Int. J. Open Problems Compt. Math., Vol. 4, No. 2, June 2011
ISSN 1998-6262; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

On Weakly Semihereditary Rings

Fuad Ali Ahmed Almahdi and Najib Mahdou

Department of Mathematics, Faculty of Science and Technology of Fez,
Box 2202,University S.M. Ben Abdellah Fez, Morocco.
e-mail: fuadalialmahdy@hotmail.com
Department of Mathematics, Faculty of Science and Technology of Fez,
Box 2202,University S.M. Ben Abdellah Fez, Morocco.
e-mail: mahdou@hotmail.com

Abstract

In this paper, we introduce a generalization of the well-known notion of a semihereditary ring which we call a weak semihereditary ring. We investigate the transfer of the weak semihereditary properties to trivial ring extensions, localizations, homomorphic image of rings, and in direct product of rings. For the pullback constructions, we give example showing that the transfer does not hold. For amalgamated duplication of a ring along an ideal, we study the transfer of weak semihereditary properties from a ring R to a ring $R \bowtie I$.

Keywords: *semihereditary rings, weak semihereditary rings, coherent rings, trivial ring extension, localization of rings, homomorphic image of rings, Pullbacks, direct product of rings, amalgamated duplication of a ring along an ideal.*

1 Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. We use "local" to refer to (not necessarily Noetherian) ring with a unique maximal ideal.

Recall, for a ring A and an A-module E, that the ring $R := A \propto E$ of pairs (a, e) whose underlying group is $A \times E$ with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$ is called trivial ring extension of A by E (also called the idealization of E over A). For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. Ideals of R need not be of this form (see [11, Example 2.5]). However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [10, Theorem 25.1(3)]. Considerable works have been concerned with trivial ring extension. Part of it has been summarized in Glaz's book [8], and Huckaba's book (where R is called the idealization of E by A) [10]. These kind of rings have been useful for solving many open problems and conjectures in both commutative and noncommutative ring theory.

The amalgamated duplication of R along an ideal I is a ring that is defined as the following subring with unit element (1, 1) of $R \times R$:
$R \bowtie I = \{(r, r + i) | r \in R, i \in I\}$.
In the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [19]. One main difference of this construction, with respect to the idealization, is that the ring $R \bowtie I$ can be a reduced ring (and it is always reduced if R is a domain).

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \propto I$. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero-divisor of the ring $R \bowtie I$. See for instance [19, 18, 16].

In this paper, we introduce and investigate a generalization of a semihereditary ring, which we call a weak semihereditary ring. A ring R is called weak semihereditary if, for every two ideals $I \subseteq J$ of R such that I is finitely generated, J projective proper ideal, then I is projective (Definition 3.1).

Naturally, every semihereditary ring is a weak semihereditary ring. In Theorem 4.1(2), we give a sufficient condition to have the converse. Also, in Theorem 4.1(3), we show that if R is a local total ring of quotients, then R is weak semihereditary.

We use Theorem 4.1 to study the transfer of the notion of a weak semihereditary ring in particular kind of trivial ring extensions (Corollaries 3.3 and 3.4).

We also use trivial ring extensions to generate suitable examples of weak semihereditary rings. Namely, in Examples 3.5 and 3.6 , we show, unlike the classical case, that there is not relation between a weak semihereditary rings and coherent.

In Proposition 3.7, we give a condition so that the descent of the notion of the weak semihereditary rings holds in extensions of rings. Namely, if R be a subring retract of T with T is a faithfully flat $R-module$, then T is weak semihereditary implies that R is weak semihereditary. However, in Example 3.8, we show that the ascent of the notion of the weak semihereditary ring does not hold in extensions of ring, and so the homomorphic image of a weak semihereditary ring is not necessarily in general weak semihereditary. And, in Example 3.9, we use the trivial ring extension to show that the condition " R is a subring retract of T with T is a faithfully flat $R-module$" in Proposition 3.7 cannot be dropped, and namely the localization of a weak semihereditary ring is not in general weak semihereditary.

