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Series Solutions of the Modified Falkner-Skan Equation

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Abstract

The homotopy analysis method is employed to derive an analytical solution to the modified non-homgenuous Falkner Skan equation which describes the fluid flow over a moving wedge with included angle $\pi\beta$.and mving with a relative velcity λ. The solution of the problem is governed by the two parameters λ and β. The slution is obtainesd for different values of the two parameters andit shows that the method is realiable and accurate.

Key words: *Falkner-Skan equation. homotopy analysis method, moving wedge, non-uniqueness of solution.*

1 Introduction:

The fluid flow past a fixed wedge with included angle $\pi\beta$ is usually governed by the standard Falkner-Skan equation:

$$f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'^2(\eta)) = 0 \tag{1}$$

with the following initial and boundary conditions:

$$f(0) = 0, \qquad f'(0) = 0, \quad \text{and} \qquad f'(\infty) = 1. \tag{2}$$

Herein, the function $f(\eta)$ is the non-dimensional stream function and η is the similarity ordinate.

Over the last few decades this equation has been studied intensively by many authors. Previous work on this equation includes the mathematical treatments due to Weyl [22], Coppel [11], and Rosenhead [18]. These works have mainly focused on obtaining existence and uniqueness results. The most significant of these works is that of Coppel [11] in which an elegant proof of the existence, uniqueness, and detailed analysis of solutions to the Falkner–Skan equation, for $\beta > 0$, with more general initial conditions are given. In addition, Coppel also shows that $\alpha = d^2 f(0)/d\eta^2$ is an increasing function of β. This work has been extended for $\beta < 0$ by Veldman and Van der Vooren [21] .Some new existence and uniqueness results for the solution of the Falkner-Skan equations are also given in [12] and [13].

In [7] and [9], a computational method for the solution of the Falkner–Skan equation on a semi-infinite domain was presented. The method used a technique known as automatic differentiation, which is neither numerical nor symbolic. Using automatic differentiation, a Taylor series solution is constructed for the initial value problems by calculating the Taylor coefficients recursively. Smith [20], Cebeci and Keller [10], and Na [17] [8]have considered other numerical treatments that use shooting and invariant imbedding. A new approach to solving this problem by shooting from ∞ (instead of from 0), using some simple analysis of the asymptotic behavior of the solution at ∞, is presented in [4]. Salama [19] develops a one-step method of order 5. The strength of the method of this paper comes from the fact that it does not use numerical differentiation or other approximations for the derivatives involved in the calculations. It uses a technique known as automatic differentiation, which is neither numerical nor symbolic, and which computes exact derivatives using recursive formulas.

In the present paper, we discuss the numerical solution for the non-classical non-homogenous Falkner-skan equation which describes the fluid flow past a moving wedge with constant velocity, $\lambda = u_w/u_\infty$; given by:

$$(1+\lambda)\, f''' + \frac{1}{2} f f'' + \beta(1 - f'^2) = 0 \tag{3}$$

with the boundary conditions

$$f(0) = 0, \quad f'(0) = -\lambda, \qquad f'(\infty) = 1. \tag{4}$$

This equation was first derived from the classical Navier-Stokes equation by Allan [2], where the non-classical similarity transformation introduced by Allan [1] was employed. The existence and non-uniqueness of solution of special cases of this problem has been discussed by several authors. Allan and Abu Saris [3] and Allan and Syam [5] discuss the case when $\beta = 0$. They show that two solutions exist on the range $0 < \lambda < \lambda_c$, where λ_c was found to be 0.3546, one solution exists for $\lambda = \lambda_c$, and no solution exists for $\lambda > \lambda_c$. Allan [2] studies the case when $\beta = -1$ and the following theorem is prved:

Theorem 1 *If $\beta = -1$ then Equations 3 with 4 has a solution only if $\lambda < -1$. Moreover, $f''(0) = \alpha = \pm\sqrt{2}$ and therefore two solutions exist.*

The rest of the paper is organized as follows: In the next section, an analytical series solution of the non-homogenous Falkner-Skan given by 3 will be discussed. The numerical results is presented in sections 4 and conclusion remarks are presented in the last section.

