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Total Domination in Circulant Graphs

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Abstract

Cayley graph is a graph constructed out of a group Γ and a generating set $A \subseteq \Gamma$. When $\Gamma = \mathbb{Z}_n$, the corresponding Cayley graph is called as a circulant graph and denoted by $Cir(n, A)$. In this paper, we attempt to find the total domination number of $Cir(n, A)$ for a particular generating set A of $\mathbb{Z}_n$ and a minimum total dominating set of $Cir(n, A)$. Further, it is proved that $Cir(n, A)$ is 2-total excellent if and only if $n = t|A| + 1$ for some integer $t > 0$.

Keywords: *circulant graph, k-total excellent graph, total domination.*

1 Introduction

The concept of Cayley graphs is useful to attempt routing problem in parallel computing. In parallel computers, more than one processor is inducted and in order to communicate between processors, a network is essential as a part of the system. For this purpose, famous Cayley networks viz., ring, torus and hypercube are used[4]. The concept of domination in Cayley graphs has been studied by various authors and one can refer to [1, 5, 3, 6]. I.J.Dejter and O.Serra[1] have obtained efficient dominating sets for Cayley graphs constructed on permutation groups. The efficient domination number for vertex transitive graphs have been obtained by J.Huang and J-M.Xu[3] whereas J.Lee[5] has obtained a necessary and sufficient condition for the existence of independent perfect domination sets in Cayley graphs. N.Obradovič, J.Peters and G.Ru*žić*[6] have studied the efficient dominating sets in circulant graphs with two chord lengths. Tamizh Chelvam and Rani[7, 8, 9] have obtained domination parameters such as domination, independent domination, total domination and connected domination for some classes of circulant graphs.

Let Γ be a group with e as the identity element of Γ. A generating set of Γ is a subset A such that every element of Γ can be expressed as a product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The Cayley graph is defined by $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, xa)/x \in V(G), a \in A\}$ and it is denoted by $Cay(\Gamma, A)$. Since A is a generating set of Γ, $Cay(\Gamma, A)$ is a connected regular graph of degree $|A|$. When $\Gamma = \mathbb{Z}_n$, the Cayley graph $Cay(\Gamma, A)$ is called as a circulant graph and denoted by $Cir(n, A)$, where A is a generating set of $\mathbb{Z}_n$.

Let $G = (V, E)$ be a graph. The open neighbourhood $N(v)$ of a vertex $v \in V(G)$, is the set of all vertices which are adjacent to v. The closed neighbourhood of v is defined by $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N(S)$ is defined to be $\underset{v\in S}{\cup} N(v)$, and the closed neighbourhood of S is defined by $N[S] = N(S) \cup S$ [2]. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$, is called a total dominating set of G if $N(S) = V(G)$. The total domination number $\gamma_t(G)$ of a graph G, is the minimum cardinality of a total dominating set in G and a corresponding total dominating set is called a γ_t-set of G [2]. A graph G is said to be total excellent if for each vertex $v \in V(G)$, there exists a γ_t-set D_t such that $v \in D_t$. A graph G is said to be k-total excellent if for every subset $S \subseteq V(G)$ with $|S| = k$, there exists a γ_t-set D_t such that $S \subseteq D_t$.

Throughout this paper, $n(\geq 3)$ is a positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, $m = \lfloor \frac{n-1}{2} \rfloor$ and k is an integer such that $1 \leq k \leq m$. The set $A \subset \mathbb{Z}_n$, is taken as $A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ when n is odd, and $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ when n is even. Since A contains at least two consecutive integers, A is a generating set of $\mathbb{Z}_n$. Hereafter, $+$ stands for addition modulo n in $\mathbb{Z}_n$.

2 Total Domination

In this section, we obtain the value of the total domination number and a corresponding γ_t-set for $G = Cir(n, A)$, where $A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ when n is odd, and $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ when n is even.