In Example 3.10, we show that, in general, the transfer of weak semihereditary notion does not hold in pullback constructions.

In Proposition4.2, we prove that if R is a commutative ring and I is a proper ideal of R. Then if (R,M) is a local total ring of quotients, then $R \bowtie I$ is a weak semihereditary ring.

In Theorem 3.13, we study the notion of weak semihereditary rings in direct products of rings.

In Example 3.15, we prove that the direct products of a weak semihereditary ring is not in general weak semihereditary.

2 Problem Formulations

Recall that a ring R is called semihereditary if every finitely generated ideal is projective. In this paper, we introduce and investigate a generalization of a semihereditary ring, which we call a weak semihereditary ring. A ring R is called weak semihereditary if, for every two ideals $I \subseteq J$ of R such that I is finitely generated, J projective proper ideal, then I is projective (Definition 3.1).

Question. Let R be a commutative ring. Is R weak semihereditary if and only if is R semihereditary, in general ?

3 Main Results

Recall that a ring R is called semihereditary if every finitely generated ideal is projective. In this paper we introduce and investigate the following generalization of semihereditary rings.

Definition 3.1 *A ring R is called weak semihereditary if, for every two ideals $I \subseteq J$ of R such that I is finitely generated, J projective proper ideal, then I is projective.*

Now, we give a sufficient condition to have equivalence between a semihereditary and weak semihereditary properties, and we show that if R is a local total ring of quotients, then R is weak semihereditary.

Theorem 3.2 *Let R be a ring. Then:*

1. *If R is a semihereditary ring, then R is a weak semihereditary ring.*
2. *If R contains a regular element, then R is a weak semihereditary ring if and only if R is a semihereditary ring.*
3. *If R is a local total ring of quotients, then R is a weak semihereditary ring.*

Proof.

(1) It is clear that if R is a semihereditary ring, then R is a weak semihereditary ring.

(2) By (1) if R is a semihereditary ring, then R is a weak semihereditary ring. Conversely, assume that R is a weak semihereditary ring and let I be a finitely generated proper ideal of R. Let $x \in R$ be a regular element of R, then $xI \subseteq xR$. Therefore, xI is a projective ideal of R, since R is a weak semihereditary ring. Hence, I is a projective ideal of R, since $xI(\cong I)$ (since x is regular), as desired.

(3) Let R be a local total ring of quotients. We claim that R is a weak semihereditary ring. Deny. Then, there exist $I \subseteq J \subseteq M$, where M is a maximal ideal of R, J is a proper projective ideal and I is a non projective finitely generated ideal of R. Then J is free (since R is local). Then $J = xR$, where x is a regular element of R. A contradiction (since R is a total ring of quotients). Then R is a weak semihereditary ring. And this completes the proof of the Theorem.

As an application of Theorem 4.1, we give the two following results which study the transfer of the notion of the weak semihereditary rings, in particular kind of trivial ring extensions.

Recall, for a ring A and an A-module E, that the ring $R := A \propto E$ of pairs (a, e) whose underlying group is $A \times E$ with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$ is called trivial ring extension of A by E (also called the idealization of E over A).

Corollary 3.3 *Let D be a domain, $K := qf(D)$, and $R := D \propto K$ be the trivial ring extension of D by K. Then:*

1. *R is never semihereditary.*
2. *If D is not a field, then R is never weak semihereditary.*
3. *If D is a field, then :*
   - *(a) R is a weak semihereditary ring .*
   - *(b) R is not a semihereditary ring.*

Proof.
(1) By [11, Theorem 2.8].

