Int. J. Open Problems Compt. Math., Vol. 4, No. 2, June 2011
ISSN 1998-6262; Copyright © ICSRS Publication, 2011
www.i-csrs.org

Application of Generalized Fractional Integral Operators to Certain Class of Multivalent Prestarlike Functions With Negative Coefficients

Jamal M. Shenan

Department of mathematics, Alazhar University-Gaza

P.O.Box 1277, Gaza ,Palestine.

e-mail: shenanjm@yahoo.com

Abstract

In this paper we introduce certain class of analytic and multivalent prestarlike functions with negative coefficients defined by a certain linear operator. Coefficient inequalities and distortion theorems in terms of the fractional integral operators involving H-function are obtained. Some convolution properties and radius of convexity for this class are also included.

Keywords: Multivalent functions, Prestarlike functions, Fractional calculus.
2000 Mathematics Subject Classification: 30C45, 26A33.

1 Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad \left(p \in \mathrm{N} = \{1,2,3,\ldots\}\right), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U=\{z:|z|<1\}$. And let T_p denote the subclass of A_p consisting of analytic and p-valent functions which can be expressed in the form

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}z^{k+p},\ \ (a_{k+p}\geq 0). \tag{1.2}$$

A function $f(z)\in A_p$ is said to be p-valent starlike of order α, if and only if

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\}>\alpha \qquad (z\in U), \tag{1.3}$$

for some $\alpha(0\leq\alpha<p)$. We denoted the class of all p-valent starlike functions of order α by $S_p(\alpha)$. Further a function $f(z)$ from A_p is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1+\frac{z f''(z)}{f'(z)}\right\}>\alpha \qquad (z\in U), \tag{1.4}$$

for some $\alpha(0\leq\alpha<p)$. We denote the class of all p-valent convex functions of order α by $C_p(\alpha)$. The classes $S_p(\alpha)$ and $C_p(\alpha)$ were first introduced by Patil and Thakare [11].
The function

$$S_\gamma(z)=z^p(1-z)^{-2(p-\gamma)} \quad \left(0\leq\gamma<p\ ;p\in\mathrm{N}\right) \tag{1.5}$$

is the well-known extreme function for the class $S_p(\gamma)$. Setting

$$C(\gamma,k)=\frac{\prod_{i=2}^{k+1}\left(2(p-\gamma)+i-2\right)}{k\,!} \quad \left(k\geq 1;0\leq\gamma<p\right), \tag{1.6}$$

then $S_\gamma(z)$ can be written in the form

$$S_\gamma(z)=z^p+\sum_{k=1}^{\infty}C(\gamma,k)z^{k+p}\ . \tag{1.7}$$

We note that $C(\gamma,k)$ is a decreasing function in γ and that

$$\lim_{k\to\infty}C(\gamma,k)=\begin{cases}\infty & \left(\gamma<\frac{2p-1}{2}\right)\\ 1 & \left(\gamma=\frac{2p-1}{2}\right)\\ 0 & \left(\gamma>\frac{2p-1}{2}\right).\end{cases} \tag{1.8}$$

Let $(f*g)(z)$ denote the Hadamard product (convolution) of two analytic functions $f(z)$ and $g(z)$, that is , if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z^{p}+\sum_{k=1}^{\infty}b_{k+p}z^{k+p}, \tag{1.9}$$

then

$$(f*g)(z)=z^{p}+\sum_{k=1}^{\infty}b_{k+p}a_{k+p}z^{k+p} \quad . \tag{1.10}$$

A function $f(z)\in A_p$ is said to be p -valent γ -prestarlike function of order $\alpha(0\le\alpha<p;0\le\gamma<p)$ if

$$(f*S_{\gamma})(z)\in S_{p}^{*}(\alpha), \tag{1.11}$$

where $S_{\gamma}(z)$ is defined by (1.7). We denote by $R_p(\gamma,\alpha)$ the class of all p -valent γ -prestarlike function of order $\alpha(0\le\alpha<p;0\le\gamma<p)$.

The class $R_p(\gamma,\alpha)$ was studied by Aouf and Siverman [1], Shenan, Salim and Marouf [18],while the class $R_p(\alpha,\alpha)=R_p(\alpha)$ is the class of p-valent prestarlike functions of order α studied by Kumar and Reddy [9] and others. For $\gamma=\frac{2p-1}{2};0\le\alpha<p$, $R_p\left(\frac{2p-1}{2},\alpha\right)=S_p(\alpha)$. Let $R_p^{*}(\gamma,\alpha)=R_p(\gamma,\alpha)\bigcap T_p$, $S_p^{*}(\alpha)=S_p(\alpha)\bigcap T_p$ and $C_p^{*}(\alpha)=C_p(\alpha)\bigcap T_p$.

The function $\phi_p(a,c;z)$ is defined by

$$\phi_p(a,c;z)=z^{p}+\sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}z^{p+k}, \tag{1.12}$$

$\left(z\in U;a,c\in R\setminus Z_0^{-}\right)$

where $(a)_k$ is the Pochammer symbol defined by

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}=\begin{cases}1; & (k=0),\\ a(a+1)(a+2)...(a+k-1); & (k\in N).\end{cases} \tag{1.13}$$

Corresponding to the function $\phi_p(a,c;z)$, Saitoh [14] introduced and studied a linear operator $L_p(a,c)$ on A_p by the following Hadamard product (or convolution):

$$L_p(a,c)f(z)=\phi_p(a,c;z)*f(z) \qquad \left(f(z)\in A_p\right). \tag{1.14}$$

For $p=1$ $L_1(a,c)$ on A was first defined by Carlson and Shaffer [3].

We can easily find from (1.12) and (1.14) and for the function $f(z)\in T_p$ that

$$L_p(a,c)f(z)=z^{p}-\sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}a_{k+p}z^{k+p}. \tag{1.15}$$

It is known [14] that

$$z\left(L_p(a,c)f(z)\right)' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z). \qquad (1.16)$$

Note that $L_p(p,p)f(z) = f(z)$, $L_p(p+1,p)f(z) = \frac{zf'(z)}{p}$,

$L_p(p+2,p)f(z) = \frac{1}{p(p+1)}\left[2zf'(z) + z^2 f''(z)\right]$, and

$L_p(n+p,1)f(z) = D^{n+p-1}f(z)$,

where D^{n+p-1} is the well-known Ruschewehy derivative of order $(n+p-1)$ introduced by Goel and Sohi [5].

