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A Note on the Asymptotic Series Solution to Airy's Inhomogeneous Equation

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Abstract

In this note, we construct the solution to an initial value problem involving Airy's inhomogeneous equation with a variable forcing function. Solution is expressed in terms of two recently introduced integral functions whose evaluations can be obtained using asymptotic series.

Keywords: *Airy's inhomogeneous equation, integral functions, asymptotic series.*

1 Introduction

Many differential equations that arise in mathematical physics can be reduced to Airy's differential equation by either a change of variables or using an appropriate transformation, [5, 6, 9, 13]. However, computations involving Airy's inhomogeneous equation with variable forcing function continue to present a challenge due to the difficulty inherent in evaluating the resulting integrals. As is well-known, solution to Airy's inhomogeneous differential equation, namely

$$y'' - xy = f(x) \quad \ldots(1)$$

is given by the following forms, [1, 3, 7, 9, 11, 13], depending on *f(x):*

i) For *f(x)* = $-1/\pi$, solution is expressed as

$$y = c_1 Ai(x) + c_2 Bi(x) + Gi(x) \qquad \ldots(2)$$

ii) For *f(x)* = $1/\pi$, solution is expressed as

$$y = c_1 Ai(x) + c_2 Bi(x) + Hi(x) \qquad \ldots(3)$$

wherein c_1 and c_2 are arbitrary constants, *Ai(x)* and *Bi(x)* are the homogeneous Airy's functions, defined respectively by

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right) dt \qquad \ldots(4)$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt + \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt \qquad \ldots(5)$$

and *Gi(x)* and *Hi(x)* are the inhomogeneous Airy's functions (or the Scorer functions, [1, 11]), defined respectively by

$$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt \qquad \ldots(6)$$

$$Hi(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt. \qquad \ldots(7)$$

Solutions to equation (1) as given by (2) and (3) involve the two functions *Gi(x)* and *Hi(x)* in addition to the homogeneous Airy's functions. In a recent article, Hamdan and Kamel [4], provided a solution to equation (1) for the two cases of *f(x)* = $\mp 1/\pi$ using a single function, namely the Nield and Kuznetsov *Ni(x)* function [10], defined by

$$Ni(x) = Ai(x)\int_0^x Bi(t)\,dt - Bi(x)\int_0^x Ai(t)\,dt. \qquad \ldots(8)$$

Hamdan and Kamel, [4], expressed the solution to equation (1) for any constant forcing function, *f(x)* = δ, in the form

$$y = c_1 Ai(x) + c_2 Bi(x) - \pi\delta Ni(x). \qquad \ldots(9)$$

Using $\delta = -1/\pi$, and $\delta = 1/\pi$ in (9), we obtain the following solutions, respectively:

$$y = c_1 Ai(x) + c_2 Bi(x) + Ni(x). \qquad \ldots(10)$$

$$y = c_1 Ai(x) + c_2 Bi(x) - Ni(x). \qquad \ldots(11)$$

The case when the forcing function in equation (1) is a variable function of *x* is a fairly old problem, [9, 11], but continues to receive considerable attention in the literature, so do computations involving the inhomogeneous Airy's functions (*cf.* [2, 3, 7, 8, 13]). In this note we revisit this problem in an attempt to offer an alternative and viable methodology to solve initial value problems involving equation (1). We express the solution in terms of the recently introduced integral function, *Ni(x)*, [10], and in terms of an integral function, *Ki(x)*, which we will introduce in this work. While we recognize there is no easy way to compute these integral functions, we resort to asymptotic series approximations of these functions to represent the solution.