(2) If D is not a field, let $d \in D\backslash\{0\}$ which is not invertible. Then $(d, 0)$ is a regular element of R, so R is not a weak semihereditary ring by Theorem 4.1(2) (since R is not a semihereditary ring by (2)),
(3) Assume that D is a field, then :

- (a) It is clear that R is a local total ring of quotients. Then by Theorem 4.1(3) R is weak semihereditary.
- (b) By (1) R is not a semihereditary ring.

Corollary 3.4 *Assume that (A, M) is a local ring and E an A-module such that $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is always weak semihereditary.*
2. *R is never semihereditary.*

Proof.
(1) R is a local total ring of quotients. Then by Theorem 4.1(3) R is a weak semihereditary ring.
(2) By [11, Theorem 2.6].

It is well-known that the semihereditary rings are coherent (see, for instance,[8, 20]). Recall that a ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \bigcap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R (see for instance[3, 8, 20]). Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Prüfer/semihereditary rings.

The following example shows that a weak semihereditary ring can not be a coherent ring.

3.5 *Let K be a field, $R := K \propto K^{\infty}$ be the trivial ring extension of K by K^{∞}. Then :*

1. *R is a weak semihereditary ring.*
2. *R is not a coherent ring.*

Proof.
(1) By Corollary 3.3(3(a)) R is a weak semihereditary ring.

(2) By [13, Theorem 2.1].

The following example shows that a coherent ring may not be a weak semihereditary ring.

3.6 *Let K be a field and $R := K[X, Y]$ the polynomial ring, where X and Y are two indeterminate elements. Then :*

1. *R is a coherent ring.*
2. *R is not a weak semihereditary ring.*

Proof.
(1) R is Noetherian, then R is a coherent ring.

(2) $w.dim(K[X, Y]) = 2$, then R is not a semihereditary ring. Then R is not a weak semihereditary ring (since R is a domain).

The following result gives condition so that the descent of the notion of a weak semihereditary rings holds in extensions of rings.

Proposition 3.7 *Let R be a subring retract of T with T is a faithfully flat $R-$module, for each ideal I of R, $IT \neq T$. Then if T is a weak semihereditary ring, then R is a weak semihereditary ring.*

Proof.
Assume that T is a weak semihereditary ring. Let $I_1 \subseteq I_2$ be a two ideals of R with I_2 is proper projective, and I_1 is a finitely generated ideal, so $I_2 \bigotimes_R T = I_2T$ is a proper projective ideal of T, since T is faithfully flat over R, and $I_1 \bigotimes_R T = I_1T$ is a finitely generated ideal of T. On the other hand, we have $I_1T \subseteq I_2T$, then I_1T is projective, since T is a weak semihereditary ring. We claim that I_1 is a projective ideal of R. Indeed, for any R-module N, we have by [5, p.118],

$$Ext_R(I_1, N \bigotimes_R T) \cong Ext_T(I_1 \bigotimes_R T, N \bigotimes_R T) = 0$$

On the other hand, N is a direct summand of $N \bigotimes_R T$ since R is a direct summand of T. Therefore, $Ext_R(I_1, N) = 0$ for every R-module N. This means that I_1 is a projective ideal of R. , as desired.

The following example shows that the homomorphic image of a weak semihereditary ring is not necessarily in general weak semihereditary.

3.8 *Let (A, M) be a $non-Prüfer$ local domain ring, $(0 \neq:)E$ is an $A-$module, $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is a weak semihereditary ring.*
2. *$A(\cong R/0 \propto E)$ is not a weak semihereditary ring.*

Proof.
(1) R is a total ring of quotients. And R is local (since A is local). Then by Theorem 4.1(3) R is a weak semihereditary ring.
(2) Since $A \cong R/(0 \propto E)$ with $(0 \propto E)$ is an ideal of R. We claim that A is not a weak semihereditary ring. Deny. Then, A is a semihereditary ring, implies Prüfer, since A is a domain, a contradiction. Then $R/(0 \propto E)$ is not a weak semihereditary ring.