The function $f(z)$ is said to be subordinate to $g(z)$ in U written $f(z) \prec g(z)$ if there exist a function $w(z)$ analytic U such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

Now making use of the operator $L_p(a,c)$ defined by (1.14) we introduce the following subclass $S_p(a,c,A,B,\gamma,\alpha)$ of p-valent γ-prestarlike function of order $\alpha(0 \le \alpha < p; 0 \le \gamma < p)$.

Definition 1. For A, B arbitrary fixed real numbers, $-1 \le B < A \le 1$, a function $f(z) \in A_p$ defined by (1.1) is said to be in the class $S_p(a,c,A,B,\gamma,\alpha)$ if it satisfies

$$\frac{aL_p(a+1,c)(f * S_\gamma)(z)}{L_p(a,c)(f * S_\gamma)(z)} \prec \frac{a + \left[aB + (A-B)(p-\alpha)\right]z}{1+Bz} \quad (z \in U), \qquad (1.17)$$

$(0 \le \alpha < p; 0 \le \gamma < p; a,c \in R \setminus Z_0^-)$

also let $T_p(a,c,A,B,\gamma,\alpha) = S_p(a,c,A,B,\gamma,\alpha) \bigcap T_p$.

Using (1.16) it is easily seen that (1.17) is equivalent to

$$\left| \frac{\dfrac{z\left(L_p(a,c)(f * S_\gamma)\right)'(z)}{L_p(a,c)(f * S_\gamma)(z)} - p}{pB + (A-B)(p-\alpha) - B\,\dfrac{z\left(L_p(a,c)(f * S_\gamma)\right)'(z)}{L_p(a,c)(f * S_\gamma)(z)}} \right| < 1 \qquad (z \in U). \qquad (1.18)$$

Note that :

(i) $T_p(p,p,1,-1,\gamma,\alpha) = R_p^*(\gamma,\alpha)$;

(ii) $T_p(p,p,1,-1,\frac{2p-1}{2},\alpha) = S_p^*(\alpha)$;

(iii) $T_p\,(p+1, p, 1, -1, \frac{2p-1}{2}, \alpha) = C_p^*(\alpha)$,

The class $T_p\,(a, c, A, B, \gamma, \alpha)$ generalizes and extends other classes studied and introduced by several research workers as Aouf and Siverman [1], Shenan, Salim and Marouf [18], Kumar and Reddy [9], Sheil-Small et al. [17], Owa and Uralegaddi [10].

2 Fractional Integral Operators

We shall be concerned with fractional integral operators involving Fox's H-function which have been recently introduced by [6] and [7].

Definition 2. Let $s \in N_0$ (set of non-negative integers) , $\beta_j \in \mathbb{R}_+$ (set of positive real numbers) and $\delta_j, \gamma_j \in \mathbb{C}$ (set of complex numbers) for $j = 1, 2, ..., s$, while $\sum_{j=1}^{s} \mathrm{Re}(\delta_j) > 0$, then the generalized fractional integral operator for the function $f(z)$ is given by

$$I_{(\beta_s);s}^{(\gamma_s);(\delta_s)} f(z) = I_{(\beta_1,\ldots,\beta_s);s}^{(\gamma_1,\ldots,\gamma_s);(\delta_1,\ldots,\delta_s)} f(z)$$

$$= \frac{1}{z}\int_0^z H_{s,s}^{s,0}\left[\frac{t}{z}\left|\begin{matrix}(1+\gamma_j+\delta_j-1/\beta_j, 1/\beta_j)_{1,s}\\ (1+\gamma_j-1/\beta_j, 1/\beta_j)_{1,s}\end{matrix}\right.\right] f(t)\,dt\ \textbf{,} \text{ for } \sum_{j=1}^{s}\mathrm{Re}(\delta_j) > 0$$

$$= f(z), \text{ for } \delta_1 = \delta_2 = ... = \delta_s = 0 \tag{2.1}$$

where $f(z)$ is analytic function in a simply connected region of the $z-$plane containing the origin and

$$H_{s,s}^{s,0}[z] = \frac{1}{2\pi i}\int_L \prod_{j=1}^{s} \frac{\Gamma\left(b_j - t/\beta_j\right)}{\Gamma\left(a_j - t/\beta_j\right)} z^t\,dt\ , \tag{2.2}$$

where

$a_j = \gamma_j + \delta_j + 1 - 1/\beta_j$ and $b_j = \gamma_j + 1 - 1/\beta_j$ $(j = 1, 2, ..., s)$.

The fractional integral operator $I_{(\beta_s);s}^{(\gamma_s);(\delta_s)} f(z)$ contains as special case, many other fractional integral operators [cf. Saigo, Raina and Kilbas(13)], here we need some of them.

(I) For $t > 0$ and $f(z)$ being analytic function in a simply connected region of the z –plane containing the origin,

$$z^{\beta} I_{1,1}^{0,\beta} f(z) = z^{\beta-1} \int_0^z H_{1,1}^{1,0}\left[\frac{t}{z}\middle|\begin{matrix}(\beta,1)\\(0,1)\end{matrix}\right] f(t)\,dt$$

$$= \frac{1}{\Gamma(\beta)} \int_0^z \frac{f(t)}{(z-t)^{1-\beta}}\,dt$$

$$= D_z^{-\beta} f(z), \tag{2.3}$$

where $\mathrm{Re}(\beta) > 0$ and the multiplicity of $(z-t)^{\beta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0, D_z^{-\beta} f(z)$ is called the Riemann-liouville fractional integral operator of order β [16].

