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Derivations On Prime Near-rings

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Abstract

Several results asserts that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. Our aim in this paper is to investigate the conditions for a near-ring to be a commutative ring. Moreover, examples proving the necessity of the primeness condition are given.

Keywords: *prime near-rings, derivation, commutativity*
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1 Introduction

A left near-ring is a set N with two operations $+$ and $\cdot$ such that $(N,+)$ is a group and $(N,\cdot)$ is a semigroup satisfying the left distributive law $x\cdot(y+z) = x\cdot y+x\cdot z$ for all $x,y,z\in N$. N is called Zero symmetric left near-rings satisfy $0\cdot x = 0$ for all $x\in N$ (recall that left distributivity yields $x\cdot 0 = 0$). Throughout this paper, unless otherwise specified, we will use the word near-ring to mean zero symmetric left near-ring and denote xy instead of $x\cdot y$. An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x,y\in N$, or equivalently, as noted in [7], that $d(xy) = d(x)y + xd(y)$ for all $x,y\in N$. According to [5], a near-ring N is said to be prime if $xNy = 0$ for $x,y\in N$ implies $x = 0$ or $y = 0$. For any $x,y\in N$ as usual $[x,y] = xy-yx$ and $x\circ y = xy+yx$ will denote the well-known Lie and Jordan products respectively. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x\in N \mid xy = yx \text{ for all } y\in N\}$.

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There is an increasing body of evidence that prime near-rings with derivations have ring like behavior, indeed, there are several results (see for example [2], [3], [4], [5]) asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. In this paper we continue the line of investigation regarding the study of prime near-rings with derivations. More precisely, we shall prove that a prime near-ring which admits a nonzero derivation satisfying certain differential identities must be a commutative ring.

2 Main results

In [6], M. N. Daif and H. E. Bell established that a prime ring R must be commutative if it admits a derivation d such that either $d([x,y]) = [x,y]$ for all x, y in K or $d([x,y]) = -[x,y]$ for all x, y in K, where K is a nonzero ideal of R. Inspired by the results of Bell and Daif, our purpose in this section is to give conditions under which a near-ring must be a commutative ring.
We begin with the following lemma which is essential in developing the proof of our main result.

Lemma 2.1 ([3], Theorem 2.1) *Let N be a prime near-ring. If N admits a nonzero derivation d for which $d(N) \subset Z(N)$, then N is a commutative ring.*

Theorem 2.2 *Let N be a prime near-ring. If N admits a nonzero derivation d such that $d([x,y]) = [x,y]$ for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that

$$d([x,y]) = [x,y] \quad \text{for all } x, y \in N. \tag{1}$$

Replacing y by xy in (1), because of $[x,xy] = x[x,y]$, we get

$$x[x,y] = d(x[x,y]) \ \text{ for all } x, y \in N.$$

Since $d(x[x,y]) = xd([x,y]) + d(x)[x,y]$, then according to (1) we obtain

$$x[x,y] = x[x,y] + d(x)[x,y]$$

and therefore $d(x)[x,y] = 0$. Hence

$$d(x)(xy - yx) = 0 \quad \text{for all } x, y \in N. \tag{2}$$

Substituting yz for y in (2), we obtain $d(x)y(xz - zx) = 0$ which leads to

$$d(x)N(xz - zx) = 0 \quad \text{for all } x, z \in N. \tag{3}$$

Since N is prime, equation (3) reduces to

$$d(x) = 0 \quad \text{or} \quad [x, z] = 0 \;\; \text{for all} \;\; x, z \in N. \tag{4}$$

From (4) it follows that for each fixed $x \in N$ we have

$$d(x) = 0 \quad \text{or} \;\; x \in Z(N). \tag{5}$$

But $x \in Z(N)$ also implies that $d(x) \in Z(N)$ and equation (5) forces

$$d(x) \in Z(N) \quad \text{for all} \;\; x \in N. \tag{6}$$

In the light of (6), $d(N) \subset Z(N)$ and using Lemma 2.1 we conclude that N is a commutative ring. This completes the proof of our theorem. ∎

Remark 1. The hypothesis of Theorem 2.2 may be weakened a bit. Indeed, one may assume that $d([x, y]) = [x, y]$ for all x, y in some nonzero semigroup right ideal U. The proof is essentially the same, but it uses Lemma 1.3 (iii) of [3].

The following example proves that the primeness hypothesis in Theorem 2.2 is necessary even in the case of arbitrary rings.

Example 1. Let R be a commutative ring which is not a zero ring and consider $N = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} |x, y \in R \right\}$. If we define $d : N \longrightarrow N$ by $d \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$, then it is straightforward to check that d is a nonzero derivation of N. On the other hand, if $a = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$, where $0 \neq r$, then $aNa = 0$ which proves that N is not prime. Moreover, d satisfies the condition

$$d([A, B]) = [A, B] \;\; \text{for all} \;\; A, B \in N,$$

but N is a noncommutative ring.