2 Homotopy Analysis Method

The following is a brief derivation of the algorithm used to solve problem (3) with (4). This algorithm is based on Liao's homotopy analysis method to obtain an analytic solution expressed as power series.

By the homotopy technique, we construct a homotopy $\hat{y}(x, \eta) : \Omega \times [0, 1] \to R$ which satisfies the equation

$$\mathcal{H}(\hat{y}, \eta) = (1 - p)\mathcal{L}_f[\hat{y}(\eta, p) - y_0(\eta)] = p\hbar_f \mathcal{N}_f [\hat{y}(\eta, p)] , \tag{5}$$

where $\hbar \neq 0$ and $\mathcal{N}_f$ is a non-linear operator given by

$$\mathcal{N}_f [\hat{y}(\eta, p)] = (1 + \lambda)\frac{\partial^3 \hat{y}}{\partial \eta^3} + \frac{1}{2}\hat{y}\frac{\partial^2 \hat{y}}{\partial \eta^2} - \beta\left(1 - \left(\frac{\partial \hat{y}}{\partial \eta}\right)^2\right) := 0$$

which is subjected to the conditions

$$\hat{y}(\eta, p)|_{\eta=0} = 0, \quad \left.\frac{\partial \hat{y}(\eta, p)}{\partial \eta}\right|_{\eta=0} = -\lambda, \quad \left.\frac{\partial \hat{y}(\eta, p)}{\partial \eta}\right|_{\eta=\infty} = 1.$$

Here the linear operator $\mathcal{L}_f$ is a properly selected auxiliary operator. Taking into account the boundary conditions, the linear operater is chosen to be

$$\mathcal{L}_f = (1 + \lambda)\frac{d^3}{d\eta^3} + \gamma\frac{d^2}{d\eta^2}.$$

where γ is a constant that will be determined later. Assuming that the solution $\hat{y}(\eta, p)$ is analytic in the variable p, one can use Taylor series to expand $\hat{y}(\eta, p)$ with respect to the embedding parameter p at $p_0 = 0$ and get

$$\hat{y}(\eta, p) = y_0(\eta) + \sum_{m=1}^{\infty} y_m(\eta)p^m, \tag{6}$$

where

$$y_m(\eta) = \left.\frac{1}{m!}\frac{\partial^m y(\eta; p)}{\partial p^m}\right|_{p=0}.$$

Obviously when $p=0$ and $p=1$, we respectively have

$$\hat{y}(\eta;0)=y_0(\eta), \qquad \hat{y}(\eta;1)=y(\eta).$$

Substituting (6) in equation (5) gives the following $m^{th}-$order deformation problems

$$\mathcal{L}_f\left[y_m(\eta)-\chi_m y_{m-1}(\eta)\right]=\hbar_f \mathcal{R}_m(\eta), \quad (m\geq 1) \tag{7}$$

with the conditions

$$y_m(0)=0, y_m'(0)=0, y_m'(\infty)=0, \tag{8}$$

where

$$\mathcal{R}_m(\eta)=(1+\lambda)f'''_{m-1}+\sum_{k=0}^{m-1}\left[\frac{1}{2}f_k f''_{m-1-k}-\beta f'_{m-1-k}f'_k\right]+\beta(1-\chi_m), \tag{9}$$

and

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1.\end{cases}$$

On the other hand, the function $y_0(\eta)$ is the solution for the linear operator:

$$\mathcal{L}_f[y_0(\eta)]=(1+\lambda)\frac{d^3y_0(\eta)}{d\eta^3}+\gamma\frac{d^2y_0(\eta)}{d\eta^2}:=0.$$

with the conditions

$$y_0(0)=0,\ \ y_0'(0)=-\lambda,\ \ y_0'(\infty)=1, \tag{10}$$

and it is given by

$$y_0(\eta)=\frac{\left(-1+e^{-\frac{\eta\gamma}{\lambda+1}}\right)(\lambda+1)^2}{\gamma}+\eta.$$