(II) With $\mathrm{Re}(\delta) > 0,\ \beta, \eta \in \mathbb{C}$ and $f(z)$ being analytic function in a simply connected region of the z –plane containing the origin,

$$z^{-\beta} I_{(1,1)\ \ ;2}^{(0,\eta-\beta);(-\beta,\delta+\beta)} f(z) = z^{-\beta-1} \int_0^z H_{2,2}^{2,0}\left[\frac{t}{z}\middle|\begin{matrix}(-\beta,1);(\delta+\eta,1)\\(0,1);(\eta-\beta,1)\end{matrix}\right] f(t)\,dt$$

$$= \frac{z^{-\delta-\beta}}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1}\ {}_2F_1\left(\delta+\beta, -\eta; \delta; 1-\frac{t}{z}\right) f(t)\,dt$$

$$= I_{0,z}^{\delta,\beta,\eta} f(z), \tag{2.4}$$

where $I_{0,z}^{\delta,\beta,\eta} f(z)$ is the well known Saigo fractional integral operator of order δ, [12] and with the order

$$f(z) = O\left(|z|^{\varepsilon}\right),\ z \to 0$$

$$\varepsilon > \max(0, \beta-\eta) - 1$$

and the multiplicity of $(z-t)^{\delta-1}$ is remove as in (I).

(III) the following form is due to Saigo, Raina and Kilbas [13], for $\mathrm{Re}(\eta) > 0$, $\upsilon, \mu, \xi, \delta \in C, \sigma \in R_+$ and $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin

$$
\begin{aligned}
&z^{\mu+\sigma(1+\eta)+\upsilon+1} I_{(\sigma,\sigma);2}^{\left(\frac{\upsilon+1}{\sigma}-1,\eta-\xi-\delta+\frac{\upsilon-1}{\sigma}-1\right);(\eta-\xi,\xi)} f(z) \\
&\qquad = z^{\mu+\sigma(1-\eta)+\upsilon} \int_0^z H_{2,2}^{2,0}\left[\frac{t}{z}\left|\begin{matrix}\left(\eta-\xi+\frac{\upsilon}{\sigma},\frac{1}{\sigma}\right),\left(\eta-\delta+\frac{\upsilon}{\sigma},\frac{1}{\sigma}\right)\\ \left(\frac{\upsilon}{\sigma},\frac{1}{\sigma}\right),\left(\eta-\xi-\delta+\frac{\upsilon}{\sigma},\frac{1}{\sigma}\right)\end{matrix}\right.\right] f(t)\,dt \\
&\qquad = \frac{\sigma z^{\mu}}{\Gamma(\eta)} \int_0^z \left(z^{\sigma}-t^{\sigma}\right)^{\eta-1} {}_2F_1\left(\xi,\delta,\eta;1-\left(t/z\right)^{\sigma}\right) t^{\upsilon} f(t)\,dt \\
&\qquad = I_{+,\upsilon}^{\eta,\sigma,\mu,\xi,\delta} f(z)\,,
\end{aligned}
\tag{2.5}
$$

with the order

$$f(z) = O\left(|z|^{\varepsilon}\right),\ z \to 0$$

$$\sigma + \varepsilon > \max\left(0, \delta + \xi - \eta\right) - 1$$

and the multiplicity of $\left(z^{\sigma} - t^{\sigma}\right)^{1-\eta}$ is remove by requiring $\log\left(z^{\sigma} - t^{\sigma}\right)$ to be real as $\left(z^{\sigma} - t^{\sigma}\right) > 0$.

For the function $f(z)$ defined by (1.1), the operator $J_{\left(\beta_s^{-1}\right);s}^{(\gamma_s);(\delta_s)}$ is defined by

$$J_{\left(\beta_s^{-1}\right);s}^{(\gamma_s);(\delta_s)} f(z) = \prod_{j=1}^{s}\left[\frac{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+p\beta_j\right)}\right] I_{\left(\beta_s^{-1}\right);s}^{(\gamma_s);(\delta_s)} f(z) \tag{2.6}$$

where $\operatorname{Re}\left(\gamma_j\right) > p\beta_j - 1, \beta_j \in R_{+}$ and $\operatorname{Re}\left(\delta_j\right) \ge 0, j = 1,....,s$.

Note that for $s = \beta_1 = 1, \gamma_1 = 0, \delta_1 = \gamma$ in (2.6) we obtain the operator

$$J^{-\gamma} f(z) = \frac{\Gamma(1+p+\gamma)}{\Gamma(1+p)} z^{-\gamma} D_z^{-\gamma} f(z) \tag{2.7}$$

Similarly for $s = 2, \beta_1 = \beta_2 = 1, \gamma_1 = 0, \gamma_2 = \eta - \phi, \delta_1 = -\phi, \delta_2 = \beta + \phi$, we have the following operator

$$J_{0,z}^{\beta,\phi,\eta} f(z) = \frac{\Gamma(1-\phi+p)\Gamma(1+\beta+\eta)}{\Gamma(1+p)\Gamma(1-\phi+\eta+p)} z^{\phi} I_{0,z}^{\beta,\phi,\eta} f(z) \tag{2.8}$$

$\min\left(p-\phi+\eta, p-\phi, p+\beta+\eta\right) > -1$

The operators $J^{-\gamma} f(z)$ and $J_{0,z}^{\beta,\phi,\eta} f(z)$ have been studied by Choi and Saigo [4].

Lemma 1. (due to Kiryakova[7]**)** If $\operatorname{Re}\left(\gamma_j\right) > \frac{-p}{\beta_j} - 1, \beta_j \in R_{+}$**,** and

$\operatorname{Re}\left(\delta_j\right) \ge 0, j = 1,....,s$ **,** then

$$I^{(\gamma_s);(\delta_s)}_{(\beta_s);s}\left[z^p\right]=\prod_{j=1}^{s}\left[\frac{\Gamma\left(1+\gamma_j+p/\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p/\beta_j\right)}\right]z^p. \tag{2.9}$$

3 Coefficient estimates

Theorem 1. *For A, B arbitrary fixed real numbers, $-1\le B<A\le 1$, a function $f(z)\in T_p$ defined by (1.2) belongs to the class $T_p(a,c,A,B,\gamma,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left[(1-B)k+(A-B)(p-\alpha)\right]\frac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}\le (A-B)(p-\alpha), \tag{3.1}$$

where

$0\le\alpha<p;0\le\gamma<p;a,c\in R\setminus Z_0^-$ and $C(\gamma,k)$ is defined by (1.6).

The result is sharp.

