Int. J. Open Problems Compt. Math., Vol. 4, No. 2, June 2011
ISSN 1998-6262; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

Some New Type of Lacunary Generalized Difference Sequence Spaces Defined by a Sequence of Moduli

Çiğdem A. Bektaş[1] and Gülcan Atıcı[2]

Department of Mathematics, Firat University, Elazig, 23119, TURKEY.
e-mail:cigdemas78@hotmail.com
Department of Mathematics, Muş Alparslan University, Muş, 49100, TURKEY.
e-mail: gatici23@hotmail.com

Abstract

In this paper, we introduce some new generalized difference sequence spaces combining lacunary sequence and a sequence of Moduli. We also examine some topological properties and establish some inclusion relations between these spaces.

Keywords: *Difference sequence, sequence of Moduli, lacunary sequence.*

1 Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, ...\}$, the set of positive integers.

The difference sequence spaces $X(\Delta)$ was introduced by Kızmaz [9] as follows:

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = \ell_\infty$, c or c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$. The notion of difference sequence spaces was generalized by Et and Çolak [5] as follows:

$$X(\Delta^m) = \{x = (x_k) : \Delta^m x \in X\}$$

for $X = \ell_\infty$, c or c_0, where $m \in \mathbb{N}$, $(\Delta^0 x_k) = (x_k)$, $(\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$.

The sequence spaces $X(\Delta^m)$ were further generalized by Et and Esi [6] to following sequence spaces. Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Then

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $(\Delta_v^0 x_k) = (v_k x_k)$, $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and $(\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$

A function $f : [0,\infty) \to [0,\infty)$ is called a modulus function if

(i) $f(t) = 0$ iff $t = 0$,

(ii) $f(t+u) \leq f(t) + f(u)$, $\forall t, u \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. A modulus may be unbounded or bounded.

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, ...$ where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by ρ_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [7] as

$$N_\theta = \{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l\}.$$

Bhardwaj and Bala [3] defined the sequence spaces

$$\begin{aligned} N_\theta [\Delta_v^m, f, p, Q] &= \{x \in w(X) : \lim_{r\to\infty} h_r^{-1} \sum_{k \in I_r} [f(q(\Delta_v^m x_k - \ell))]^{p_k} = 0, \\ &\qquad \text{for some } \ell \in X\}, \\ N_\theta [\Delta_v^m, f, p, Q]_0 &= \{x \in w(X) : \lim_{r\to\infty} h_r^{-1} \sum_{k \in I_r} [f(q(\Delta_v^m x_k))]^{p_k} = 0\}, \\ N_\theta [\Delta_v^m, f, p, Q]_\infty &= \{x \in w(X) : \sup_r h_r^{-1} \sum_{k \in I_r} [f(q(\Delta_v^m x_k))]^{p_k} < \infty\} \end{aligned}$$

for all $q \in Q$.

Also, the sequence spaces defined by lacunary sequence and modulus function were introduced and studied by Çolak [4], Khan and Lohani [8] and many others.

Let U be the set of all real sequences $u = (u_k)$ such that $u_k > 0$ for all $k \in \mathbb{N}$.

We use the following inequality throughout this paper

$$|a_k + b_k|^{p_k} \le D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{1}$$

where a_k and b_k are complex numbers, $D = \max\left(1, 2^{G-1}\right)$ and $H = \sup_k p_k < \infty$ [10].

2 Main Results

Definition 2.1 *Let $F = (f_k)$ be a sequence of moduli, $p = (p_k)$ be a sequence of strictly positive real numbers, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q and $u \in U$. By $w(X)$ we shall denote the space of all sequences defined over X. Now we define the following sequence spaces:*

$$\begin{aligned}
N_\theta^\infty\left(\Delta_v^m, F, p, q, u\right) &= \{x \in w(X) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} < \infty\}, \\
N_\theta\left(\Delta_v^m, F, p, q, u\right) &= \{x \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \left[f_k\left(q\left(\Delta_v^m x_k - \ell\right)\right)\right]^{p_k} = 0, \\
&\qquad\qquad \text{for some } \ell\}, \\
N_\theta^0\left(\Delta_v^m, F, p, q, u\right) &= \{x \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} = 0\}.
\end{aligned}$$

For $p_k = 1$ and $u_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $N_\theta^\infty\left(\Delta_v^m, F, q\right)$, $N_\theta\left(\Delta_v^m, F, q\right)$ and $N_\theta^0\left(\Delta_v^m, F, q\right)$.

