Some New Type of Lacunary Generalized Difference Sequence Spaces Defined by a Sequence of Moduli

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1 Introduction

Let $w$ be the set of all sequences of real or complex numbers and $\ell_\infty$, $c$ and $c_0$ be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, ...\}$, the set of positive integers.

The difference sequence spaces $X (\Delta)$ was introduced by Kızmaç [9] as follows:

$$X (\Delta) = \{ x = (x_k) : \Delta x \in X \}$$

for $X = \ell_\infty$, $c$ or $c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$. The notion of difference sequence spaces was generalized by Et and Çolak [5] as follows:

$$X (\Delta^m) = \{ x = (x_k) : \Delta^m x \in X \}$$
for $X = \ell_{\infty}$, $c$ or $c_0$, where $m \in \mathbb{N}$, $(\Delta^0 x_k) = (x_k), (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$.

The sequence spaces $X (\Delta^m)$ were further generalized by Et and Esi [6] to following sequence spaces. Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Then

$$X (\Delta^m_v) = \{x = (x_k) : (\Delta^m_v x_k) \in X\}$$

for $X = \ell_{\infty}$, $c$ or $c_0$, where $(\Delta^0_v x_k) = (v_k x_k), (\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and $(\Delta^m_v x_k) = (\Delta^{m-1}_v x_k - \Delta^{m-1}_v x_{k+1})$ and so that

$$\Delta^m_v x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$  

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if

(i) $f(t) = 0$ iff $t = 0$,

(ii) $f(t + u) \leq f(t) + f(u)$, $\forall t, u \geq 0$,

(iii) $f$ is increasing,

(iv) $f$ is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f (|x - y|)$, it follows from condition (iv) that $f$ is continuous on $[0, \infty)$. A modulus may be unbounded or bounded.

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, \ldots$ where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio $k_r/k_{r-1}$ will be denoted by $\rho_r$. The space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. [7] as

$$N_\theta = \{x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l\}.$$  

Bhardwaj and Bala [3] defined the sequence spaces

$$N_\theta [\Delta^m_v, f, p, Q] = \{x \in w (X) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} [f (q (\Delta^m_v x_k - \ell))]^{p_k} = 0, \text{ for some } \ell \in X\},$$

$$N_\theta [\Delta^m_v, f, p, Q]_0 = \{x \in w (X) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} [f (q (\Delta^m_v x_k))]^{p_k} = 0\},$$

$$N_\theta [\Delta^m_v, f, p, Q]_{\infty} = \{x \in w (X) : \sup_{r} h_r^{-1} \sum_{k \in I_r} [f (q (\Delta^m_v x_k))]^{p_k} < \infty\}$$

for all $q \in Q$.

Also, the sequence spaces defined by lacunary sequence and modulus function were introduced and studied by Çolak [4], Khan and Lohani [8] and many others.

Let $U$ be the set of all real sequences $u = (u_k)$ such that $u_k > 0$ for all $k \in \mathbb{N}$. 

We use the following inequality throughout this paper

\[ |a_k + b_k|^{p_k} \leq D \{ |a_k|^{p_k} + |b_k|^{p_k} \} \]  

(1)

where \( a_k \) and \( b_k \) are complex numbers, \( D = \max \{ 1, 2^{G-1} \} \) and \( H = \sup_k p_k < \infty \) [10].

2 Main Results

**Definition 2.1** Let \( F = (f_k) \) be a sequence of moduli, \( p = (p_k) \) be a sequence of strictly positive real numbers, \( X \) be a seminormed space over the field \( \mathbb{C} \) of complex numbers with the seminorm \( q \) and \( u \in U \). By \( w(X) \) we shall denote the space of all sequences defined over \( X \). Now we define the following sequence spaces:

\[
N_\theta^\infty (\Delta^m_v, F, p, q, u) = \{ x \in w(X) : \sup_r \frac{1}{r} \sum_{k \in \mathbb{N}} u_k \{ f_k (q (\Delta^m_v x_k)) \}^{p_k} < \infty \},
\]

\[
N_\theta (\Delta^m_v, F, p, q, u) = \{ x \in w(X) : \lim_{r \to \infty} \frac{1}{r} \sum_{k \in \mathbb{N}} u_k \{ f_k (q (\Delta^m_v x_k - \ell)) \}^{p_k} = 0, \quad \text{for some } \ell \},
\]

\[
N_\theta^0 (\Delta^m_v, F, p, q, u) = \{ x \in w(X) : \lim_{r \to \infty} \frac{1}{r} \sum_{k \in \mathbb{N}} u_k \{ f_k (q (\Delta^m_v x_k)) \}^{p_k} = 0 \}.
\]

For \( p_k = 1 \) and \( u_k = 1 \) for all \( k \in \mathbb{N} \), we write these spaces as \( N_\theta^\infty (\Delta^m_v, F, q) \), \( N_\theta (\Delta^m_v, F, q) \) and \( N_\theta^0 (\Delta^m_v, F, q) \).