Proof. Let the function $f(z)\in T_p$ defined by (1.2), then from (1.7) and (1.15) we have

$$L_p(a,c)(f*S_\gamma)(z)=z^p-\sum_{k=1}^{\infty}\frac{(a)_k}{(c)_k}C(\gamma,k)a_{k+p}z^{k+p}, \tag{3.2}$$

Assume that the inequality (3.1) holds true and let $|z|=1$, then from (1.18) we have

$$\left|\frac{z\left(L_p(a,c)(f*S_\gamma)\right)'(z)}{L_p(a,c)(f*S_\gamma)(z)}-p\right|-\left|pB+(A-B)(p-\alpha)-B\frac{z\left(L_p(a,c)(f*S_\gamma)\right)'(z)}{L_p(a,c)(f*S_\gamma)(z)}\right|$$

$$=\left|-\sum_{k=1}^{\infty}k\frac{(a)_k}{(c)_k}C(\gamma,k)a_{k+p}z^{k+p}\right|$$

$$-\left|(A-B)(p-\alpha)z^p+\sum_{k=1}^{\infty}\left(Bk-(A-B)(p-\alpha)\right)\frac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}z^{k+p}\right|$$

$$\le\sum_{k=1}^{\infty}\left[(1-B)k+(A-B)(p-\alpha)\right]\frac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}-(A-B)(p-\alpha)\le 0.$$

Hence by the principle of maximum modulus, $f(z)\in T_p(a,c,A,B,\gamma,\alpha)$.

Conversely, Assume that $f(z)$ defined by (3.1) is in the class $T_p(a,c,A,B,\gamma,\alpha)$, then from (3.2) we have

$$\left|\frac{\dfrac{z\left(L_p(a,c)(f*S_\gamma)\right)'(z)}{L_p(a,c)(f*S_\gamma)(z)}-p}{pB+(A-B)(p-\alpha)-B\,\dfrac{z\left(L_p(a,c)(f*S_\gamma)\right)'(z)}{L_p(a,c)(f*S_\gamma)(z)}}\right|$$

$$=\left|\frac{-\sum_{k=1}^{\infty}k\,\dfrac{(a)_k}{(c)_k}C(\gamma,k)a_{k+p}z^{k+p}}{(A-B)(p-\alpha)z^p+\sum_{k=1}^{\infty}\left(Bk-(A-B)(p-\alpha)\right)\dfrac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}z^{k+p}}\right|<1.$$

Since $|\mathrm{Re}(z)|\le|z|$ for all z, we have

$$\mathrm{Re}\left[\frac{\sum_{k=1}^{\infty}k\,\dfrac{(a)_k}{(c)_k}C(\gamma,k)a_{k+p}z^{k+p}}{(A-B)(p-\alpha)z^p+\sum_{k=1}^{\infty}\left(Bk-(A-B)(p-\alpha)\right)\dfrac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}z^{k+p}}\right] \tag{3.3}$$

$<1.$

Choose the values of z on the real axis so that $\dfrac{z\left(L_p(a,c)(f*S_\gamma)\right)'(z)}{L_p(a,c)(f*S_\gamma)(z)}$ is real. Upon clearing the denominator of (3.3) and letting $z\to 1$ through real values we get

$$\sum_{k=1}^{\infty}k\,\frac{(a)_k}{(c)_k}C(\gamma,k)a_{k+p}$$

$$\le(A-B)(p-\alpha)+\sum_{k=1}^{\infty}\left[Bk-(A-B)(p-\alpha)\right]\frac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}\ ,$$

which implies the inequality (3.1). Sharpness of the result follows by setting

$$f(z)=z^p-\frac{(A-B)(p-\alpha)}{\left|(1-B)k+(A-B)(p-\alpha)\right|\dfrac{(a)_k}{(c)_k}C(\gamma,k)a_{p+k}}z^{p+k}\quad(k\ge 1). \tag{3.4}$$

4 Distortion Theorem for the class $T_p(a,c,A,B,\gamma,\alpha)$.

Theorem 2. *Let* $\beta_j \in \mathbb{N}, \gamma_j, \delta_j \in \mathbb{R}_+ \left(j = 1,2,3,...,s\right)$ *such that* $\operatorname{Re}(\gamma_j) > -\operatorname{Re}(p\beta_j) - 1$, *and*

$$\prod_{j=1}^{s}\left[\frac{\left(1+\gamma_j+(p+1)\beta_j\right)_{\beta_j}}{\left(1+\gamma_j+\delta_j+(p+1)\beta_j\right)_{\beta_j}}\right] \le 1, \tag{4.1}$$

if the function $f(z)$ *defined by (1.2) is in the class* $T_p(a,c,A,B,\gamma,\alpha)$, *then*

$$\left|I_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z)\right| \ge \prod_{j=1}^{s}\frac{\Gamma\left(1+\gamma_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)}|z|^p$$
$$\times\left\{1-\frac{c(A-B)(p-\alpha)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right]}\prod_{j=1}^{s}\frac{\left(1+\gamma_j+p\beta_j\right)_{\beta_j}}{\left(1+\gamma_j+\delta_j+p\beta_j\right)_{\beta_j}}|z|\right\} \tag{4.2}$$

and

$$\left|I_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z)\right| \le \prod_{j=1}^{s}\frac{\Gamma\left(1+\gamma_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)}|z|^p$$
$$\times\left\{1+\frac{c(A-B)(p-\alpha)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right]}\prod_{j=1}^{s}\frac{\left(1+\gamma_j+p\beta_j\right)_{\beta_j}}{\left(1+\gamma_j+\delta_j+p\beta_j\right)_{\beta_j}}|z|\right\}. \tag{4.3}$$

$(0 \le \alpha < p; 0 \le \gamma < p; a,c \in R \setminus Z_0^-; -1 \le B < A \le 1)$.