For $f_k(x) = x$ for every k, $p_k = 1$ and $u_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $N_\theta^\infty\left(\Delta_v^m, q\right)$, $N_\theta\left(\Delta_v^m, q\right)$ and $N_\theta^0\left(\Delta_v^m, q\right)$.

Theorem 2.2 *Let $F = (f_k)$ be a sequence of moduli. Then $N_\theta^0\left(\Delta_v^m, F, p, q, u\right) \subset N_\theta\left(\Delta_v^m, F, p, q, u\right) \subset N_\theta^\infty\left(\Delta_v^m, F, p, q, u\right)$ and the inclusions are strict.*

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x \in N_\theta\left(\Delta_v^m, F, p, q, u\right)$. By definition of modulus function and inequality (1), we have

$$u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} \le D u_k \left[f_k\left(q\left(\Delta_v^m x_k - \ell\right)\right)\right]^{p_k} + D u_k \left[f_k\left(q(\ell)\right)\right]^{p_k}.$$

Now we may choose an integer K_ℓ such that $q(\ell) \le K_\ell$. Hence, we have

$$u_k \left[f_k \left(q \left(\Delta_v^m x_k \right) \right) \right]^{p_k} \le D u_k \left[f_k \left(q \left(\Delta_v^m x_k - \ell \right) \right) \right]^{p_k} + \max \left[1, \left((K_\ell) f_k(1) \right)^H \right].$$

Therefore $x \in N_\theta^\infty \left(\Delta_v^m, F, p, q, u \right)$.

To show the inclusions are strict consider the following example.

Let $f_k(x) = x$, $p_k = 1$, $v_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$, $q(x) = |x|$ and $\theta = (2^r)$. Then, the sequence $x = (k^m)$ belongs to $N_\theta \left(\Delta_v^m, F, p, q, u \right)$ but does not belong to $N_\theta^0 \left(\Delta_v^m, F, p, q, u \right)$ and the sequence $x = \left((-1)^k \right)$ belongs to $N_\theta^\infty \left(\Delta_v^m, F, p, q, u \right)$ but does not belong to $N_\theta \left(\Delta_v^m, F, p, q, u \right)$. Therefore the inclusions are strict.

Theorem 2.3 *The sets* $N_\theta^0 \left(\Delta_v^m, F, p, q, u \right)$, $N_\theta \left(\Delta_v^m, F, p, q, u \right)$ *and* $N_\theta^\infty \left(\Delta_v^m, F, p, q, u \right)$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. Let $x, y \in N_\theta^0 \left(\Delta_v^m, F, p, q, u \right)$ and $\alpha, \beta \in \mathbb{C}$. Then exists positive integers N_α and M_β such that $|\alpha| \le N_\alpha$ and $|\beta| \le M_\beta$. From the definition of modulus function and Δ_v^m, we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(q \left(\Delta_v^m \left(\alpha x_k + \beta y_k \right) \right) \right) \right]^{p_k}$$

$$\begin{aligned} &= \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(q \left(\alpha \Delta_v^m x_k + \beta \Delta_v^m y_k \right) \right) \right]^{p_k} \\ &\le D \left(N_\alpha \right)^H \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(q \left(\Delta_v^m x_k \right) \right) \right]^{p_k} + D \left(M_\beta \right)^H \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(q \left(\Delta_v^m y_k \right) \right) \right]^{p_k} \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Thus $N_\theta^0 \left(\Delta_v^m, F, p, q, u \right)$ is a linear space. The others can be treated similarly.