Theorem 2.3 *Let N be a prime near-ring. If N admits a nonzero derivation d such that $d([x, y]) = -[x, y]$ for all $x, y \in N$, then N is a commutative ring.*

Proof. From

$$d([x, xy]) = -[x, xy] \quad \text{for all} \;\; x, y \in N$$

it follows that

$$d(x[x, y]) = -x[x, y] \quad \text{for all} \;\; x, y \in N.$$

Thus

$$d(x)[x,y]+xd([x,y])=d(x)[x,y]+x(-[x,y])=d(x)[x,y]-x[x,y]=-x[x,y]$$

and therefore

$$d(x)[x,y]=0 \quad \text{for all } \ x,y\in N.$$

The rest of the proof is as in the proof of Theorem 2.2. ∎

Remark 2. Note that again the $d([x,y]) = -[x,y]$ hypothesis need only hold on a nonzero semigroup right ideal.

The conclusion of Theorem 2.2 remains valid if we replace the product $[x,y]$ by $x\circ y$. In fact, we obtain the following result:

Theorem 2.4 *Let N be a prime near-ring. If N admits a nonzero derivation d such that $d(x\circ y)=x\circ y$ for all $x,y\in N$, then N is a commutative ring.*

Proof. By the hypotheses, we have

$$d(x\circ y)=xy+yx \quad \text{for all } \ x,y\in N. \tag{7}$$

Replacing y by xy in (7), we get

$$d(x\circ(xy))=x^2y+xyx \quad \text{for all } \ x,y\in N. \tag{8}$$

Since $x\circ(xy)=x(x\circ y)$, then (7) yields $d(x\circ(xy))=x(x\circ y)+d(x)(x\circ y)$. Hence equation (8) reduces to

$$x(x\circ y)+d(x)(x\circ y)=x^2y+xyx \quad \text{for all } \ x,y\in N. \tag{9}$$

As $x^2y+xyx=x(x\circ y)$, then (9) assures that

$$d(x)(x\circ y)=0 \quad \text{for all } \ x,y\in N$$

which leads to

$$d(x)xy=-d(x)yx \quad \text{for all } \ x,y\in N. \tag{10}$$

Substituting yz for y in (10), we find that

$$-d(x)yzx=d(x)xyz=(-d(x)yx)z=d(x)y(-x)z \quad \text{for all } \ x,y,z\in N. \tag{11}$$

Since $-d(x)yzx=d(x)yz(-x)$, then (11) becomes

$$d(x)yz(-x)=d(x)y(-x)z \quad \text{for all } \ x,y,z\in N. \tag{12}$$

Taking $-x$ instead of x in (12) gives

$$d(-x)yzx = d(-x)yxz \quad \text{for all } \ x, y, z \in N$$

so that $d(-x)y(zx - xz) = 0$ and therefore

$$d(-x)N[z, x] = 0 \quad \text{for all } \ x, z \in N. \tag{13}$$

By primeness, equation (13) assures that for each $x \in N$, either $x \in Z(N)$ or $d(-x) = 0$. Accordingly,

$$d(x) = 0 \quad \text{or} \quad [x, z] = 0 \ \text{ for all } \ x, z \in N. \tag{14}$$

Since equation (14) is the same as equation (4), arguing as in the proof of Theorem 2.2 we conclude that N is a commutative ring. ∎

The following example proves that the primeness hypothesis in Theorem 2.4 is necessary even in the case of arbitrary rings.

Example 2. Let S be any ring. Next, let us consider the ring $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} | x, y, z \in S \right\}$. Define a map $d : N \longrightarrow N$ such that $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If we set $a = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $0 \neq s$, then $aNa = 0$ proving that N is not prime. Moreover, it can be easily seen that d is a nonzero derivation such that

$$d(A \circ B) = A \circ B \quad \text{for all } \ A, B \in N,$$

but N is a noncommutative ring.

Theorem 2.5 *Let N be a prime near-ring. If N admits a nonzero derivation d such that $d(x \circ y) = -(x \circ y)$ for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that

$$d(x \circ y) = -(x \circ y) \quad \text{for all } \ x, z \in N. \tag{15}$$

Replacing y by xy in (15) we get

$$d(x(x \circ y)) = -x(x \circ y) \quad \text{for all } \ x, y \in N. \tag{16}$$

Since

$$d(x(x \circ y)) = d(x)(x \circ y) + xd(x \circ y) = d(x)(x \circ y) + x(-(x \circ y)) = d(x)(x \circ y) - x(x \circ y)$$

then equation (16) reduces to

$$d(x)(x \circ y) - x(x \circ y) = -x(x \circ y) \quad \text{for all} \quad x, y \in N,$$

in such a way that

$$d(x)(x \circ y) = 0 \quad \text{for all} \quad x, y \in N.$$

Therefore, the rest of the proof is as in the proof of Theorem 2.4. ∎

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