2 Problem Formulation and Method of Solution

Consider the initial value problem composed of solving equation (1) subject to

$$y(0) = \alpha \qquad \ldots(12)$$

$$y'(0) = \beta \qquad \ldots(13)$$

where α and β are known constants, and the function *f(x)* in equation (1) is a continuous function of *x*.
General solution to equation (1) is the sum of the complementary function, y_c, and a particular solution, y_p. The complementary function is the solution to the homogeneous Airy's equation, and is given by

$$y = c_1 Ai(x) + c_2 Bi(x) \qquad \ldots(14)$$

In order to find y_p we use the method of variation of parameters and assume the particular solution of the form:

$$y_p = u_1 Ai(x) + u_2 Bi(x) \qquad \ldots(15)$$

The functions u_1 and u_2 are given by the following forms, respectively, which we arrive at while utilizing the Wronskian of the Airy's homogeneous functions, [1], namely $1/\pi$:

$$u_1 = \pi \int_0^x f(t) Bi(t) dt \qquad \ldots(16)$$

$$u_2 = -\pi \int_0^x f(t) Bi(t) dt \qquad \ldots(17)$$

Using (16) and (17) in (15), we obtain:

$$y_p = \pi\left\{Ai(x)\int_0^x f(t)Bi(t)dt - Bi(x)\int_0^x f(t)Ai(t)dt\right\}. \quad \ldots(18)$$

Now, the integrals appearing in (18) are expressed in the following forms:

$$\int_0^x f(t)Bi(t)\,dt = f(x)\int_0^x Bi(t)\,dt - \int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt \quad \ldots(19)$$

$$\int_0^x f(t)Ai(t)\,dt = f(x)\int_0^x Ai(t)\,dt - \int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt. \quad \ldots(20)$$

Now, defining the integral function $Ki(x)$ by:

$$Ki(x) = Ai(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - Bi(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt \quad \ldots(21)$$

and using (10) and (21) in (18), we can express the particular solution as:

$$y_p = \pi\, Ki(x) - \pi f(x)Ni(x). \quad \ldots(22)$$

General solution to equation (1) is then obtained from (14) and (22) as:

$$y = c_1 Ai(x) + c_2 Bi(x) + \pi Ki(x) - \pi f(x)Ni(x). \quad \ldots(23)$$

Now, in order to obtain the solution to equation (1) satisfying initial conditions (12) and (13), we need the derivative of $Ni(x)$ and $Ki(x)$, and the values $Ni(0)$, $Ki(0)$, $N'i(0)$ and $K'i(0)$.

From (10) we obtain, [10]

$$N'i(x) = A'i(x)\int_0^x Bi(t)\,dt - N'i(x)\int_0^x Ai(t)\,dt \quad \ldots(24)$$

and the following zeroes of $Ni(x)$ and $N'i(x)$ are then obtained from (10) and (24):

$$Ni(0) = N'i(0) = 0. \quad \ldots(25)$$

Derivative of the function $Ki(x)$ is obtained from (21) and takes the following form, after some simplification:

$$K'i(x) = A'i(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - B'i(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt + f'(x)Ni(x) \quad \ldots(26)$$

The following zeroes are then obtained from (21) and (26), with the help of (25):

$$Ki(0) = K'i(0) = 0. \qquad \ldots(27)$$

Now, a solution satisfying initial conditions (12) and (13) is obtained from (23) with the help of (25) and (27), and takes the form:

$$y = \pi\left\{\frac{3^{1/6}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{3^{1/6}\Gamma(\frac{2}{3})}\right\}Ai(x) + \pi\left\{\frac{\alpha}{3^{1/3}\Gamma(\frac{1}{3})} + \frac{\beta}{3^{2/3}\Gamma(\frac{2}{3})}\right\}Bi(x) + \pi Ki(x) - \pi f(x)Ni(x). \qquad \ldots(28)$$

It is worth noting that due to the zeroes in (25) and (27), the arbitrary constants c_1 and c_2 in equation (23) will always take the values expressed in (28) for any forcing function *f(x)*.