Lemma 2.4 *Let* $F = (f_k)$ *be a sequence of moduli and let* $0 < \delta < 1$. *Then for each* $x > \delta$ *we have* $f_k(x) \le 2 f_k(1) \delta^{-1} x$ *[11].*

Theorem 2.5 *Let* $F = (f_k)$ *be a sequence of moduli. Then*

$$N_\theta \left(\Delta_v^m, q \right) \subset N_\theta \left(\Delta_v^m, F, q \right).$$

Proof. Let $x \in N_\theta(\Delta_v^m, q)$. Then we have

$$\varphi_r = \frac{1}{h_r} \sum_{k \in I_r} q(\Delta_v^m x_k - \ell) \to 0 \text{ as } r \to \infty, \text{ for some } \ell.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \varepsilon$ for every t with $0 \le t \le \delta$. Then we can write

$$\frac{1}{h_r} \sum_{k \in I_r} f_k(q(\Delta_v^m x_k - \ell))$$

$$\begin{aligned} &= \frac{1}{h_r} \sum_{k \in I_r, q(\Delta_v^m x_k - \ell) \le \delta} f_k(q(\Delta_v^m x_k - \ell)) + \frac{1}{h_r} \sum_{k \in I_r, q(\Delta_v^m x_k - \ell) > \delta} f_k(q(\Delta_v^m x_k - \ell)) \\ &\le \frac{1}{h_r}(h_r \varepsilon) + \frac{1}{h_r} 2 f_k(1) \delta^{-1} h_r \varphi_r. \end{aligned}$$

Therefore $x \in N_\theta(\Delta_v^m, F, q)$.

Theorem 2.6 *Let $F = (f_k)$ be a sequence of moduli, if $\lim\limits_{t \to \infty} \frac{f_k(t)}{t} = \gamma > 0$, then*

$$N_\theta(\Delta_v^m, q) = N_\theta(\Delta_v^m, F, q).$$

Proof. We need to show that $N_\theta(\Delta_v^m, F, q) \subset N_\theta(\Delta_v^m, q)$. Let $\gamma > 0$ and $x \in N_\theta(\Delta_v^m, F, q)$. Since $\gamma > 0$, we have $f_k(t) \ge \gamma t$ for all $t \ge 0$. Hence we have

$$\frac{1}{h_r} \sum_{k \in I_r} f_k(q(\Delta_v^m x_k - \ell)) \ge \frac{1}{h_r} \sum_{k \in I_r} \gamma(q(\Delta_v^m x_k - \ell)) = \frac{1}{h_r} \gamma \sum_{k \in I_r} (q(\Delta_v^m x_k - \ell)).$$

Therefore we have $x \in N_\theta(\Delta_v^m, q)$. Hence $N_\theta(\Delta_v^m, F, q) \subset N_\theta(\Delta_v^m, q)$. On the other hand, by Theorem 2.5 we have $N_\theta(\Delta_v^m, q) \subset N_\theta(\Delta_v^m, F, q)$. Thus $N_\theta(\Delta_v^m, q) = N_\theta(\Delta_v^m, F, q)$.

Theorem 2.7 *Let $m \ge 1$ be a fixed integer, then*
(i) $N_\theta^0(\Delta_v^{m-1}, F, p, q, u) \subset N_\theta^0(\Delta_v^m, F, p, q, u)$,
(ii) $N_\theta(\Delta_v^{m-1}, F, p, q, u) \subset N_\theta(\Delta_v^m, F, p, q, u)$,
(iii) $N_\theta^\infty(\Delta_v^{m-1}, F, p, q, u) \subset N_\theta^\infty(\Delta_v^m, F, p, q, u)$
and the inclusions are strict.

Proof. The proof of the inclusions follows from the following inequality

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} &\leq D\frac{1}{h_r}\sum_{k\in I_r} u_k\left[f_k\left(q\left(\Delta_v^{m-1} x_k\right)\right)\right]^{p_k}\\ &+D\frac{1}{h_r}\sum_{k\in I_r} u_k\left[f_k\left(q\left(\Delta_v^{m-1} x_{k+1}\right)\right)\right]^{p_k}.\end{aligned}$$

To show that the inclusions are strict, let $f_k(x)=x$, $p_k=1$, $v_k=1$, $u_k=1$ for all $k\in\mathbb{N}$, $q(x)=|x|$, $\theta=(2^r)$ and $x=(k^m)$. Then $x\in N_\theta^\infty(\Delta_v^m,F,p,q,u)$, but $x\notin N_\theta^\infty(\Delta_v^{m-1},F,p,q,u)$. If $x=(k^m)$, then $\Delta^m x=(-1)^m m!$ and $\Delta^{m-1}x=(-1)^{m+1}m!\left(k+\frac{m-1}{2}\right)$.