We know that the localization of a semihereditary ring is semihereditary. But the following example shows that the localization of a weak semihereditary ring is not in general weak semihereditary.

3.9 *Let $B = K[[X_1, X_2, X_3]] = K + M$ be a power series ring over a field K, where X_1, X_2 and X_3 are indeterminate elements, and $M = (X_1, X_2, X_3)$ the maximal ideal of B. Let $0 \neq E$ be an B-module such that $ME = 0$ and let $R := B \propto E$ be the trivial ring extension of B by E. Let S be a multiplicative subset of R given by $S = \{(X_1, 0)^n / n \in \mathbf{N}\}$ and S_0 is the multiplicative subset of B given by $S_0 = \{X_1^n / n \in \mathbf{N}\}$. Then:*

1. *R is a weak semihereditary ring.*
2. *$S^{-1}R$ is not a weak semihereditary ring.*

Proof.

(1) R is a total ring of quotients. And R is local (since B is local). Then R is a weak hereditary ring by Theorem 4.1(3).

(2) $S^{-1}R = S_0^{-1}(B \propto E) = S_0^{-1}(B) \propto S_0^{-1}(E) = S_0^{-1}(B)$, since $S_0^{-1}(E) = 0$ ($X_1 E \subseteq ME = 0$, so $S_0^{-1} X_1 E = 0$, then $S_0^{-1} E = 0$). Then, $S^{-1}R = S_0^{-1}B = (S_0^{-1}K[[X_1]])[[X_2, X_3]]$, so $wdim(S^{-1}R) = wdim(S_0^{-1}K[[X_1]]) + 2 \geq 2$. Then $S^{-1}R$ is not a semihereditary ring. Then $S^{-1}(R)$ is not a weak semihereditary ring, since $S^{-1}(R)(\cong S_0^{-1}(B)$ is a domain.

The following example shows that, in general, the transfer of a weak semihereditary ring notion does not hold in Pullback constructions. We adopt the following riding assumptions and notations: T is a domain of the form $T = K + M$, where K is a field and M is a non-zero maximal ideal of T; D is a subring of K such that $qf(D) = K$; and $R = D + M$ (For more details, see [4, 6, 7, 8, 19].

3.10 *Let D be a Prüfer domain which is not a field, $K = qf(D)$, and let the following Pullback:*

$$\begin{array}{ccc} R = D \propto K & \longrightarrow & T = K \propto K \\ \downarrow & & \downarrow \\ D \cong R/0 \propto K & \longrightarrow & K \end{array}$$

T is a weak semihereditary ring (since it is a local total ring of quotient), D is a Prüfer domain, but R is not weak semihereditary by Corollary 2.3(ii).

Now we study the notion of the weak semihereditary rings in amalgamated duplication of a ring along an ideal I.

Proposition 3.11 *Let R be a commutative ring and let I be a proper ideal of R. If (R, M) is a local total ring of quotients, then $R \bowtie I$ is a weak semihereditary ring.*

Proof.
Since R is a local ring, then $R \bowtie I$ is a local ring by [17, Corollary 6]. Our aim is to show that $R \bowtie I$ is a total ring of quotients. We wish to show that each element $(r, r+i)$ of $R \bowtie I$ is invertible or zero-divisor element. Two cases are then possible.

Case 1. $r \notin M$. In this case, r is invertible in R and then $(r, r+i) \notin M \bowtie I$. Hence, $(r, r+i)$ is invertible in $R \bowtie I$ (since $R \bowtie I$ is a local ring, where $M \bowtie I$ is a maximal ideal of the local ring $R \bowtie I$).

Case 2. $r \in M$, then r is a zero-divisor element of R (since R is a total ring of quotients, that is, every element is either a unit or a zero divisor). Then by [16, Proposition 2.2] we have $(r, r+i)$ is zero-divisor element of $R \bowtie I$. Then $R \bowtie I$ is a local total ring of quotients. Hence $R \bowtie I$ is a weak semihereditary ring by Theorem 4.1(3). This completes the proof of the Proposition.