Lemma 2.1 *Let $n(\geq 3)$ be an odd integer, $m = \frac{n-1}{2}$ and $G = Cir(n, A)$, where $A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. Then $\gamma_t(G) = \lceil \frac{n}{2k} \rceil$.*

Proof. Let $\ell = \lceil \frac{n}{2k} \rceil$. Since G is $2k$-regular, from the definition of the total domination number, it follows that $\lceil \frac{n}{2k} \rceil \leq \gamma_t(G)$. Let $x = m + k + 1$ and $D_t = \{x, x + 2k, x + 2(2k), \ldots, x + (\ell - 1)2k\}$. One may note at this moment that some of the elements of D_t exceed n and hence addition modulo n is operated. Note that, $|D_t| = \ell$. Since $\ell = \lceil \frac{n}{2k} \rceil$, $n = (\ell-1)2k+j$ for some j with $1 \leq j \leq 2k$. Thus $V(G)$ can be partitioned into ℓ intervals $I_1 = [1, 2k], I_2 = [2k + 1, 2(2k)], I_3 = [2(2k) + 1, 3(2k)], \ldots, I_{\ell-1} = [(\ell - 2)2k + 1, (\ell - 1)2k]$ and $I_\ell = [(\ell - 1)2k + 1, n(= 0)]$. Note that, $|I_i| = 2k$ for all i with $1 \leq i \leq \ell - 1$ and $1 \leq |I_\ell| \leq 2k$. Since $n - m = m + 1$, one can write the generating set A as $A = \{m - (k - 1), m - (k - 2), \ldots, m, m + 1, \ldots, m + k\}$.

For any i with $0 \leq i \leq \ell-2$, we have $x+i(2k) \in D_t$ and $I_{i+1} = [i(2k)+1, (i+1)2k]$. Since $(x+i(2k))+(m-(k-1)) \equiv i(2k)+1 \ (mod\ n)$ and A is a set of $2k$ consecutive integers with least element $m-(k-1)$, we have $N(x+i(2k)) = I_{i+1}$. Also, $(x + (\ell - 1)(2k)) + (m - (k - 1)) \equiv (\ell - 1)(2k) + 1 \ (mod\ n)$ and so $I_\ell \subseteq N(x + (\ell - 1)2k)$. Hence, $N(D_t) = N(\{x, x + 2k, x + 2(2k), \ldots, x + (\ell - 1)2k\}) \subseteq I_1 \cup I_2 \cup \ldots \cup I_\ell = V(G)$ and so D_t is a total dominating set of G. Thus $\gamma_t(G) \leq \lceil \frac{n}{2k} \rceil$ and so $\gamma_t(G) = \lceil \frac{n}{2k} \rceil$.

Lemma 2.2 *Let $n(\geq 3)$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $G = Cir(n, A)$, where $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. Then $\gamma_t(G) = \lceil \frac{n}{2k+1} \rceil$.*

Proof. As $|A| = 2k+1$, let $\ell = \lceil \frac{n}{|A|} \rceil = \lceil \frac{n}{2k+1} \rceil$. Since n is even and $m = \lfloor \frac{n-1}{2} \rfloor$, we have $m = \frac{n}{2} - 1$. Since $\frac{n}{2} = m + 1$ and $n - m = m + 2$, one can write the generating set A as $A = \{m-(k-1), m-(k-2), \ldots, m, m+1, \ldots, m+k+1\}$. Since G is a $2k + 1$-regular graph, we have $\lceil \frac{n}{2k+1} \rceil \leq \gamma_t(G)$. Let $x = m + k + 2$ and $D_t = \{x, x+(2k+1), x+2(2k+1), \ldots, x+(\ell-1)(2k+1)\}$. By partitioning the vertices of G into ℓ intervals $I_1 = [1, 2k + 1], I_2 = [(2k + 1) + 1, 2(2k + 1)], \ldots, I_{\ell-1} = [(\ell-2)(2k+1)+1, (\ell-1)(2k+1)]$ and $I_\ell = [(\ell-1)(2k+1)+1, n(= 0)]$, one can prove the remaining part of the proof as in the proof of Lemma 2.1.