For $z \in U$ *and the equality in (4.2) and (4.3) are attained by the function*

$$f(z) = z^p - \frac{c(A-B)(p-\alpha)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right]}z^{p+1}. \tag{4.4}$$

Proof. Making use of (1.2) and Lemma 1, we obtain

$$I_{(\beta^{-1}{}_s);s}^{(\gamma_s);(\delta_s)} f(z) = \prod_{j=1}^{s}\left[\frac{\Gamma\left(1+\gamma_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)}\right]z^p$$
$$-\sum_{k=1}^{\infty}a_{k+p}\prod_{j=1}^{s}\left[\frac{\Gamma\left(1+\gamma_j+\left(p+k\right)\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+\left(p+k\right)\beta_j\right)}\right]z^{p+k}. \tag{4.5}$$

From (2.6) we have

$$J_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z) = \prod_{j=1}^{s}\left[\frac{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+p\beta_j\right)}\right]I_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z)$$

$$=z^{p}-\sum_{k=1}^{\infty}\psi\left(k\right)a_{k+p}z^{k+p}, \tag{4.6}$$

where

$$\psi\left(k\right)=\prod_{j=1}^{s}\left[\frac{\left(1+\gamma_{j}+p\beta_{j}\right)_{\beta_{j}k}}{\left(1+\gamma_{j}+\delta_{j}+\beta_{j}p\right)_{\beta_{j}k}}\right]\qquad\left(k=1,2,\ldots\right). \tag{4.7}$$

Under the assumption of the Theorem we see that $\psi\left(k\right)$ is non-increasing on k, i.e.

$$0<\psi\left(k\right)\le\psi\left(1\right)=\prod_{j=1}^{s}\left[\frac{\left(1+\gamma_{j}+p\beta_{j}\right)_{\beta_{j}}}{\left(1+\gamma_{j}+\delta_{j}+\beta_{j}p\right)_{\beta_{j}}}\right]. \tag{4.8}$$

Now employing (4.8) and Theorem 1 in (4.6), we get

$$\left|J_{\left(\beta_{s}^{-1}\right);s}^{(\gamma_{s});(\delta_{s})}f\left(z\right)\right|\ge|z|^{p}-\psi\left(1\right)|z|^{p+1}\sum_{k=1}^{\infty}a_{k+p}$$

$$\ge|z|^{p}-\prod_{j=1}^{s}\frac{\Gamma\left(1+\gamma_{j}+p\beta_{j}\right)_{\beta_{j}}}{\Gamma\left(1+\gamma_{j}+\delta_{j}+p\beta_{j}\right)_{\beta_{j}}}\frac{\left(A-B\right)\left(p-\alpha\right)}{\left[1-B+(A-B)(p-\alpha)\right]\frac{a}{c}C\left(\gamma,1\right)}|z|^{p+1},$$

Which implies the assertion (4.2) of Theorem 2.
Also, we have

$$\left|J_{\left(\beta_{s}^{-1}\right);s}^{(\gamma_{s});(\delta_{s})}f\left(z\right)\right|$$

$$\le|z|^{p}+\prod_{j=1}^{s}\frac{\Gamma\left(1+\gamma_{j}+p\beta_{j}\right)_{\beta_{j}}}{\Gamma\left(1+\gamma_{j}+\delta_{j}+p\beta_{j}\right)_{\beta_{j}}}\frac{\left(A-B\right)\left(p-\alpha\right)}{\left[1-B+(A-B)(p-\alpha)\right]\frac{a}{c}C\left(\gamma,1\right)}|z|^{p+1}.$$

Which implies the assertion (4.3) of Theorem 2.

Corollary 1. *Let the function $f\left(z\right)$ defined by (1.2) be in the class $T_{p}(a,c,A,B,\gamma,\alpha)$, then*

$$\left|D_{z}^{-\beta}f\left(z\right)\right|\ge\frac{\Gamma\left(1+p\right)}{\Gamma\left(1+\gamma+p\right)}|z|^{p+\beta}$$

$$\left\{1-\frac{c\left(A-B\right)\left(p-\alpha\right)(p+1)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right](p+\beta+1)}|z|\right\}, \tag{4.9}$$

and

$$\left|D_z^{-\beta} f(z)\right| \le \frac{\Gamma(1+p)}{\Gamma(1+\gamma+p)}|z|^{p+\beta}$$
$$\left\{1+\frac{c(A-B)(p-\alpha)(p+1)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right](p+\beta+1)}|z|\right\} \quad (4.10)$$

$(\beta > 0, 0 \le \alpha < p; 0 \le \gamma < p; a,c \in R \setminus Z_0^-; -1 \le B < A \le 1)$

For $z \in u$ and the equality in (4.9) and (4.10) are attained by the function given by (4.4).

Proof. Setting $s=1, \gamma_1 = 0, \delta_1 = \beta$ and $\beta_1 = 1$ in Theorem 3 and using **(**2.3) we obtain the result.

Corollary 2 . *Let the function $f(z)$ defined by (1.2) be in the class $T_p(a,c,A,B,\gamma,\alpha)$, then under the assumption ,*

$$\frac{\beta(\eta+\delta)}{\delta} - p \le 2$$

$$\left|I_{0,z}^{\delta,\beta,\eta} f(z)\right| \ge \frac{\Gamma(1+p)\Gamma(p-\beta+\eta+1)}{\Gamma(1-\beta+p)\Gamma(p+\delta+\eta+1)}|z|^{p-\beta}$$
$$\left\{1-\frac{c(A-B)(p-\alpha)(p+1)(p-\beta+\eta+1)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right](p-\beta+1)p+\delta+\eta+1}|z|\right\}, (4.11)$$

and

$$\left|I_{0,z}^{\delta,\beta,\eta} f(z)\right| \le \frac{\Gamma(1+p)\Gamma(p-\beta+\eta+1)}{\Gamma(1-\beta+p)\Gamma(p+\delta+\eta+1)}|z|^{p-\beta}$$
$$\left\{1+\frac{c(A-B)(p-\alpha)(p+1)(p-\beta+\eta+1)}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right](p-\beta+1)p+\delta+\eta+1}|z|\right\}. (4.12)$$

$(0 \le \alpha < p; 0 \le \gamma < p; a,c \in R \setminus Z_0^-; -1 \le B < A \le 1)$

For $z \in U_0$, where

$$U_0 = \begin{cases} U & \beta \le p \\ U - \{0\} & \beta > p \end{cases}$$

The equality in (4.11) and (4.12) are attained by the function given by (4.4).

