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Some New Weighted Erdős–Mordell Type Inequalities

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Abstract

In this short note, we establish a geometric inequality conjectured by Liu [2] and obtain some new weighted Erdős–Mordell type inequalities. At the end, an open problem is posed.

Keywords: *Erdős–Mordell type inequality, triangle, Cauchy-Schwarz inequality.*

1 Introduction and Main Results

Throughout this paper we consider a triangle ABC with an interior point P. Denote by R_1, R_2, R_3 the distances from P to the vertices A, B, C, and by r_1, r_2, r_3 the distances from P to the sides BC, CA, AB, respectively. Let w_1, w_2, w_3 be the lengths of the bisectors of angles BPC, CPA, APB, and R_a, R_b, R_c the radii of the circles PBC, PCA, PAB, respectively (see Figure 1).

In 2005, Jian Liu [2] posed the following conjecture.

Conjecture 1.1. *If $x, y, z \in \mathbb{R}$, then*

$$x^2\sqrt{R_2+R_3}+y^2\sqrt{R_3+R_1}+z^2\sqrt{R_1+R_2} \geq 2(yz\sqrt{r_1}+zx\sqrt{r_2}+xy\sqrt{r_3}). \quad (1)$$

In this short note, we give two proofs of the conjecture. The first is by establishing Theorem 1.2 and the second is by making a direct connection to the well-known weighted Erdős-Mordell type inequality (16).

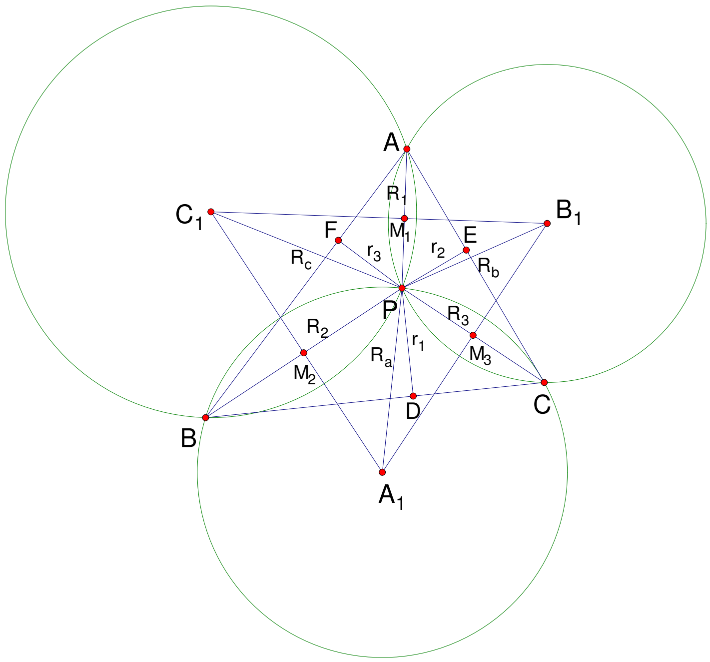

Figure 1:

Theorem 1.2. *If $x, y, z \in \mathbb{R}$, $u, v, w > 0$, and $\alpha, \beta, \gamma \in (0, \frac{\pi}{2})$ with $\alpha + \beta + \gamma = \pi$, then*

$$\begin{aligned} &x^2\sqrt{v+w} + y^2\sqrt{w+u} + z^2\sqrt{u+v} \\ \geq &2\left(yz\sqrt[4]{vw}\sqrt{\cos\alpha} + zx\sqrt[4]{wu}\sqrt{\cos\beta} + xy\sqrt[4]{uv}\sqrt{\cos\gamma}\right). \end{aligned} \tag{2}$$

2 Preliminary Results

In order to prove our main results, we need the following two lemmas.

Lemma 2.1. (see [3]) *Let p_i, q_i $(i = 1, 2, 3)$ be real numbers. Then the following inequality*

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy. \tag{3}$$

holds for all real numbers x, y, z if and only if

$$p_i \geq 0 \quad (i = 1, 2, 3), \qquad 4p_2p_3 \geq q_1^2, \qquad 4p_3p_1 \geq q_2^2, \qquad 4p_1p_2 \geq q_3^2,$$

