \cong 2$
Now will solve this problem by stage B (spline method) and stage C (wavelet method). We will get the following numerical solution in table (4).

Table (4)
Numerical solution for problem (2)

X	y-Numerical	y-Exact	Error
0.0000	4.00221	4.00000	0.00221
0.7333	3.98881	3.98857	0.0.00024
1.4667	3.97773	3.97721	0.00051
2.2000	3.96565	3.96592	0.00027
2.9333	3.95532	3.954469	0.00063
3.6667	3.94275	3.94352	0.00077
4.4000	3.93130	3.93242	0.00111
5.1333	3.91985	3.92138	0.00152
5.8667	3.90974	3.91040	0.00066
6.6000	3.89695	3.89949	0.00253
7.3333	3.88551	3.88864	0.00313
8.0667	3.87406	3.87785	0.00379
8.8000	3.86261	3.86712	0.00451
9.5333	3.85116	3.85645	0.00530
10.2667	3.83971	3.84585	0.00614
11.0000	3.82826	3.83530	0.00704



Fig(4): Comparison between the numerical solutions and exact solutions of problem (2)

6 Conclusions

In this research, we have developed a new algorithm for solving the BVP with **an** infinite number of boundary conditions using wavelet method. This algorithm consists of three stages; A , B and C. In the first stage "A" the approximate value of (∞) was found; in the second stage "B" the approximate value of y'(a) was found and finally, in the third stage "C", the new algorithm finds the boundary values which are different from the way used in the original paper of [17]. Two examples have been taken and a comparison was made between the exact solutions and the numerical solutions and our numerical results were good and acceptable.

5 Open Problem

For this paper we may replace the spline method by either Neural Network methods or Fuzzy Neural methods. These will open two new open problems for this paper.

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