For \( f_k (x) = x \) for every \( k \), \( p_k = 1 \) and \( u_k = 1 \) for all \( k \in \mathbb{N} \), we write these spaces as \( N_\theta^\infty (\Delta^m_v, q) \), \( N_\theta (\Delta^m_v, q) \) and \( N_\theta^0 (\Delta^m_v, q) \).

**Theorem 2.2** Let \( F = (f_k) \) be a sequence of moduli. Then \( N_\theta^0 (\Delta^m_v, F, p, q, u) \subset N_\theta (\Delta^m_v, F, p, q, u) \subset N_\theta^\infty (\Delta^m_v, F, p, q, u) \) and the inclusions are strict.

**Proof.** The first inclusion is obvious. We establish the second inclusion. Let \( x \in N_\theta (\Delta^m_v, F, p, q, u) \). By definition of modulus function and inequality (1), we have

\[
u_k \{ f_k (q (\Delta^m_v x_k)) \}^{p_k} \leq D u_k \{ f_k (q (\Delta^m_v x_k - \ell)) \}^{p_k} + D u_k \{ f_k (q (\ell)) \}^{p_k}.
\]
Now we may choose an integer $K_{\ell}$ such that $q(\ell) \leq K_{\ell}$. Hence, we have

$$u_k [f_k (q (\Delta^m_{\ell} x_k))]^{pk} \leq Du_k [f_k (q (\Delta^m_{\ell} x_k - \ell))]^{pk} + \max \left[ 1, ((K_{\ell}) f_k (1))^H \right].$$

Therefore $x \in N^0_{\theta} (\Delta^m_{\ell}, F, p, q, u)$.

To show the inclusions are strict consider the following example.

Let $f_k (x) = x$, $p_k = 1$, $v_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$, $q(x) = |x|$ and $\theta = (2^r)$. Then, the sequence $x = (k^m)$ belongs to $N^0_{\theta} (\Delta^m_{\ell}, F, p, q, u)$ but does not belong to $N_{\theta}^0 (\Delta^m_{\ell}, F, p, q, u)$ and the sequence $x = ((-1)^k)$ belongs to $N_{\theta}^\infty (\Delta^m_{\ell}, F, p, q, u)$ but does not belong to $N_{\theta} (\Delta^m_{\ell}, F, p, q, u)$. Therefore the inclusions are strict.

**Theorem 2.3** The sets $N^0_{\theta} (\Delta^m_{\ell}, F, p, q, u)$, $N_{\theta} (\Delta^m_{\ell}, F, p, q, u)$ and $N^\infty_{\theta} (\Delta^m_{\ell}, F, p, q, u)$ are linear spaces over the complex field $\mathbb{C}$.

**Proof.** Let $x, y \in N^0_{\theta} (\Delta^m_{\ell}, F, p, q, u)$ and $\alpha, \beta \in \mathbb{C}$. Then exists positive integers $N_{\alpha}$ and $M_{\beta}$ such that $|\alpha| \leq N_{\alpha}$ and $|\beta| \leq M_{\beta}$. From the definition of modulus function and $\Delta^m_{\ell}$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k (q (\Delta^m_{\ell} (\alpha x_k + \beta y_k)))]^{pk}$$

$$= \frac{1}{h_r} \sum_{k \in I_r} u_k [f_k (q (\alpha \Delta^m_{\ell} x_k + \beta \Delta^m_{\ell} y_k))]^{pk}$$

$$\leq D (N_{\alpha})^H \frac{1}{h_r} \sum_{k \in I_r} u_k [f_k (q (\Delta^m_{\ell} x_k))]^{pk} + D (M_{\beta})^H \frac{1}{h_r} \sum_{k \in I_r} u_k [f_k (q (\Delta^m_{\ell} y_k))]^{pk}$$

$$\to 0 \text{ as } r \to \infty.$$

Thus $N^0_{\theta} (\Delta^m_{\ell}, F, p, q, u)$ is a linear space. The others can be treated similarly.

**Lemma 2.4** Let $F = (f_k)$ be a sequence of moduli and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f_k (x) \leq 2 f_k (1) \delta^{-1} x$ [11].