and

$$4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3.$$

Lemma 2.2. *If* $\alpha+\beta+\gamma=\pi$, *then*

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1. \tag{4}$$

Proof. For $\alpha+\beta+\gamma=\pi$, we have

$$\begin{aligned}
&\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma\\
=&\frac{3}{2}+\frac{1}{2}(\cos 2\alpha+\cos 2\beta+\cos 2\gamma)+2\cos\alpha\cos\beta\cos\gamma\\
=&\frac{3}{2}+\frac{1}{2}[\cos 2(\beta+\gamma)+\cos 2\beta+\cos 2\gamma]+2\cos\alpha\cos\beta\cos\gamma\\
=&\frac{3}{2}+\frac{1}{2}[2\cos^2(\beta+\gamma)-1+2\cos(\beta+\gamma)\cos(\beta-\gamma)]+2\cos\alpha\cos\beta\cos\gamma\\
=&1+\cos(\beta+\gamma)[\cos(\beta+\gamma)+\cos(\beta-\gamma)]+2\cos\alpha\cos\beta\cos\gamma\\
=&1+2\cos(\beta+\gamma)\cos\beta\cos\gamma+2\cos\alpha\cos\beta\cos\gamma\\
=&1-2\cos\alpha\cos\beta\cos\gamma+2\cos\alpha\cos\beta\cos\gamma=1,
\end{aligned}$$

completing the proof. □

3 The Proof of Theorem 1.2

Proof. It is obvious that

$$\begin{cases}\sqrt{v+w}>0,\\ \sqrt{w+u}>0,\\ \sqrt{u+v}>0.\end{cases} \tag{5}$$

For $\alpha,\beta,\gamma\in(0,\frac{\pi}{2})$, we also have

$$\begin{cases}4\sqrt{(w+u)(u+v)}>4\sqrt{vw}>4\sqrt{vw}\cos\alpha,\\ 4\sqrt{(u+v)(v+w)}>4\sqrt{wu}>4\sqrt{wu}\cos\beta,\\ 4\sqrt{(v+w)(w+u)}>4\sqrt{uv}>4\sqrt{uv}\cos\gamma.\end{cases} \tag{6}$$

By the Cauchy-Schwarz inequality and Lemma 2.2, we get

$$\begin{aligned}
&[\sqrt{(v+w)vw}\cos\alpha+\sqrt{(w+u)wu}\cos\beta+\sqrt{(u+v)uv}\cos\gamma\\
&\qquad+2\sqrt{uvw\cos\alpha\cos\beta\cos\gamma}]^2\\
&\le[(v+w)vw+(w+u)wu+(u+v)uv+2uvw]\\
&\quad\cdot[\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma]\\
&=(u+v)(v+w)(w+u).
\end{aligned}$$

Hence

$$\begin{aligned}4\sqrt{(v+w)(w+u)(u+v)}\ge{}&4\sqrt{(v+w)vw}\cos\alpha+4\sqrt{(w+u)wu}\cos\beta\\ &+4\sqrt{(u+v)uv}\cos\gamma+8\sqrt{uvw\cos\alpha\cos\beta\cos\gamma}\end{aligned} \tag{7}$$

By Lemma 2.1 and (5)–(7), we conclude that inequality (2) holds.

The proof of Theorem 1.2 is completed. □

4 Applications of Theorem 1.2

Let $\alpha = \frac{1}{2}\angle BPC$, $\beta = \frac{1}{2}\angle CPA$, and $\gamma = \frac{1}{2}\angle APB$. Substituting (R_1, R_2, R_3) for (u, v, w) in Theorem 1.2 and using the known inequalities

$$r_1 \le w_1 \le \sqrt{R_2R_3}\cos\alpha,\ r_2 \le w_2 \le \sqrt{R_3R_1}\cos\beta,\ \text{and}\ r_3 \le w_3 \le \sqrt{R_1R_2}\cos\gamma,$$

we obtain the following two corollaries.

Corollary 4.1. *If $x, y, z \in \mathbb{R}$, then*

$$\begin{aligned} &x^2\sqrt{R_2+R_3} + y^2\sqrt{R_3+R_1} + z^2\sqrt{R_1+R_2} \\ \ge&2\ (yz\sqrt{w_1} + zx\sqrt{w_2} + xy\sqrt{w_3}). \end{aligned} \tag{8}$$

Corollary 4.2. *If $x, y, z \in \mathbb{R}$, then*

$$x^2\sqrt{R_2+R_3}+y^2\sqrt{R_3+R_1}+z^2\sqrt{R_1+R_2} \ge 2(yz\sqrt{r_1}+zx\sqrt{r_2}+xy\sqrt{r_3}). \tag{9}$$

Taking $x = y = z = 1$ in (8) and (9), we get

Corollary 4.3.

$$\sqrt{R_2+R_3} + \sqrt{R_3+R_1} + \sqrt{R_1+R_2} \ge 2(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3}) \tag{10}$$

and

Corollary 4.4.

$$\sqrt{R_2+R_3} + \sqrt{R_3+R_1} + \sqrt{R_1+R_2} \ge 2(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}). \tag{11}$$