Finally, the general form of the $m^{th}-$order problem (7) is

$$y_m^{(3)}(\eta)+\gamma y_m^{(2)}(\eta)=\chi_m\left(y_{m-1}^{(3)}(\eta)+\gamma y_{m-1}^{(2)}(\eta)\right)+\hbar_f\mathcal{R}_m(\eta), \quad (m\geq 1). \tag{11}$$

It is very important to emphasize that equation (11) is a linear differential equation in which the right hand side term depends on the first $(m-1)^{th}$-order approximations $(y_0,\cdots,y_{m-1})$. Clearly, the original non-linear problem (3) is converted into an infinite sequence of linear sub-problems governed by equations (11) which is solved using Mathematica. The solution for the 1^{st}-order approximation, $y_1(\eta)$, is found to be

$$\begin{aligned} y_1(\eta) &= -\frac{h(\lambda+1)^2\left(8\gamma^2+3\lambda^2-2\lambda+2\beta\left(\lambda^2-6\lambda-7\right)-5\right)}{8\gamma^3}+\frac{e^{-\frac{x\gamma}{\lambda+1}}h(\lambda+1)}{4\gamma^3} \\ &\times \left[-\eta^2\gamma^2+2x\left(2\gamma^2+\lambda^2-1\right)\gamma+2\beta(\lambda+1)\left(\lambda^2-2\lambda-4\eta\gamma-3\right)\right. \\ &+ \left.(\lambda+1)\left(4\gamma^2+\lambda^2-2\lambda-3\right)\right]-\frac{e^{-\frac{2x\gamma}{\lambda+1}}h(2\beta-1)(\lambda+1)^4}{8\gamma^3}. \end{aligned}$$

The formulas for f_m ($m \geq 2$) are too much involved and will not be included here.

3 Convergence discussion

For $h = -1$, and fixed value of γ, analytical treatment of the convergence of the Homotopy analysis method was proved by several authors [16] , [6]. For the problem under consideration, the convergence of the solution series (6) depends strongly on the auxiliary parameter h and γ. To ensure the convergence of the series solution (6) we follow the procedures suggested by Liao [16] and Yao [23]. Therefore, the valid region of the parameters h and γ, can be obtained by plotting $f''(0) \sim (h, \gamma)$ curve as shown in figure (**??**). The values of β is taken to be $\beta = 0.5$ and the value of λ is taken to be $\lambda = 0.2$. A wide valid zone is evident in this figure ensuring convergence of the series. It is clearly seen that series solution (6) converges when $(h, \gamma) \in [0.6, 1.2] \times [2.5, \infty)$.

The choice f the parameter γ is chosen such that the results are in god agreement with the numerical slution f the classical Blasius problem when the values of $\lambda = 0$, and $\beta = 0$. The prper value f γ is chosen t be $\gamma = 1.5$. Cmparison f the currtent results with the numercial results fr Blasuis prblem are displayed in Figure 2.

4 Results and discussion

The existence and uniqueness of solution of the modified Falkner–Skan equation (3) have been investigated in the literature by many authors. For example,in [4] the solution is studied over the range of $|\beta| \leq 1$ and for various values of λ. It is shown that for each β in this range there exists a critical value $\lambda_c(\beta)$ such that two solutions exist for $0 < \lambda < \lambda_c(\beta)$, and one solution exists for $\lambda = \lambda_c(\beta)$, and no solution exists for $\lambda < \lambda_c(\beta)$. When $\beta = -1$, two exact solutions were derived for $\lambda < -1$. The case $\beta = 0$ was also discussed in [1] and [5]. In [1] , the effect of the parameter λ on the boundary layer thickness δ is presented. it was shown that for $-1 < \lambda < 0$ the boundary layer thickness