**Theorem 2.5** Let $F = (f_k)$ be a sequence of moduli. Then

$$N_{\theta} (\Delta^m_{\ell}, q) \subset N_{\theta} (\Delta^m_{\ell}, F, q).$$
**Proof.** Let \( x \in N_\theta (\Delta_v^m, q) \). Then we have

\[
\varphi_r = \frac{1}{h_r} \sum_{k \in I_r} q (\Delta_v^m x_k - \ell) \to 0 \text{ as } r \to \infty, \text{ for some } \ell.
\]

Let \( \varepsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( f_k (t) < \varepsilon \) for every \( t \) with \( 0 \leq t \leq \delta \). Then we can write

\[
\frac{1}{h_r} \sum_{k \in I_r} f_k (q (\Delta_v^m x_k - \ell))
\]

\[
= \frac{1}{h_r} \sum_{k \in I_r, q (\Delta_v^m x_k - \ell) \leq \delta} f_k (q (\Delta_v^m x_k - \ell)) + \frac{1}{h_r} \sum_{k \in I_r, q (\Delta_v^m x_k - \ell) > \delta} f_k (q (\Delta_v^m x_k - \ell))
\]

\[
\leq \frac{1}{h_r} (h_r \varepsilon) + \frac{1}{h_r} 2f_k (1) \delta^{-1} h_r \varphi_r.
\]

Therefore \( x \in N_\theta (\Delta_v^m, q) \).

**Theorem 2.6** Let \( F = (f_k) \) be a sequence of moduli, if \( \lim_{t \to \infty} \frac{f_k(t)}{t} = \gamma > 0 \), then

\[
N_\theta (\Delta_v^m, q) = N_\theta (\Delta_v^m, F, q).
\]

**Proof.** We need to show that \( N_\theta (\Delta_v^m, F, q) \subset N_\theta (\Delta_v^m, q) \). Let \( \gamma > 0 \) and \( x \in N_\theta (\Delta_v^m, F, q) \). Since \( \gamma > 0 \), we have \( f_k (t) \geq \gamma t \) for all \( t \geq 0 \). Hence we have

\[
\frac{1}{h_r} \sum_{k \in I_r} f_k (q (\Delta_v^m x_k - \ell)) \geq \frac{1}{h_r} \sum_{k \in I_r} \gamma (q (\Delta_v^m x_k - \ell)) = \frac{1}{h_r} \gamma \sum_{k \in I_r} (q (\Delta_v^m x_k - \ell)).
\]

Therefore we have \( x \in N_\theta (\Delta_v^m, q) \). Hence \( N_\theta (\Delta_v^m, F, q) \subset N_\theta (\Delta_v^m, q) \). On the other hand, by Theorem 2.5 we have \( N_\theta (\Delta_v^m, q) \subset N_\theta (\Delta_v^m, F, q) \). Thus \( N_\theta (\Delta_v^m, q) = N_\theta (\Delta_v^m, F, q) \).

**Theorem 2.7** Let \( m \geq 1 \) be a fixed integer, then

(i) \( N_\theta^0 (\Delta_v^{m-1}, F, p, q, u) \subset N_\theta^0 (\Delta_v^m, F, p, q, u) \),

(ii) \( N_\theta (\Delta_v^{m-1}, F, p, q, u) \subset N_\theta (\Delta_v^m, F, p, q, u) \),

(iii) \( N_\theta^\infty (\Delta_v^{m-1}, F, p, q, u) \subset N_\theta^\infty (\Delta_v^m, F, p, q, u) \)

and the inclusions are strict.
Proof. The proof of the inclusions follows from the following inequality

$$\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(\Delta^m x_k))]^{p_k} \leq D \frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(\Delta^{m-1} x_k))]^{p_k} + D \frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(\Delta^{m-1} x_{k+1}))]^{p_k}.$$  

To show that the inclusions are strict, let \( f_k(x) = x, p_k = 1, v_k = 1, u_k = 1 \) for all \( k \in \mathbb{N}, q(x) = |x|, \theta = (2^r) \) and \( x = (k^m) \). Then \( x \in N^\infty \phi(\Delta^m, F, p, q, u) \), but \( x \notin N^\infty \phi^+(\Delta^{m-1}, F, p, q, u) \). If \( x = (k^m) \), then \( \Delta^m x = (-1)^m m! \) and \( \Delta^{m-1} x = (-1)^{m+1} m! (k + \frac{m-1}{2}) \).