Remark 2.3 In the above theorem, each interval (except I_ℓ) contains exactly $2k + 1$ vertices and $1 \leq |I_\ell| \leq 2k + 1$. From this, one can find that the vertices $1, 2, \ldots, (2k + 1) - j$ are dominated by both $x(= m + k + 2)$ and $x + (\ell - 1)(2k + 1)$, and they are the only vertices dominated twice by the vertices of D_t, specified in Lemma 2.2.

Remark 2.4 Since the elements of $V(G)$ are group elements, $D_t + v$ is a γ_t-set for all $v \in V(G)$. This implies that G is total excellent. In particular,

$D_t^{''} = D_t + (n-x) = \{0,(2k+1),2(2k+1),\ldots,(\ell-1)(2k+1)\}$ when n is even and $D_t^{'} = D_t + (n-x) = \{0,2k,2(2k),\ldots,(\ell-1)2k\}$ when n is odd, are also γ_t-sets of $G = Cir(n,A)$ with respective to the generating sets $\{\frac{n}{2}, m, n-m, m-1, n-(m-1),\ldots,m-(k-1), n-(m-(k-1))\}$ and $\{m, n-m, m-1, n-(m-1),\ldots,m-(k-1), n-(m-(k-1))\}$ respectively.

3 2-Total Excellent Circulant Graphs

In this section, 2-total excellent circulant graphs are characterized.

Lemma 3.1 *Let $n(\geq 3)$ be an odd integer, $m = \frac{n-1}{2}$ and $G = Cir(n,A)$, where $A = \{m, n-m, m-1, n-(m-1),\ldots,m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. If $n = t(2k)+1$ for some integer $t > 0$, then G is 2-total excellent.*

Proof. Let $\ell = \lceil \frac{n}{2k} \rceil = t+1$ and $x = m+k+1$. Then by Lemma 2.1, $\gamma_t(G) = \ell$. Further, $D_t = \{x, x+2k, x+2(2k),\ldots,x+(\ell-1)2k\}$ is a γ_t-set of G. Actually, the vertex $x+i(2k) \in D_t$ dominate all the vertices in the interval $I_{i+1} = [i(2k)+1,(i+1)(2k)]$ for all i with $0 \leq i \leq \ell-1$ and $V(G) = I_1 \cup I_2 \cup \ldots \cup I_\ell$. Note that, the interval I_ℓ contains only one element $n = 0 = t(2k)+1$. Since the cancellation law is valid for the elements of $\mathbb{Z}_n = V(G)$, to prove G is 2-total excellent, it is enough if we prove that for any $d \in V(G)(d \neq x)$, there exists a γ_t-set D_1 such that $\{x,d\} \subseteq D_1$.

Let $d(\neq x) \in V(G)$. If $d \in D_t$, then nothing to prove. On the other hand $d \notin D_t$. Since $x+(\ell-1)2k+1 \equiv x(mod\ n)$, there exist no element between $x+(\ell-1)2k$ and x. Thus d lies between $x+i(2k)$ and $x+(i+1)2k$ for some i with $0 \leq i \leq \ell-2$. From this, $d = x+i(2k)+j$ for some j with $1 \leq j \leq 2k-1$.

Having i fixed, consider the set $D_1 = \{x, x+2k, x+2(2k),\ldots,x+i(2k), d, x+(i+1)2k+1, x+(i+2)2k+1,\ldots,x+(\ell-2)2k+1\}$. Clearly, $|D_1| = \ell$. Now, one can partition the vertices of G into $(\ell-1)$ intervals $J_1 = [1,2k], J_2 = [2k+1,2(2k)],\ldots,J_i = [(i-1)2k+1,i(2k)], J_{i+1} = [i(2k)+1,(i+1)2k+1], J_{i+2} = [(i+1)2k+2,(i+2)2k+1],\ldots,J_{\ell-1} = [(\ell-2)(2k)+2,n(=0)]$. Except J_{i+1}, all J_j's contains exactly $2k$ elements and $|J_{i+1}| = 2k+1$. As in Lemma 2.1, the vertex $x+g(2k) \in D_1$ dominate all the vertices in the interval J_{g+1} for $0 \leq g \leq i-1$. Also, the vertex $x+i(2k)$ dominate all the elements between $i(2k)+1$ and $(i+1)2k$.