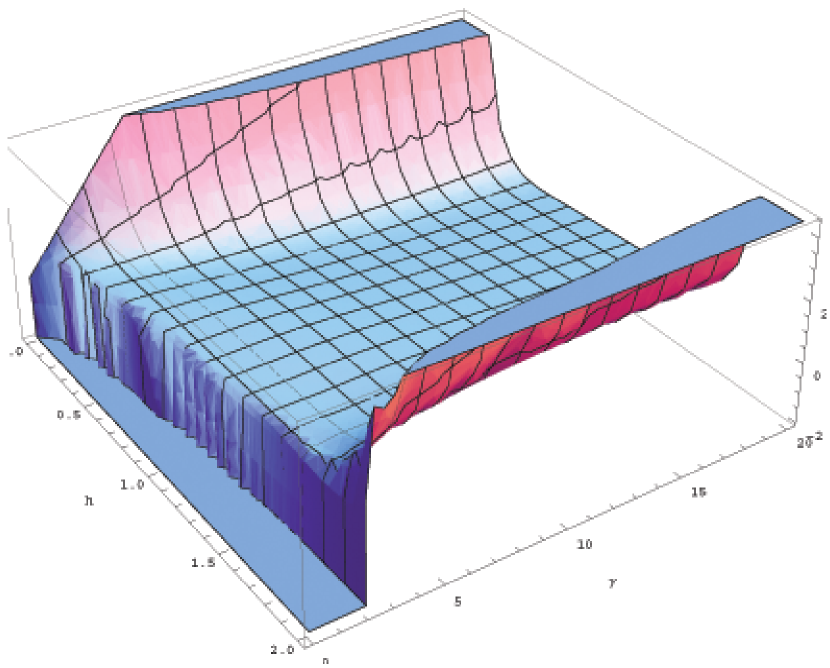


Figure 1: $h-\gamma$ curve

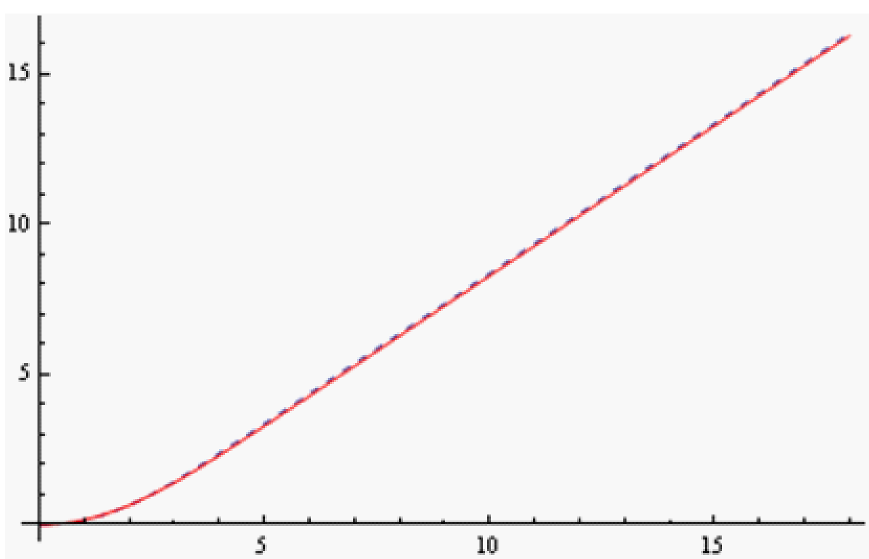
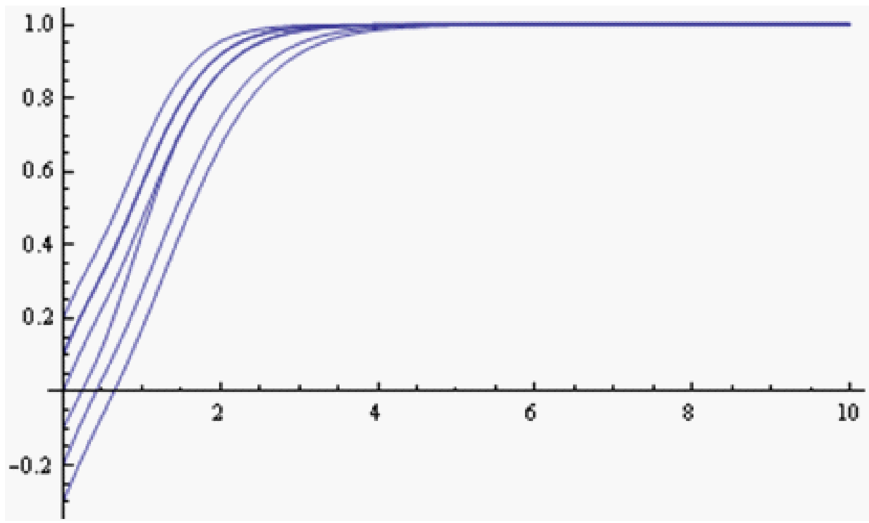