Proof. Setting $s=2,\beta_1=\beta_2=1,\gamma_1=0,\gamma_2=\eta-\beta,\delta_1=-\beta,\delta_2=\delta+\beta$ in Theorem 2 and using **(**2.4**)** we obtain the result.

Corollary 3 . *Let the function $f\ (z)$ defined by (1.2) be in the class $T_p\,(a,c,A,B,\gamma,\alpha)$, then*

$$\left|I_{+,\upsilon}^{\eta,1/\sigma,\mu,\xi,\delta}f\ (z)\right|\geq\frac{\Gamma\left[\sigma\left(\upsilon+p+1\right)\right]\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi-\delta\right]}{\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi\right]\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\delta\right]}$$

$$\times|z|^{p-\left\{\left(\frac{1-\eta}{\sigma}\right)-p-\mu-\upsilon-1\right\}}\times\left\{1-\frac{c\,(A-B)(p-\alpha)\left[\sigma\left(\upsilon+p+1\right)\right]_\sigma}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right]}\right. \tag{4.13}$$

$$\left.\frac{\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi-\delta\right]_\sigma}{\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi\right]_\sigma\left[\sigma\left(\upsilon+p+1\right)+\eta-\delta\right]_\sigma}|z|\right\}$$

$$\left|I_{+,\upsilon}^{\eta,1/\sigma,\mu,\xi,\delta}f\ (z)\right|\leq\frac{\Gamma\left[\sigma\left(\upsilon+p+1\right)\right]\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi-\delta\right]}{\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi\right]\Gamma\left[\sigma\left(\upsilon+p+1\right)+\eta-\delta\right]}$$

$$\times|z|^{p-\left\{\left(\frac{1-\eta}{\sigma}\right)-p-\mu-\upsilon-1\right\}}\times\left\{1+\frac{c\,(A-B)(p-\alpha)\left[\sigma\left(\upsilon+p+1\right)\right]_\sigma}{2a(p-\gamma)\left[1-B+(A-B)(p-\alpha)\right]}\right. \tag{4.14}$$

$$\left.\frac{\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi-\delta\right]_\sigma}{\left[\sigma\left(\upsilon+p+1\right)+\eta-\xi\right]_\sigma\left[\sigma\left(\upsilon+p+1\right)+\eta-\delta\right]_\sigma}|z|\right\}$$

where σ is a positive integer and $z\in U_0$, where

$$U_0=\begin{cases}U & \left(\dfrac{1-\eta}{\sigma}\right)-p-\mu-\upsilon-1\leq p\\ U-\{0\} & \left(\dfrac{1-\eta}{\sigma}\right)-p-\mu-\upsilon-1>p\end{cases}$$

The equality in (4.13) and (4.14) are attained by the function given by (4.4).

Proof. Setting
$s=2,\beta_1=\beta_2=\sigma,\gamma_1=\sigma\left(\upsilon+1\right)-1,\gamma_2=\eta-\gamma-\delta+\left(\sigma\left(\upsilon+1\right)-1\right),\delta_1=\eta-\gamma,\delta_2=\gamma$
in Theorem 2 and using **(**2.5**)** we obtain the result.

Corollary 4 . *Under the assumption of Theorem 2, let the function $f\ (z)$ defined by (1.2) be in the class $R_p^*(\gamma,\alpha)$, then*

$$\left|I_{\left(\beta_s^{-1}\right);s}^{(\gamma_s);(\delta_s)} f(z)\right| \geq \prod_{j=1}^{s} \frac{\Gamma\left(1+\gamma_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)} |z|^p$$
$$\times \left\{1-\frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)} \prod_{j=1}^{s} \frac{\left(1+\gamma_j+p\beta_j\right)_{\beta_j}}{\left(1+\gamma_j+\delta_j+p\beta_j\right)_{\beta_j}} |z|\right\} \quad (4.15)$$

and

$$\left|I_{\left(\beta_s^{-1}\right);s}^{(\gamma_s);(\delta_s)} f(z)\right| \leq \prod_{j=1}^{s} \frac{\Gamma\left(1+\gamma_j+p\beta_j\right)}{\Gamma\left(1+\gamma_j+\delta_j+p\beta_j\right)} |z|^p$$
$$\times \left\{1+\frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)} \prod_{j=1}^{s} \frac{\left(1+\gamma_j+p\beta_j\right)_{\beta_j}}{\left(1+\gamma_j+\delta_j+p\beta_j\right)_{\beta_j}} |z|\right\}. \quad (4.16)$$

The equality in (4.15) and (4.16) are attained by the function given by

$$f(z) = z^p - \frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)} z^{p+1}.$$

Proof. Setting $a = p, c = p, A = 1, B = -1$ in Theorem2 we get the result.

Remark 1. Setting $a = p, c = p, A = 1, B = -1, \gamma = \frac{2p-1}{2}$ in Theorem2 we get the corresponding result by Banerji and Shenan[2] (Theorem2.1, p.113).

Remark 2. Setting $a = p+1, c = p, A = 1, B = -1, \gamma = \frac{2p-1}{2}$ and $\lambda = 1$ in Theorem2 we get the corresponding result by Banerji and Shenan[2] (Theorem2.2, p.115). Several other particular cases studied by different authors can be obtained from Theorem 2 by specializing the parameters $s, \beta_s, \delta_s, \gamma_s, \alpha, \gamma, A, B, a$ and c see for example [19] and [2].

