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# On Certain Class of Harmonic Univalent Functions Defined By Generalized Derivative Operator

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**Abstract**

***A complex-valued functions that are univalent and sense preserving in the unit disk U can be written in the form*** $f(z)=h(z)+\overline{g(z)}$***, where*** $h(z)$ ***and*** $g(z)$ ***are analytic in*** $U$ ***. In [7], authors introduced the operator*** $D_{m,\lambda}^{n}$ ***which defined by convolution involving the polylogarithms functions. Using this operator, we introduce the class*** $SHP_{\lambda}(\alpha,\beta,n,m,k)$ ***by generalized derivative operator of harmonic univalent functions. We give sufficient coefficient conditions for normalized harmonic functions in the class*** $SHP_{\lambda}(\alpha,\beta,n,m,k)$***. These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.***

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## 1. Introduction

Let $U$ denote the open unit disk and $S_H$ denote the class of all complex valued harmonic, sense preserving univalent functions $f(z)$ in $U$ normalized by $f(0)=0$, $f_z(0)=1$. Each $f(z)\in S_H$ can be expressed as

$$f(z)=h(z)+\overline{g(z)} \tag{1.1}$$

where

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k \ , \quad g(z)=\sum_{k=1}^{\infty}b_k z^k \ , |b_1|<1$$

are analytic in $U$. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense preserving in $U$ is that $|h'(z)|>|g'(z)|$ in $U$. Clunie and Sheil- Small studied $S_H$ together with some geometric subclasses of $S_H$.

A function of the form (1.1) is harmonic starlike [8] for $|z|=r<1$, if

$$\frac{\partial}{\partial\theta}\left(\arg\left(f(re^{i\theta})\right)\right)=\operatorname{Re}\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\}>0,$$

Silverman [9], proved that the coefficient conditions

$$\sum_{k=2}^{\infty}k\left(|a_k|+|b_k|\right)\le 1 \quad and \quad \sum_{k=2}^{\infty}k^2\left(|a_k|+|b_k|\right)\le 1$$

are necessary and sufficient conditions for functions $f(z)=h(z)+\overline{g(z)}$ to be harmonic starlike with negative coefficient and harmonic convex with negative coefficient respectively. Different authors in [1, 2, 4, 5, 6,7,12], studied $S_H$ together with some geometric subclasses of $S_H$.

For $f(z)=h(z)+\overline{g(z)}$ given by (1.1), we define the derivative operator introduced by Shaqsi and Darus [7] of $f(z)$ as

$$D^n_{m,\lambda}f(z)=D^n_{m,\lambda}h(z)+(-1)^n\overline{D^n_{m,\lambda}g(z)} \tag{1.2}$$

where

$$D_{m,\lambda}^{n}h(z)=z+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}C(m,k)a_k z^k,$$

$$D_{m,\lambda}^{n}g(z)=\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}C(m,k)b_k z^k,\qquad |b_1|<1,\qquad C(m,k)=\binom{k+m-1}{m}.$$

Let $SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ denote the family of harmonic functions $f(z)$ of the form (1.1) such that

$$\operatorname{Re}\left\{(1-\alpha)\frac{D_{m,\lambda}^{n}h(z)+(-1)^n\overline{D_{m,\lambda}^{n}g(z)}}{z}+\alpha\left[D_{m,\lambda}^{n}h(z)+(-1)^n\overline{D_{m,\lambda}^{n}g(z)}\right]'\right\}\geq\beta \qquad (1.3)$$

where $D_{m,\lambda}^{n}h(z),\ D_{m,\lambda}^{n}g(z)$ is defined by (1.2).

If the co-analytic part of $f(z)=h(z)+\overline{g(z)}$ is identically zero, $n$=0 and $m$=0 then the family $SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ turns out to be the class $F_\lambda\left(\alpha\right)$ introduced by Bhoosnurmath and Swamy [2] for the analytic case.

We further denote by $THP_\lambda\left(\alpha,\beta,n,m,k\right)$ the subclass of $SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ such that the functions $h(z)$ and $g(z)$ in $f(z)=h(z)+\overline{g(z)}$ are of the form

$$h(z)=z-\sum_{k=2}^{\infty}a_k z^k\ ,\quad g(z)=(-1)^n\sum_{k=1}^{\infty}b_k z^k\ ,\ |b_1|<1 \qquad (1.4)$$

It is clear that the class $THP_\lambda\left(\alpha,\beta,n,m,k\right)$ includes a variety of well-known subclasses of $S_H$.

In this paper, we will give the sufficient condition for functions $f(z)=h(z)+\overline{g(z)}$ where $h(z)$ and $g(z)$ given by (1.1) to be in the class $SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ and it is shown that the coefficient condition is also necessary for the functions in the class $THP_\lambda\left(\alpha,\beta,n,m,k\right)$. Coefficient bounds, Distortion bounds, extreme points, convolution conditions, convex combination of this class are obtained.