Figure 2: Cmparizon between the Numerical slution of the Blasuis problem and the Homotopy analysis solution (4 terms only)



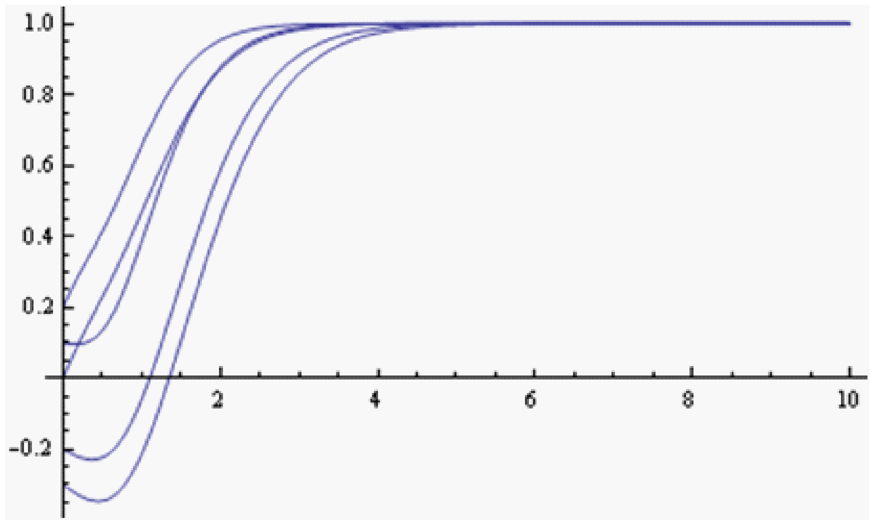
is less when the present model is used, and it is larger for $0 < \lambda \leq \lambda_c$. It was also shown that the boundary layer thickness δ increases exponentially as $\lambda \to \lambda_c$; which gives, from the physical point of view, a partial explanation for the nonexistence of solutions beyond this value.

In the present paper, Mathematica software package was employed to program the homotopy analysis method as applied to the modified Fakner Skan equation. The set of parameters used are as follows: $A(p) = p$, $B(p) = p$, and $\hbar = -1$. The parameter γ is used to adjust the rate of convergence of the method. The value of this parameter uses is $\gamma = -1$. We derived the first few terms (8 terms) of the homotopy analysis method for different values of the parameters λ and β.The value of the initial condition $f''(0) = \alpha$ was solved for using the fact that as $\eta \to \infty$, $f'(\eta) = 1$.where $f(\eta) \cong \sum_{k=0}^{k=4} f_k(\eta)$. Figures 1 and 2 respectively display the velocity profiles $f'(\eta)$ for $\beta = 0.1$ and $\beta = 0.2$ and various values of the wall velocity λ.These values are taken to be positive and negative. The behavior of the velocity profiles are the same. The curve will asymptotically approaches 1 as $\eta \to \infty$.

The non-uniqueness of the solution can be obtained by solving the equation

$$\lim_{\eta \to \infty} f'(\eta, \alpha) = 1$$

for α.



5 Conclusion

In this article, the homotopy analysis method was employed to derive an analytical solution to the modified Falkner-Skan equation. The equation describes the fluid flow over a moving edge with included angle $\pi\beta$ and moving with relative velocity λ. Few terms of the series solution were obtained and it is shown that the method is very efficient to handle these kind of problems. The solution converges very fast. Unlike other methods, no restriction was made on the interval of investigation and it is shown that the solution is valid on the interval $[0, \infty]$.

6 Open problem:

The problem considered in this articles has many features including multi solutions for different values of the parameteres β and λ. The method used in this articles shed aome lights on the non-uniqueness of solutions. The question that still need to be ansered is how these solutions can be discovered using this method or any other method. It is also very important to explain the physical meaning of each solution.

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