5 Convolution Results.

For the functions

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \quad \left(a_{k+p,j} \geq 0, j = 1, 2, ..., n\right), \quad (5.1)$$

the modified Hadmard product (or convolution) is defined by

$$\left(f_1 * f_2 * ... * f_n\right)(z) = z^p - \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} ... a_{k+p,n} z^{k+p} \quad \left(a_{k+p,j} \geq 0, j = 1, 2, ..., n\right).$$

For positive real numbers $r_1, r_2, ..., r_n$ the generalized hadmard product is defined by

$$\left(f_1 \Delta f_2 \Delta ... \Delta f_n\right)(r_1, r_2, ..., r_n; z) = z^p - \sum_{k=1}^{\infty} \left(a_{k+p,1}\right)^{r_1} \left(a_{k+p,2}\right)^{r_2} ... \left(a_{k+p,n}\right)^{r_n} z^{k+p} \quad ,$$

or

$$\Delta_j f_j (r_j; z) = z^p - \sum_{k=1}^{\infty} \prod_{j=1}^{n} \left(a_{k+p,j}\right)^{r_j} z^{k+p}. \tag{5.2}$$

If we take $r_1 = r_2 = ... = r_n = 1$, then we have

$$\left(f_1 \Delta f_2 \Delta ... \Delta f_n\right)(1,1,...,1; z) = \left(f_1 * f_2 * ... * f_n\right)(z).$$

It is also noted that if $f_j = f$, $j = 1, 2, ..., n$ and $\sum_{j=1}^{n} \frac{1}{r_j} = 1$, then

$$\Delta_j f \left(\frac{1}{r_j}; z\right) \equiv f(z)$$

Theorem 3. *Let $f_j(z)$, $j = 1, 2, ..., n$ be defined by (4.1) belong to the class $T_p(a, c, A, B, \gamma, \alpha_j)$ for each j . Then*

$$\Delta_j f_j \left(\frac{1}{r_j}; z\right) \in T_p(a, c, A, B, \gamma, \sigma),$$

where

$$\sigma = \min_{k \ge 1} \left\{ p - \frac{1}{A-B} \left[\frac{k(1+\beta)(a)_k C(\gamma, k)}{(c)_k \prod_{j=1}^{n} \left\{ \frac{\left[(1-B)k + (A-B)(p-\alpha_i)\right](a)_k C(\gamma, k)}{(A-B)(p-\alpha_i)(c)_k} \right\}^{\frac{1}{r_j}} - (a)_k C(\gamma, k)} \right] \right\}. \tag{5.3}$$

Proof. since $f_j(z) \in T_p(a, c, A, B, \gamma, \alpha_j), j = 1, 2, ..., n$, then by Theorem2 we have

$$\sum_{k=1}^{\infty} \frac{\left[(1-B)k + (A-B)(p-\alpha_i)\right](a)_k C(\gamma, k)}{(A-B)(p-\alpha_i)(c)_k} a_{k+p,j} \le 1 \qquad \text{(j=1,2,...,n)}, \tag{5.4}$$

hence

$$\left\{\sum_{k=1}^{\infty}\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right](a)_k C(\gamma,k)}{(A-B)(p-\alpha_i)(c)_k}a_{k+p,j}\right\}^{\frac{1}{r_j}}\leq 1. \tag{5.5}$$

Thus, using Holder's inequality, we get

$$\sum_{k=1}^{\infty}\prod_{j}\left\{\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right](a)_k C(\gamma,k)}{(A-B)(p-\alpha_i)(c)_k}\right\}^{\frac{1}{r_j}}\prod_{j}\left(a_{k+p,j}\right)^{\frac{1}{r_j}}\leq 1, \tag{5.6}$$

since

$$\Delta_j f_j\left(\frac{1}{r_j};z\right)=z^p-\sum_{k=1}^{\infty}\prod_{j=1}^{n}\left(a_{k+p,j}\right)^{\frac{1}{r_j}}z^{k+p},$$

we need to show

$$\sum_{k=1}^{\infty}\frac{\left[(1-B)k+(A-B)(p-\sigma)\right](a)_k C(\gamma,k)}{(A-B)(p-\sigma)(c)_k}\prod_{j=1}^{n}\left(a_{k+p,j}\right)^{\frac{1}{r_j}}\leq 1. \tag{5.7}$$

Now in view of (5.6) the inequality (5.7) holds if

$$\frac{\left[(1-B)k+(A-B)(p-\sigma)\right](a)_k C(\gamma,k)}{(A-B)(p-\sigma)(c)_k}$$

$$\leq\prod_{j=1}^{n}\left\{\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right](a)_k C(\gamma,k)}{(A-B)(p-\alpha_i)(c)_k}\right\}^{\frac{1}{r_j}}$$

which simplifies to (5.3) , hence we get the result.

Theorem 4. *Let $f_j(z)$, $j=1,2,...,n$ be defined by (4.1) belong to the class $T_p(a,c,A,B,\gamma,\alpha_j)$ for each j . Then the function $h(z)$ defined by*

$$h(z)=z^p-\frac{1}{\sum_{j=1}^{n}r_j}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}r_j\left|a_{p+k,j}\right|\right)z^{p+k}, \tag{5.8}$$

with $r_j>0$ for $j=1,2,...,n$, belongs to the class $T_p(a,c,A,B,\gamma,\psi)$, where

$$\psi = \min_{k\geq 1}\left\{ p - \frac{k(1+B)}{\sum_{j=1}^{n}\left\{\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right]}{(p-\alpha_i)}\right\}-(A-B)}\right\}. \tag{5.9}$$

Proof. since $f_j(z) \in T_p(a,c,A,B,\gamma,\alpha), j=1,2,...,n$, then from (5.4) we have

$$\sum_{k=1}^{\infty}\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right](a)_k C(\gamma,k)}{(A-B)(p-\alpha_i)(c)_k} a_{k+p,j} r_j \leq r_j$$

or

$$\frac{1}{\sum_{j=1}^{n} r_j}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right](a)_k C(\gamma,k)}{(A-B)(p-\alpha_i)(c)_k} a_{k+p,j} r_j\right)\leq 1. \tag{5.10}$$

To show that $h(z)$ belongs $T_p(a,c,A,B,\gamma,\psi)$, we need to show

$$\frac{1}{\sum_{j=1}^{n} r_j}\sum_{k=1}^{\infty}\left(\frac{\left[(1-B)k+(A-B)(p-\psi)\right](a)_k C(\gamma,k)\sum_{j=1}^{n} r_j a_{k+p,j}}{(A-B)(p-\psi)(c)_k}\right)\leq 1. \tag{5.11}$$

Now in view of (5.10) the inequality (5.11) holds if

$$\sum_{j=1}^{n}\frac{\left[(1-B)k+(A-B)(p-\alpha_i)\right]}{(p-\alpha_i)} \leq \frac{\left[(1-B)k+(A-B)(p-\psi)\right]}{(p-\psi)}$$

which simplifies to (5.9) , hence we get the result.

