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Dual Spacelike Biharmonic Curves with Timelike Principal Normal According to Dual Bishop Frames in the Dual Lorentzian Space $\mathbb{D}_1^3$

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Abstract

In this paper, we study dual spacelike biharmonic curves with timelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with timelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

Keywords: *Biharmonic curve, curvature, dual space curve, dual Bishop frame.*

1 Introduction

Dual numbers were introduced by W. K. Clifford [1] as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the points of dual unit sphere $\mathbb{S}^2$ and the directed lines in $\mathbb{R}^3$.

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

Harmonic maps $f:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\text{trace}\nabla df. \tag{1.2}$$

The bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [2], showing that the Euler-Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)-\text{trace}R^N(df,\tau(f))df \\ &=0,\end{aligned} \tag{1.4}$$

where J^f is the Jacobi operator of f . The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study dual spacelike biharmonic curves with timelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with timelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

2 Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

W.K. Clifford, in [1], introduced dual numbers with the set

$$\mathbb{D}=\{\hat{x}=x+\varepsilon x^*: x,x^*\in\mathbb{R}\}.$$

The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$ for $\varepsilon \neq 0$. Thereafter, A good amount of reserach work has been done on dual numbers, dual functions and as well as dual curves [7]. Then, dual angle is introduced, which is defined as $\hat{\theta} = \theta + \varepsilon\theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. The set $\mathbb{D}$ of dual numbers is a commutative ring with the operations (+) and (.). The set $\mathbb{D}^3 = \mathbb{D}\times\mathbb{D}\times\mathbb{D} = \{\hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^* \in \mathbb{E}^3\}$ is a module over the ring $\mathbb{D}$, [8].

Let us denote

$$\hat{a} = a + \varepsilon a^* = (a_1, a_2, a_3) + \varepsilon(a_1^*, a_2^*, a_3^*)$$

and

$$\hat{b} = b + \varepsilon b^* = (b_1, b_2, b_3) + \varepsilon(b_1^*, b_2^*, b_3^*).$$

The Lorentzian inner product of $\hat{a}$ and $\hat{b}$ defined by

$$\left\langle \hat{a}, \hat{b} \right\rangle = \left\langle a, b \right\rangle + \varepsilon(\left\langle a, b^* \right\rangle + \left\langle a^*, b \right\rangle).$$

We call the dual space $\mathbb{D}^3$ together with Lorentzian inner product as dual Lorentzian space and show by $\mathbb{D}_1^3$. We call theelements of $\mathbb{D}_1^3$ the dual vectors. For $\hat{\varphi} \neq 0$, the norm $\|\hat{\varphi}\|$ of is defined by $\|\hat{\varphi}\| = \sqrt{\left|\left\langle \hat{\varphi}, \hat{\varphi} \right\rangle\right|}$. A dual vector $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is called dual space-like vector if $\left\langle \hat{\varphi}, \hat{\varphi} \right\rangle > 0$ or $\varepsilon = 0$, dual time-like vector if $\left\langle \hat{\varphi}, \hat{\varphi} \right\rangle < 0$, and dual null (light-like) vector if $\left\langle \hat{\varphi}, \hat{\varphi} \right\rangle = 0$ for $\hat{\varphi} \neq 0$. Therefore, an arbitrary dual curve, which is a differentiable mapping onto $\mathbb{D}_1^3$, can locally be dual space-like, dual time-like or dual null, if its velocity vectoris respectively, dual space-like, dual time-like or dual null.

Besides, for the dual vectors $\hat{a}, \hat{b} \in \mathbb{D}_1^3$, Lorentzian vector product of dual vectors is defined by

$$\hat{a} \times \hat{b} = a \times b + \xi(a^* \times b + a \times b^*),$$

where $a \times b$ is the classical Lorentzian cross product .

3 Dual Biharmonic Curves in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon\gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with timelike principal normal by the arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{t}$ is defined, and the principal normal is $\hat{n} = \frac{1}{\hat{\kappa}}\nabla_{\hat{t}}\hat{t}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\| \nabla_{\hat{t}} \hat{t} \right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{b} = \hat{t} \times \hat{n}$. Hence, the triple $\{\hat{t}, \hat{n}, \hat{b}\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned} \nabla_{\hat{t}} \hat{t} &= \hat{\kappa} \hat{n}, \\ \nabla_{\hat{t}} \hat{n} &= \hat{\kappa} \hat{t} + \hat{\tau} \hat{b}, \\ \nabla_{\hat{t}} \hat{b} &= \hat{\tau} \hat{n}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon \tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. In addition,

$$\begin{aligned} &g(\hat{t}, \hat{t}) = 1, g(\hat{n}, \hat{n}) = -1, g(\hat{b}, \hat{b}) = 1, \\ &g(\hat{t}, \hat{n}) = g(\hat{t}, \hat{b}) = g(\hat{n}, \hat{b}) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\hat{t}} \hat{t} &= \hat{k}_1 \hat{m}_1 - \hat{k}_2 \hat{m}_2, \\ \nabla_{\hat{t}} \hat{m}_1 &= \hat{k}_1 \hat{t}, \\ \nabla_{\hat{t}} \hat{m}_2 &= \hat{k}_2 \hat{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g(\hat{t}, \hat{t}) = 1, g(\hat{m}_1, \hat{m}_1) = -1, g(\hat{m}_2, \hat{m}_2) = 1, \\ &g(\hat{t}, \hat{m}_1) = g(\hat{t}, \hat{m}_2) = g(\hat{m}_1, \hat{m}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\hat{t}, \hat{m}_1, \hat{m}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures. where $\hat{\theta}(s) = \arctan \dfrac{\hat{k}_2}{\hat{k}_1}$, $\tau(s) = \hat{\theta}'(s)$ and $\hat{\kappa}(s) = \sqrt{\left| \hat{k}_2^2 - \hat{k}_1^2 \right|}$.