3.12 *Let $A := \mathbf{Z}/(2^i\mathbf{Z})$, where $i \geq 2$ be an integer, and let $R := A \propto A$ the trivial ring extension of A by A, and let I be a proper ideal of R. Then:*

(1) *A is a local total ring of quotients with maximal ideal $M = 2A$ by [2, example 3.6(1)]. In particular, A is a weak semihereditary ring by Theorem 4.1*(3).

(2) *R is a local total ring of quotients by [2, example 3.6(2)]. In particular, R is a weak semihereditary ring by Theorem 4.1*(3).

(3) *$R \bowtie I$ is a local total ring of quotients by Proposition 4.2. In particular, $R \bowtie I$ is a weak semihereditary ring by Theorem 4.1*(3).

Now we study the notion of the weak semihereditary rings in direct products of rings.

Theorem 3.13 *Let $(R_i)_{i=1,2,\ldots,n}$ be a family of rings and let $R := \prod_{i=1}^{n} R_i$. Then, if R is a weak semihereditary, then R_i for each $i = 1, \ldots, n$.*

We need the following Lemma before proving Theorem 3.13.

Lemma 3.14 *([15, Lemma 2.5]) Let $(R_i)_{i=1,2}$ be a family of rings and let E_i an R_i- module for $i = 1, 2$. Then:*

1. *$E_1 \prod E_2$ is a finitely generated $R_1 \prod R_2 -$ module if and only if E_i is a finitely generated R_i- module for i = 1, 2.*

2. *$E_1 \prod E_2$ is a projective $R_1 \prod R_2 -$ module if and only if E_i is a projective $R_i -$ module for i = 1, 2.*

Proof of Theorem 3.13. We prove the result for $i = 1, 2$, and the Theorem will be established by induction on n.
Assume that $(R_1 \times R_2)$ is a weak semihereditary ring. We wish to show that R_1 is weak semihereditary (it is the same for R_2). Let $I_1 \subseteq J_1$ be two ideals of R_1 such that J_1 is a projective proper ideal and I_1 is a finitely generated ideal of R_1. So $I_1 \times R_2$ is a finitely generated ideal of $R_1 \times R_2$ and $J_1 \times R_2$ is a projective proper ideal of $R_1 \times R_2$ by Lemma 3.14. And we have $I_1 \times R_2 \subseteq J_1 \times R_2$ (since $I_1 \subseteq J_1$), so $I_1 \times R_2$ is projective (since $R_1 \times R_2$ is weak semihereditary), then I_1 is projective by Lemma 3.14. This completes the proof of the Theorem.

We know that the direct products of a semihereditary ring is semihereditary. But the following example shows that the direct products of a weak semihereditary ring is not in general weak semihereditary.

3.15 *Let $R_1 = \mathbf{Z}$ and let $R_2 = K \propto K$ be two rings with K is a field. Then $R_1 \times R_2$ is not weak semihereditary.*

Proof.
It is clear that $R_1 = \mathbf{Z}$ is a semi hereditary ring, so $R_1 = \mathbf{Z}$ is a weak semihereditary ring.
$R_2 = K \propto K$ is a non-semihereditary ring weak semihereditary by Corollary 3.3. On the other hand, $p\mathbf{Z} \times (0 \propto K) \subseteq p\mathbf{Z} \times R_2$ with $p\mathbf{Z} \times (0 \propto K)$ is a finitely generated ideal of $R_1 \times R_2$ and $p\mathbf{Z} \times R_2$ is projective proper ideal of $R_1 \times R_2$ by Lemma 3.14, but $p\mathbf{Z} \times (0 \propto K)$ is not projective of $R_1 \times R_2$ since $(0 \propto K)$ is not a projective ideal of R_2. Then $R_1 \times R_2$ is not weak semihereditary.