6 Integral transform of the class $T_p(a,c,A,B,\gamma,\alpha)$.

The generalized Komatu integral operator $L^{\delta}_{c,p} : T_p \to T_p$ is defined for $\delta > 0$ and $c > -p$ as (see, e.g. [8,15])

$$H(z) = L^{\delta}_{c,p} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^{c}}\int_{0}^{z} t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1} f(t)\,dt. \tag{6.1}$$

Notice that for $c = 1$ we get the integral operator introduced by Jung, Kim and Srivastava[5].

Theorem 5. *Let* $f(z) \in T_p(a,c,A,B,\gamma,\alpha)$. *Then* $\mathrm{L}^{\delta}_{\mathrm{c,p}} f(z) \in T_p(a,c,A,B,\gamma,\alpha)$

Proof . By using the definition of $L^{\delta}_{c,p} f(z)$ we have

$$L^{\delta}_{c,p} f(z) = \frac{(c+1)^{\delta}}{\Gamma(\delta)} \int_0^1 \left(\log \tfrac{1}{t}\right)^{\delta-1} t^{c+p-1} \left(z^p - \sum_{k=1}^{\infty} a_{k+p} t^k z^{k+p} \right) dt \tag{6.2}$$

Simplifying by using the definition of gamma function we get

$$L^{\delta}_{c,p} f(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k} \right)^{\delta} a_{k+p} z^{k+p}. \tag{6.3}$$

Now $L^{\delta}_{c,p} f(z) \in T_p(a,c,A,B,\gamma,\alpha)$ if

$$\sum_{k=1}^{\infty} \frac{\left[(1-B)k + (A-B)(p-\alpha)\right] \frac{(a)_k}{(c)_k} C(\gamma,k)}{(A-B)(p-\alpha)} \left(\frac{c+p}{c+p+k} \right)^{\delta} a_k \le 1. \tag{6.4}$$

From Theorem 1 we have $f(z) \in T_p(a,c,A,B,\gamma,\alpha)$ if and only if

$$\sum_{k=1}^{\infty} \frac{\left[(1-B)k + (A-B)(p-\alpha)\right] \frac{(a)_k}{(c)_k} C(\gamma,k)}{(A-B)(p-\alpha)} a_{k+p} \le 1. \tag{6.5}$$

Thus in view of (5.5) and the fact that $\left(\frac{c+p}{c+p+k} \right) < 1 \ for \ K \ge 1$, (6.4) holds true , and hence $L^{\delta}_{c,p} f(z) \in T_p(a,c,A,B,\gamma,\alpha)$.

7 Open Problem

One can define the same class in this paper by using another operator for example Cho, Kwon , Srivastava operator which generalized the operator defined in this class and hence new results can be obtained.

References.

[1] M. K. Aouf and H. Silverman, subclasses of p-valent and prestarlike functions, *Int. J. Contemp. Math. Sciences*, 2(2007), No. 8, 357 - 372.

[2] P. K. Baneji and G. M. Shenan, On investigation of certain class of starlike and convex functions in terms of generalized fractional integral operators, *J. Indian Acd. Math.* 23 (2001), No. 1, 109-120.

[3] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal*., **15** (1984), 737–745.

[4] J. H. Choi and M. Saigo, Starlike and Convex function of complex order Involving a certain fractional integral operator. *Fukuoka Univ. Sci. Rep.* 2(1998), No. 28, 29-40.

[5] R. M. Goel and N.S. Sohi A new criterion for p-valent functions, *Proc. Amer. Math. Soc.*, **78**(1980), 353–357.

[6] S. L. Kalla and V. S. Kiryakova, On H-function generalized fractional calculus based upon compositions of Erdelyi-kober operators in Lp. *Math. Japonica* (1990), 1151-1177.

[7] V. S. Kiryakova, Fractional integration operators involving Fox's $H_{m,m}^{m,0}$ function, C. R. Belug. *Acad. Sci.*, 41(11) (1988), 11-14.

[8] S. M. Khairnar and M. More, On a subclass of multivalent $\beta-$uniformly starlike and convex functions defined by a linear operator. *IAENG Int. J. Appl. Math.* 39:3, IJAM 3-39-06 (2009).

[9] G. A. Kumar and Gl. Reddy, Certain class of prestarlike functions, *J. Math. Res.Exposition* 12(1992), no.3, 402 -412.

[10] S. Owa and B.A. Uralegaddi, A class of functions $\alpha-$prestarlike of order β, Bull. Korean Math. Soc. 21(1984), 77 - 85.

[11] D. A. Patil and N. K. Thakare, On convex hulls and extreme points of pvalent starlike and convex classes with applications, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* (N.S.), 27 (1983), no. 75, 145 - 160.

[12] M. Saigo, Aremark on integral operators involving the Gauss hyper geometric Functions, *Math. Rep. College General Ed. Kyushu Univ.* (1978), no.11, 135- 143.

[13] M. Saigo, R. K. Raina, and A. A. Kilbas :On generalized fractional operators and their composition with axiymmetric differential operators of the potential theory on spaces $F_{p,\mu}$ and $F'_{p,\mu}$ *Fukuoka Univ. Sci. Rep.* 23(1993), 133-154.

[14] H. Saitoh, A Linear operator and applications of the first order differential subordinations, *Math. Japon.* 44(1996), 31-38.

[15] T. O. Salim, A Class of Multivalent Functions Involving a Generalized Linear Operator and Subordination, *Int. J. Open Problems Complex Analysis*, 2(2010), No. 2, 82-94.

[16] S. G. Samko, Kilbas, A. A. and O.I. Marichev :On Fractional Integrals and Derivatives,theory and applications, (1993). Gorden and Breach, New York/Philadelphia/London/Paris/Montrev/Toronto/Melbourne.

[17] T. Sheil - Small, H. Silverman and E. M. Silvia, Convolution multipliersand starlike functions, *J. Analyse Math.* 41 (1982), 181 - 192.

[18] G. M. Shenan, T. O. Salim and M. S. Marouf, A certain class of multivalent prestarlike functions involving the Srivastava -Saigo – Owa fractional integral operator, *Kyungpook Math. J.* 44(2004), 353 - 362.

[19] H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. *Math. Anal. Appl.*,131(1988), 412-420.