Thus, Bishop curvatures are defined by

$$\begin{aligned} \hat{k}_1 &= \hat{\kappa}(s) \sinh \hat{\theta}(s), \\ \hat{k}_2 &= \hat{\kappa}(s) \cosh \hat{\theta}(s). \end{aligned} \tag{3.5}$$

Theorem 3.1. *Let $\hat{\gamma} : I \to \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with timelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} \hat{k}_1^2 - \hat{k}_2^2 &= \hat{\Upsilon}, \\ \hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 &= 0, \\ -\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 &= 0, \end{aligned} \tag{3.6}$$

where $\hat{\Upsilon}$ is dual constant of integration.

Proof. From (1.4), we get the biharmonic equation of $\hat{\gamma}$

$$\tau_2(\hat{\gamma}) = \nabla_{\hat{t}}^3\hat{t} - R(\hat{t}, \nabla_{\hat{t}}\hat{t})\hat{t} = 0. \tag{3.7}$$

Next, using the Bishop equations (3.3) we obtain

$$\nabla_{\hat{t}}^3\hat{t} = \left(3\hat{k}_1'\hat{k}_1 - 3\hat{k}_2'\hat{k}_2\right)\hat{t} + \left(\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1\right)\hat{m}_1 + \left(-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2\right)\hat{m}_2. \tag{3.8}$$

Thus, (3.7) and (3.8) imply

$$\begin{aligned} &\left(3\hat{k}_1'\hat{k}_1 - 3\hat{k}_2'\hat{k}_2\right)\hat{t} + \left(\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1\right)\hat{m}_1 + \left(-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2\right)\hat{m}_2 \\ &- R(\hat{t}, \nabla_{\hat{t}}\hat{t})\hat{m}_2 = 0. \end{aligned} \tag{3.9}$$

In $\mathbb{D}^3$, the Riemannian curvature is zero, we have

$$\left(3\hat{k}_1'\hat{k}_1 - 3\hat{k}_2'\hat{k}_2\right)\hat{t} + \left(\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1\right)\hat{m}_1 + \left(-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2\right)\hat{m}_2 = 0. \tag{3.10}$$

From Bishop frame, we have

$$3\hat{k}_1'\hat{k}_1 - 3\hat{k}_2'\hat{k}_2 = 0. \tag{3.11}$$

Also, from (3.11) we get

$$\hat{k}_1^2 - \hat{k}_2^2 = \hat{\Upsilon}, \tag{3.12}$$

where $\hat{\Upsilon}$ is dual constant of integration.
Using (3.10), we get

$$\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 = 0, \tag{3.13}$$

$$-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 = 0. \tag{3.14}$$

The proof is completed.

Lemma 3.2. *Let $\hat{\gamma} : I \to \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with timelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} \hat{k}_1^2 - \hat{k}_2^2 &= \hat{\Upsilon}, \\ \hat{k}_1'' + \hat{k}_1\hat{\Upsilon} &= 0, \\ \hat{k}_2'' - \hat{k}_2\hat{\Upsilon} &= 0, \end{aligned} \tag{3.15}$$

where $\hat{\Upsilon} = \Upsilon + \varepsilon\Upsilon^*$ *is constant of integration.*

Lemma 3.3. *Let* $\hat{\gamma} : I \to \mathbb{D}_1^3$ *be a non-geodesic spacelike dual curve with timelike principal normal parametrized by arc length.* $\hat{\gamma}$ *is a non-geodesic spacelike dual biharmonic curve if and only if*

$$k_1^2 - k_2^2 = \Upsilon, \tag{3.16}$$

$$k_1 k_1^* - k_2 k_2^* = \Upsilon^*. \tag{3.17}$$

Proof. Using (3.12), we have (3.16) and (3.17).

Corollary 3.4. *Let* $\hat{\gamma} : I \to \mathbb{D}_1^3$ *be a non-geodesic spacelike dual curve with timelike principal normal parametrized by arc length. If* $\hat{\Upsilon} = 0,$ *then*

$$\begin{aligned} &k_1^2 = k_2^2, \\ &k_1 k_1^* = k_2 k_2^*, \\ &\hat{k}_1 = \hat{C}_1 \hat{s} + \hat{C}_2, \\ &\hat{k}_2 = \hat{C}_3 \hat{s} + \hat{C}_4, \end{aligned} \tag{3.18}$$

where $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4$ *are dual constants of integration.*

Proof. Suppose that $\hat{\Upsilon} = 0$. Substituting $\Upsilon = \Upsilon^* = 0$ in Lemma 3.3, we have

$$\begin{aligned} &k_1^2 = k_2^2, \\ &k_1 k_1^* = k_2 k_2^*. \end{aligned}$$

On the other hand, using second and third equation of (3.2), we get

$$\hat{k}_1'' = 0, \hat{k}_2'' = 0.$$

If we integrate above equation, we have

$$\begin{aligned} &\hat{k}_1 = \hat{C}_1 \hat{s} + \hat{C}_2, \\ &\hat{k}_2 = \hat{C}_3 \hat{s} + \hat{C}_4, \end{aligned}$$

where $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4$ are dual constants of integration. The proof is completed.

5 Open Problem

This work, we study dual spacelike biharmonic curves with timelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with timelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

Moreover the resarcher can characterize curvature and torsion of dual spacelike biharmonic curves with spacelike principal normal.

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