4 Conclusion

These are the main results of the paper.

Theorem 4.1 *Let R be a ring. Then:*

1. *If R is a semihereditary ring, then R is a weak semihereditary ring.*

2. *If R contains a regular element, then R is a weak semihereditary ring if and only if R is a semihereditary ring.*

3. *If R is a local total ring of quotients, then R is a weak semihereditary ring.*

Proposition 4.2 *Let R be a commutative ring and let I be a proper ideal of R. If (R, M) is a local total ring of quotients, then $R \bowtie I$ is a weak semihereditary ring.*

5 Open Problem

Question. Let R be a commutative ring and let I be a proper ideal of R. Is R weak semihereditary if and only if is $R \bowtie I$ weak semihereditary, in general ?

ACKNOWLEDGEMENTS. The authors thank the referee for his/her careful reading of this work.

References

[1] C. Bakkari, S. Kabbaj, and N. Mahdou, "Trivial extensions defined by Prüfer conditions", *J. Pure and Applied Algebra*, (2010), 53–60.

[2] C. Bakkari, N. Mahdou and H. Mouanis; "Prüfer-Like Conditions in Subrings Retract and Applications", *Comm Algebra*, 37 (1) (2009) 47–55.

[3] N. Bourbaki, *Algebre homologique*, Chapitre 10, Masson, Paris, (1980).

[4] W. Brewer, E. Rutter, "D+M constructions with general overrings", *Michigan Math. J*, 23 (1976), 33–42.

[5] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, (1956).

[6] D.E. Dobbs, S. Kabbaj, N. Mahdou, M. Sobrani, "When is D + M n-coherent and an (n, d)-domain?", *Lecture Notes in Pure and Appl. Math., Dekker*, 205 (1999), 257–270.

[7] S. Gabelli, E. Houston, "Coherentlike conditions in pullbacks", *Michigan Math. J*, 44 (1997), 99–123.

[8] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, 1371, Springer- Verlag, Berlin, (1989).

[9] S. Glaz, "The weak global dimension of Gaussian rings", *Proc. Amer. Math. Soc*, 133 (9) (2005) 2507–2513.

[10] J.A. Huckaba, *Commutative Rings with Zero-Divisors*, Marcel Dekker, New York, (1988).

[11] S. Kabbaj and N. Mahdou, "Trivial extension defined by coherent-like conditions", *Comm. Algebra* , 32(10)(2004), 3937–3953.

[12] S. Kabbaj, N. Mahdou, "Trivial extensions of local rings and a conjecture of Costa", *Lecture Notes in Pure and Appl. Math., Dekker*, 231 (2003), 301–311.

[13] N. Mahdou, "on 2-von neumann regular rings", *Comm. Algebra* , 33 (2005), 3489—3496.

[14] N. Mahdou, "On weakly finite conductor rings", *Comm. Algebra* , 10 (2004), 4027—4036.

[15] N. Mahdou, "On Costa's conjecture", *Comm. Algebra* ,29 (2001), 2775–2785.

[16] H. R. Maimani and S. Yassemi, "Zero-divisor graphs of amalgamated duplication of a ring along an ideal". *J. Pure Appl. Algebra* 212 (2008), no. 1, 168–174.

[17] M. D'Anna, "A construction of Gorenstein rings", *J. Algebra*, 306 (2006), 507—519.

[18] M. D'Anna and M. Fontana, "The amalgamated duplication of a ring along a multiplicative-canonical ideal", *Ark. Mat*, 45 (2007), no. 2, 241–252.

[19] M. D'Anna and M. Fontana, "An amalgamated duplication of a ring along an ideal: the basic properties", *J. Algebra Appl*, 6 (2007), no. 3, 443–459.

[20] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, (1979).

[21] W. V. Vasconcelas, "Finiteness in projective ideals", *J. Algebra*, 25(1973), 269–278.