Theorem 2.8 *Let $\theta=(k_r)$ be a lacunary sequence. If $1<\liminf_r \rho_r<\limsup_r \rho_r<\infty$, then $N_\sigma(\Delta_v^m,F,p,q,u)=N_\theta(\Delta_v^m,F,p,q,u)$, where*

$$N_\sigma(\Delta_v^m,F,p,q,u)=\left\{x\in w(X):\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(q\left(\Delta_v^m x_k-\ell\right)\right)\right]^{p_k}=0\right\}$$

for some l.

Proof. Let $\liminf_r \rho_r>1$, then there exists $\delta>0$ such that $\rho_r=\frac{k_r}{k_{r-1}}\geq 1+\delta$ for all $r\geq 1$. Furthermore we have $\frac{k_r}{h_r}\leq\frac{(1+\delta)}{\delta}$ and $\frac{k_{r-1}}{h_r}\leq\frac{1}{\delta}$, for all $r\geq 1$. Then we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i} &= \frac{1}{h_r}\sum_{i=1}^{k_r}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}-\frac{1}{h_r}\sum_{i=1}^{k_{r-1}}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}\\ &= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{i=1}^{k_r}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}\right)\\ &\quad -\frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}\right).\end{aligned}$$

Now suppose that $\limsup_r \rho_r<\infty$ and let $\varepsilon>0$ be given. Then there exists j_0 such that for every $i\geq j_0$

$$A_j=\frac{1}{h_j}\sum_{i\in I_j}u_i\left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}<\varepsilon.$$

We can also choose a number $K>0$ such that $A_j<K$ for all j. If $\limsup_r \rho_r<\infty$, then there exists a number $\beta>0$ such that $\rho_r<\beta$ for all r. Now let n be any integer with $k_{r-1}<n\leq k_r$. Then

$$\frac{1}{n}\sum_{i=1}^{n} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}$$

$$\begin{aligned}
&\leq k_{r-1}^{-1}\sum_{i=1}^{k_r} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i} \\
&= k_{r-1}^{-1}\{\sum_{i\in I_1} u_i \left[fi\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i} + ... + \sum_{i\in I_r} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}\} \\
&= k_{r-1}^{-1}\{\sum_{j=1}^{j_0}\sum_{i\in I_j} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i} + \sum_{j=j_0+1}^{r}\sum_{i\in I_j} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i}\} \\
&\leq k_{r-1}^{-1}\sum_{j=1}^{j_0}\sum_{i\in I_j} u_i \left[f_i\left(q\left(\Delta_v^m x_i\right)\right)\right]^{p_i} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \\
&= k_{r-1}^{-1}\left\{h_1A_1 + h_2A_2 + ... + h_{j_0}A_{j_0}\right\} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \\
&\leq k_{r-1}^{-1}\left(\sup_{1\leq i\leq j_0} A_j\right)k_{j_0} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \\
&< Kk_{r-1}^{-1}k_{j_0} + \varepsilon\beta.
\end{aligned}$$

Thus $x \in N_\sigma\left(\Delta_v^m, F, p, q, u\right)$.

3 Open Problem

The aim of this paper is to introduce and study the new sequence spaces $N_\theta^\infty\left(\Delta_v^m, F, p, q, u\right)$, $N_\theta\left(\Delta_v^m, F, p, q, u\right)$ and $N_\theta^0\left(\Delta_v^m, F, p, q, u\right)$, which arise from the notions of generalized difference sequence space, lacunary sequence, a sequence of Moduli. We propose to study various some topological properties and establish some inclusion relations between these spaces.

But we didn't prove inclusion relation $N_\theta\left(\Delta_v^m, p, q, u\right) \subset N_\theta\left(\Delta_v^m, F, p, q, u\right)$. Therefore it is left as an open problem.

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