3 Asymptotic series approximation

Solution (28) mandates the evaluation of the functions *Ai(x)*, *Bi(x)*, *Ni(x)* and *Ki(x)*. In what follows, we will use asymptotic series approximations for these functions. Asymptotic series approximations to the homogeneous Airy functions are given by the following forms, [13]:

$$Ai(x) \approx \frac{\exp(-\mu)}{2\sqrt{\pi}\,x^{1/4}}\left[1 + \frac{3(5)}{1!(-216\mu)} + \frac{5.7.9.11}{2!(-216\mu)^2} + \cdots\right] \qquad \ldots(29)$$

$$Bi(x) \approx \frac{\exp(\mu)}{\sqrt{\pi}\,x^{1/4}}\left[1 + \frac{3(5)}{1!(216\mu)} + \frac{5.7.9.11}{2!(216\mu)^2} + \cdots\right] \qquad \ldots(30)$$

$$\int_0^x Ai(t)dt \approx \frac{1}{3} - \frac{\exp(-\mu)}{2\sqrt{\pi}x^{3/4}}\left[1 - \frac{41}{48x^{3/2}} + \frac{9241}{4608x^3} - \cdots\right] \qquad \ldots(31)$$

$$\int_0^x Bi(t)dt \approx \frac{\exp(\mu)}{\sqrt{\pi}x^{3/4}}\left[1 + \frac{41}{48x^{3/2}} + \frac{9241}{4608x^3} + \cdots\right] \qquad \ldots(32)$$

where $\mu = \frac{2}{3}x^{3/2}$.

Truncating series (29)-(32) after the first term, we develop the following asymptotic approximations to $Ni(x)$ and $Ki(x)$, valid for large x:

$$Ni(x) \approx -\frac{1}{3}Bi(x) = -\frac{\exp(\mu)}{3\sqrt{\pi}x^{1/4}} \qquad \ldots(33)$$

$$Ki(x) = \frac{\exp(-\mu)}{2\sqrt{\pi}\,x^{1/4}}\int_0^x \left\{\frac{\exp(\varphi)}{\sqrt{\pi}t^{3/4}}\right\} f'(t)dt - \frac{\exp(\mu)}{3\sqrt{\pi}\,x^{1/4}}f(x) \qquad \ldots(34)$$

where $\varphi = \frac{2}{3}t^{3/2}$.

For the sake of illustration, suppose we use the forcing function $f(x) = x^{9/4}$ in (28), then upon substituting (29), (30), (33) and (34) in (28) we obtain, after some simplification, the following solution that is valid for large *x*:

$$y = \sqrt{\pi}x^{-1/4}\left\{\frac{3^{1/6}\alpha}{\Gamma\left(\frac{1}{3}\right)} - \frac{\beta}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}\right\}\frac{\exp(-\mu)}{2} + \sqrt{\pi}x^{-1/4}\left\{\frac{\alpha}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} + \frac{\beta}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\right\}\exp(\mu)$$
$$+\frac{x^{-1/4}}{2}[1 - \exp(-\mu)] + \frac{(\pi-1)\exp(\mu)}{3\sqrt{\pi}}x^2. \quad \ldots(35)$$

It is clear from (34) that one's ability to find $Ki(x)$ hinges on one's ability to evaluate the integral $\int_0^x \left\{\frac{\exp(\varphi)}{\sqrt{\pi}t^{3/4}}\right\} f'(t)dt$. For the chosen forcing function,$f(x) = x^{9/4}$, this has been a rather simple matter. More general forcing functions may be handled through series representation of the integrand followed by term-by-term integration.

4 Conclusion

In this note, we provided a promising methodology for handling an initial value problem associated with Airy's inhomogeneous equation. The method hinges on the two integral functions *Ni(x)* and *Ki(x)*. The function *Ni(x)* has recently been introduced by Nield and Kuznetson, [10], and the *Ki(x)* function has been defined in this work.

5 Open Problem

The proposed methodology has been applied to the case of forcing function being a power function composed of a single term. However, the following two important problems remain open and require further investigation:

a) When the forcing function is of a more general type, is there a more efficient method that requires lesser computations than term-by-term integration of the resulting series?

b) It is well-known that numerical computations of the Scorer functions, and the Airy functions, is rather involved. Does the introduction of the Ni(x) and Ki(x) reduce the amount of computations?

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