**Theorem 2.8** Let \( \theta = (k_r) \) be a lacunary sequence. If \( 1 < \liminf \rho_r < \limsup \rho_r < \infty \), then \( N_\sigma(\Delta^m, F, p, q, u) = N^\infty \phi(\Delta^m, F, p, q, u) \), where

$$N_\sigma(\Delta^m, F, p, q, u) = \left\{ x \in w(X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} u_k [f_k(q(\Delta^m x_k - \ell))]^{p_k} = 0 \right\}$$

for some \( l \).

Proof. Let \( \liminf \rho_r > 1 \), then there exists \( \delta > 0 \) such that \( \rho_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta \) for all \( r \geq 1 \). Furthermore we have \( \frac{k_r}{k_{r-1}} \leq \frac{(1+\delta)}{\delta} \) and \( \frac{k_{r-1}}{k_{r-2}} \leq \frac{1}{\delta} \), for all \( r \geq 1 \). Then we may write

$$\frac{1}{h_r} \sum_{i \in I_r} u_i [f_i(q(\Delta^m x_i))]^{p_i} = \frac{1}{h_r} \sum_{i=1}^{k_r} u_i [f_i(q(\Delta^m x_i))]^{p_i} - \frac{1}{h_r} \sum_{i=1}^{k_{r-1}} u_i [f_i(q(\Delta^m x_i))]^{p_i} = \frac{k_r}{h_r} \left( k_{r-1} \sum_{i=1}^{k_r} u_i [f_i(q(\Delta^m x_i))]^{p_i} \right) - \frac{k_{r-1}}{h_r} \left( k_{r-1} \sum_{i=1}^{k_{r-1}} u_i [f_i(q(\Delta^m x_i))]^{p_i} \right).$$

Now suppose that \( \limsup \rho_r < \infty \) and let \( \varepsilon > 0 \) be given. Then there exists \( j_0 \) such that for every \( i \geq j_0 \)

$$A_j = \frac{1}{h_j} \sum_{i \in I_j} u_i [f_i(q(\Delta^m x_i))]^{p_i} < \varepsilon.$$

We can also choose a number \( K > 0 \) such that \( A_j < K \) for all \( j \). If \( \limsup \rho_r < \infty \), then there exists a number \( \beta > 0 \) such that \( \rho_r < \beta \) for all \( r \). Now let \( n \) be any integer with \( k_{r-1} < n \leq k_r \). Then
\[ \frac{1}{n} \sum_{i=1}^{n} u_i [f_i (q (\Delta^m_i x_i))]^{p_i} \]
\[ \leq k_{r-1}^{-1} \sum_{i=1}^{k_r} u_i [f_i (q (\Delta^m_i x_i))]^{p_i} \]
\[ = k_{r-1}^{-1} \left\{ \sum_{i \in I_1} u_i [f (q (\Delta^m_i x_i))]^{p_i} + \ldots + \sum_{i \in I_r} u_i [f (q (\Delta^m_i x_i))]^{p_i} \right\} \]
\[ = k_{r-1}^{-1} \left\{ \sum_{j=1}^{j_0} \sum_{i \in I_{j}} u_i [f_i (q (\Delta^m_i x_i))]^{p_i} + \sum_{j=j_0+1}^{r} \sum_{i \in I_{j}} u_i [f_i (q (\Delta^m_i x_i))]^{p_i} \right\} \]
\[ \leq k_{r-1}^{-1} \sum_{j=1}^{j_0} \sum_{i \in I_{j}} u_i [f_i (q (\Delta^m_i x_i))]^{p_i} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \]
\[ = k_{r-1}^{-1} \left\{ h_1 A_1 + h_2 A_2 + \ldots + h_{j_0} A_{j_0} \right\} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \]
\[ \leq k_{r-1}^{-1} \left( \sup_{1 \leq i \leq j_0} A_j \right) k_{j_0} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \]
\[ < K k_{r-1}^{-1} k_{j_0} + \varepsilon \beta. \]

Thus \( x \in N_\sigma (\Delta^m_v, F, p, q, u) \).

3 Open Problem

The aim of this paper is to introduce and study the new sequence spaces \( N_\theta^\infty (\Delta^m_v, F, p, q, u) \), \( N_\theta (\Delta^m_v, F, p, q, u) \) and \( N_0^0 (\Delta^m_v, F, p, q, u) \), which arise from the notions of generalized difference sequence space, lacunary sequence, a sequence of Moduli. We propose to study various some topological properties and establish some inclusion relations between these spaces.

But we didn’t prove inclusion relation \( N_\theta (\Delta^m_v, p, q, u) \subset N_\theta (\Delta^m_v, F, p, q, u) \). Therefore it is left as an open problem.

References