Consider the element $m+k-j+1$. If $j = k$, then $m+k-j+1 = m+1 \in A$. If $j < k$, then $1 \leq k-j \leq k-1 \Rightarrow 1 \leq k-j+1 \leq k$. Since $m+1, m+2,\ldots,m+k \in A$, we have $m+(k-j+1) \in A$. If $j > k$, then $1 \leq j-k \leq k-1 \Rightarrow 0 \leq j-k-1 \leq k-2$. Since $m, m-1, m-2,\ldots,m-(k-1) \in A$, we have $m-(j-k-1) \in A$. Thus in all cases, $m+k-j+1 \in A$ and $d+(m+k-j+1) = (x+i(2k)+j)+(m+k-j+1) \equiv (m+k+1+i(2k)+$

$m+k+1)(mod\ n) \equiv (2m+1+(i+1)2k+1)(mod\ n) \equiv ((i+1)2k+1)(mod\ n)$. This means that d dominates the vertex $(i+1)2k+1$. Hence, the vertices $x+i(2k)$ and d together dominate all the vertices in the interval J_{i+1}.

Also, for g with $i+1 \leq g \leq \ell-2$, we have $x+g(2k)+1 \in D_1$ and $N(x+g(2k)+1) = [g(2k)+2,(g+1)(2k)+1] = J_{g+1}$. This means that $J_{i+2}, J_{i+3}, \ldots, J_{\ell-1}$ are also dominated by D_1. Hence, G is 2-total excellent.

Theorem 3.2 *Let $n(\geq 3)$ be an odd integer, $m = \frac{n-1}{2}$ and $G = Cir(n,A)$, where $A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. If $n = t(2k)+j$ for some integers $t(>0)$ and j with $1 \leq j \leq 2k$, then G is 2-total excellent if and only if $j = 1$.*

Proof. Suppose $j = 1$. Then $n = t(2k)+1$ and hence by Lemma 3.1, G is 2-total excellent.

Conversely, assume that G is 2-total excellent. Suppose $j \neq 1$. Then $n = t(2k)+j$ for some j with $1 < j \leq 2k$. Consider the two vertices x and $x+1$, where $x = m+k+1$. Since G is 2-total excellent, there exists a γ_t-set D_t such that $\{x, x+1\} \subseteq D_t$. Then by Lemma 2.1, $|D_t| = \ell = \lceil \frac{n}{2k} \rceil$ and $\ell = t+1$. Also, the vertex x dominate all the vertices in the interval $[1, 2k]$ and hence $x+1$ dominate all the vertices in the interval $[2, 2k+1]$. Thus the two vertices x and $x+1$ together dominate exactly $2k+1$ vertices of G. Since G is a $2k$-regular graph, the remaining $\ell-2$ vertices of D_t can dominate at most $(\ell-2)2k$ vertices. Hence, D_t can dominate at most $(\ell-2)2k+2k+1$ vertices whereas $(\ell-2)2k+2k+1 = (\ell-1)2k+1 = t(2k)+1 < n$, a contradiction to D_t is a total dominating set. Hence, $j = 1$.

Lemma 3.3 *Let $n(\geq 3)$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $G = Cir(n,A)$, where $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. If $n = t(2k+1)+1$ for some integer $t(>0)$, then G is 2-total excellent.*

Proof. Let $\ell = \lceil \frac{n}{2k+1} \rceil = t+1$ and $x = m+k+2$. By Lemma 2.2, $\gamma_t(G) = \lceil \frac{n}{2k+1} \rceil$ and $D_t = \{x, x+(2k+1), x+2(2k+1), \ldots, x+(\ell-1)(2k+1)\}$ is a γ_t-set of G. From the proof of the Lemma 2.2, the vertex $x+i(2k+1) \in D_t$ dominate all the vertices in the interval $I_{i+1} = [i(2k+1)+1, (i+1)(2k+1)]$ for all i with $0 \leq i \leq \ell-1$ and $V(G) = I_1 \cup I_2 \cup \ldots \cup I_\ell$. Note that, the interval I_ℓ contains exactly one vertex $n(=0)$. To prove the result, it is enough to prove that for given $d \in V(G)(d \neq x)$, there exists a γ_t-set D_1 such that $\{x, d\} \subseteq D_1$.