In [2, 5], it is shown that

$$\begin{aligned} &x^2\sqrt{R_2+R_3} + y^2\sqrt{R_3+R_1} + z^2\sqrt{R_1+R_2} \\ \ge&\sqrt{2}(yz\sqrt{r_2+r_3} + zx\sqrt{r_3+r_1} + xy\sqrt{r_1+r_2}) \end{aligned} \tag{12}$$

for $x, y, z \in \mathbb{R}$. With inequality (12), the transformations in triangle [4, pp. 293-295] and the formulas $R_a = \frac{R_2R_3}{2r_1}$, etc., we obtain that

Corollary 4.5. *If $x, y, z \in \mathbb{R}$, then*

$$\begin{aligned} &x^2\sqrt{R_b+R_c} + y^2\sqrt{R_c+R_a} + z^2\sqrt{R_a+R_b} \\ \ge&yz\sqrt{R_2+R_3} + zx\sqrt{R_3+R_1} + xy\sqrt{R_1+R_2}. \end{aligned} \tag{13}$$

Putting (13) and (8) together we have

Corollary 4.6. *If $x, y, z \geq 0$, then*

$$\begin{aligned} & x^2\sqrt{R_b+R_c}+y^2\sqrt{R_c+R_a}+z^2\sqrt{R_a+R_b} \\ \geq & 2(x\sqrt{yz}\sqrt{w_1}+y\sqrt{zx}\sqrt{w_2}+z\sqrt{xy}\sqrt{w_3}) \\ \geq & 2(x\sqrt{yz}\sqrt{r_1}+y\sqrt{zx}\sqrt{r_2}+z\sqrt{xy}\sqrt{r_3}). \end{aligned} \tag{14}$$

This falls short of proving another conjecture of Liu [2]:

Conjecture 4.7. *If $x, y, z \in \mathbb{R}$, then*

$$x^2\sqrt{R_b+R_c}+y^2\sqrt{R_c+R_a}+z^2\sqrt{R_a+R_b} \geq 2(yz\sqrt{r_1}+zx\sqrt{r_2}+xy\sqrt{r_3}). \tag{15}$$

5 Alternative Proofs and Relations to Other Inequalities

It is well known (see [4, p. 318]) that if $x, y, z \in \mathbb{R}$, then

$$x^2R_1+y^2R_2+z^2R_3 \geq 2(yzr_1+zxr_2+xyr_3). \tag{16}$$

Hence by Lemma 2.1,

$$R_1R_2R_3 \geq R_1r_1^2+R_2r_2^2+R_3r_3^2+2r_1r_2r_3, \tag{17}$$

or equivalently

$$\frac{r_1^2}{R_2R_3}+\frac{r_2^2}{R_3R_1}+\frac{r_3^2}{R_1R_2}+2\frac{r_1r_2r_3}{R_1R_2R_3} \leq 1.$$

Therefore by the Cauchy-Schwarz inequality,

$$\begin{aligned} & [r_1\sqrt{R_2+R_3}+r_2\sqrt{R_3+R_1}+r_3\sqrt{R_1+R_2}+2\sqrt{r_1r_2r_3}]^2 \\ \leq & [(R_2+R_3)R_2R_3+(R_3+R_1)R_3R_1+(R_1+R_2)R_1R_2+2R_1R_2R_3] \\ & \cdot\left[\frac{r_1^2}{R_2R_3}+\frac{r_2^2}{R_3R_1}+\frac{r_3^2}{R_1R_2}+2\frac{r_1r_2r_3}{R_1R_2R_3}\right] \\ \leq & (R_1+R_2)(R_2+R_3)(R_3+R_1). \end{aligned}$$

We obtain (9) again by applying Lemma 2.1 and observing that

$$R_2R_3 \geq r_1^2,\ R_3R_1 \geq r_2^2\ ,\ \text{and } R_1R_2 \geq r_3^2.$$

Similarly, (8) can be obtained from the well-known inequality (see [4, p. 318])

$$x^2R_1+y^2R_2+z^2R_3 \geq 2(yzw_1+zxw_2+xyw_3). \tag{18}$$

6 Open Problem

At the end, we pose an open problem.

Problem 6.1. *If $x, y, z \in \mathbb{R}$, then*

$$\begin{aligned} & x^2\sqrt{R_2+R_3}+y^2\sqrt{R_3+R_1}+z^2\sqrt{R_1+R_2} \\ \geq & \sqrt{2}(yz\sqrt{w_2+w_3}+zx\sqrt{w_3+w_1}+xy\sqrt{w_1+w_2}). \end{aligned} \tag{19}$$

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