Let $d(\neq x) \in V(G)$. If $d \in D_t$, then nothing to prove. Otherwise, d lies between $x+i(2k+1)$ and $x+(i+1)(2k+1)$ for some $0 \leq i \leq \ell-2$. Since $x+(\ell-1)(2k+1)+1 \equiv x(mod\ n)$, there exist no element between $x+(\ell-1)(2k+1)$ and x. Hence, $d = x+i(2k+1)+j$ for some j with $1 \leq j \leq 2k$. As in the proof of Lemma 3.1, one can prove that $D_1 = \{x, x+$

$(2k+1), \ldots, x+i(2k+1), d, x+(i+1)(2k+1)+1, \ldots, x+(\ell-2)(2k+1)+1\}$ is a γ_t-set.

Theorem 3.4 *Let $n(\geq 3)$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $G = Cir(n, A)$, where $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. If $n = t(2k+1)+j$ for some integers $t(> 0)$ and j with $1 \leq j \leq 2k+1$, then G is 2-total excellent if and only if $j = 1$.*

Proof. Suppose $j = 1$. Then $n = t(2k+1)+1$ and hence by Lemma 3.3, G is 2-total excellent.

Conversely, let G be 2-total excellent. Suppose $j \neq 1$. Then $n = t(2k+1)+j$ for some integer j with $1 < j \leq 2k+1$. Consider the two vertices x and $x+1$, where $x = m+k+2$. As in the proof of Theorem 3.2, one can prove that there exist no γ_t-set D_t such that $\{x, x+1\} \subseteq D_t$.

Lemma 3.5 *Let $n(\geq 3)$ be an odd integer, $m = \frac{n-1}{2}$ and $G = Cir(n, A)$, where $A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. Then G is not q-total excellent for $q \geq 3$.*

Proof. Consider the three consecutive integers $x, x+1$ and $x+2$, where $x = m+k+1$. Suppose there exists a γ_t-set D_t such that $\{x, x+1, x+2\} \subseteq D_t$. Then by Lemma 2.1, $|D_t| = \ell = \lceil \frac{n}{2k} \rceil$ and $n = (\ell-1)2k+j$ for some integer j with $1 \leq j \leq 2k$. As in Lemma 2.1, x dominate all the vertices in the interval $[1, 2k]$ and hence the vertices $x+1, x+2$ dominate the intervals $[2, 2k+1], [3, 2k+2]$ respectively. This means that the three vertices $x, x+1$ and $x+2$ together dominate exactly $2k+2$ vertices of G. Therefore, the remaining $\ell-3$ vertices of D_t can dominate at most $(\ell-3)2k$ vertices of G and so D_t can dominate at most $(\ell-3)2k+2k+2$ vertices of G whereas $(\ell-3)2k+2k+2 = (\ell-2)2k+2 < (\ell-1)2k+j = n$, a contradiction to D_t is a total dominating set. Therefore, G is not 3-total excellent and hence G is not q-total excellent for all $q \geq 3$.

Remark 3.6 In similar to the proof of Lemma 3.5, one can prove the following Lemma. In fact one can take $x = m+k+2$ and proceed as above.

Lemma 3.7 *Let $n(\geq 3)$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $G = Cir(n, A)$, where $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\}$ with $1 \leq k \leq m$. Then G is not q-total excellent for $q \geq 3$.*

4 Open Problems

The domination numbers for two circulant graphs constructed out of the same group, may not be equal even when the number of elements in the corresponding generating sets are same. For example, $\gamma(Cir(10, \{1, 2, 8, 9\})) = 2$ and $\gamma(Cir(10, \{1, 4, 6, 9\})) = 3$.

1. Find the total domination number for an arbitrary circulant graph.
2. Find a γ_t-set for an arbitrary circulant graph.
3. Find bounds for domination number as well as total domination number for circulant graphs.

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