## 2. Main Results

We begin by proving some sharp coefficient inequality contained in the following theorem

**Theorem 1.** Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.1) furthermore, let

$$\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right) \le 2-\beta, \tag{2.1}$$

where $a_1 = 1,\ \lambda,\alpha \ge 0, \lambda+\alpha \ge 1,\ 0 \le \beta < 1.$

Then $f(z)$ is harmonic univalent, sense preserving in $U$ and $f(z) \in SHP_{\lambda}\left(\alpha,\beta,n,m,k\right).$

**Proof:** For $|z_1| \le |z_2| < 1$, we have by equation (2.1)

$$\begin{aligned}
|f(z_1) - f(z_2)| &\ge |h(z_1)-h(z_2)| - |g(z_1)-g(z_2)| \\
&\ge |z_1 - z_2|\left(1-\sum_{k=2}^{\infty}k|a_k||z_2|^{k-1} - \sum_{k=1}^{\infty}k|b_k||z_2|^{k-1}\right) \\
&= |z_1 - z_2|\left(1-\sum_{k=2}^{\infty}k\left(|a_k|+|b_k|\right)|z_2|^{k-1} + |b_1|\right) \\
&\ge |z_1 - z_2|\left(1-\sum_{k=2}^{\infty}k\left(|a_k|+|b_k|\right) + |b_1|\right) \qquad \textit{by } |z_2|<1 \\
&\ge |z_1 - z_2|\left(1-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right)+|b_1|\right) \\
&\ge |z_1 - z_2|\left(1-\left[1-\beta-|b_1|+|b_1|\right]\right) = \beta\left(z_1 - z_2\right) \ge 0.
\end{aligned}$$

Hence, $f(z)$ is univalent in $U$. $f(z)$ is sense preserving in $U$. This is because

$$\begin{aligned}
|h'(z)| &\ge 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1} \\
&> 1-\sum_{k=2}^{\infty}k|a_k|
\end{aligned}$$

$$> 1-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|a_k|$$
$$\geq \beta+\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|b_k|$$
$$\geq \sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|b_k||z|^{k-1}$$
$$> \sum_{k=1}^{\infty}k|b_k||z|^{k-1}\geq |g'(z)|.$$

Now, we show that $f(z)\in SHP_{\lambda}(\alpha,\beta,n,m,k)$. Using the fact that $\operatorname{Re} w\geq\beta$ if and only if $|1-\beta+w|\geq|1+\beta-w|$, it suffices to show that

$$\left|1-\beta+(1-\alpha)\frac{D^{n}_{m,\lambda}h(z)+(-1)^{n}\overline{D^{n}_{m,\lambda}g(z)}}{z}+\alpha\left[D^{n}_{m,\lambda}h(z)+(-1)^{n}\overline{D^{n}_{m,\lambda}g(z)}\right]'\right|$$
$$-\left|1+\beta-(1-\alpha)\frac{D^{n}_{m,\lambda}h(z)+(-1)^{n}\overline{D^{n}_{m,\lambda}g(z)}}{z}-\alpha\left[D^{n}_{m,\lambda}h(z)+(-1)^{n}\overline{D^{n}_{m,\lambda}g(z)}\right]'\right|\geq 0 \qquad (2.2)$$
$$=\left|2-\beta+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)a_kz^{k-1}+(-1)^{n}\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)b_kz^{k-1}\right|$$
$$-\left|\beta-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)a_kz^{k-1}-(-1)^{n}\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)b_kz^{k-1}\right|$$
$$\geq 2\left[(1-\beta)-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|a_k||z|^{k-1}\right.$$
$$\left.-\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|b_k||z|^{k-1}\right]$$
$$>2\left[(1-\beta)-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|a_k|\right.$$
$$\left.+\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)|b_k|\right]>0.$$

This last expression is non-negative by (2.1) and so the proof is complete.
The harmonic mappings

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\beta}{\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)}x_kz^k+\overline{\sum_{k=1}^{\infty}\frac{1-\beta}{\left[1+(k-1)\lambda\right]^{n}(1-\alpha+k\alpha)C(m,k)}y_kz^k} \qquad (2.3)$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$ show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ because $\sum_{k=1}^{\infty}\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)\left(|a_k|+|b_k|\right)\Big]=1+\left(1-\beta\right)\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\right)=2-\beta.$

**Theorem 2.** Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.4), then $f(z)\in THP_\lambda\left(\alpha,\beta,n,m,k\right)$. If and only if

$$\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)\left(|a_k|+|b_k|\right)\le 2-\beta, \tag{2.4}$$

where $a_1=1,\ \lambda,\alpha\ge 0,\lambda+\alpha\ge 1,\ 0\le\beta<1$

**Proof:** The 'if part' follows from Theorem 1 upon noting that the functions $h(z)$ and $g(z)$ in $f(z)\in SHP_\lambda\left(\alpha,\beta,n,m,k\right)$ are of the form (1.4), then $f(z)\in THP_\lambda\left(\alpha,\beta,n,m,k\right)$.

For the 'only if' part, we show that if $f(z)\in THP_\lambda\left(\alpha,\beta,n,m,k\right)$ then the condition (2.4) holds. Note that a necessary and sufficient condition for $f(z)=h(z)+\overline{g(z)}$ given by (1.4) be in $THP_\lambda\left(\alpha,\beta,n,m,k\right)$ is that

$$\operatorname{Re}\left\{\left(1-\alpha\right)\frac{D^n_{m,\lambda}h(z)+(-1)^n\overline{D^n_{m,\lambda}g(z)}}{z}+\alpha\left[D^n_{m,\lambda}h(z)+(-1)^n\overline{D^n_{m,\lambda}g(z)}\right]'\right\}>\beta$$

or, equivalently

$$\operatorname{Re}\left\{1-\sum_{k=2}^{\infty}\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)|a_k|z^{k-1}-\sum_{k=1}^{\infty}\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)|b_k|z^{k-1}\right\}>\beta.$$

If we choose z to be real and $z\to 1^-$, we get

$$1-\sum_{k=2}^{\infty}\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)|a_k|-\sum_{k=1}^{\infty}\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha+k\alpha\right)C(m,k)|b_k|\ge\beta$$

this is precisely the assertion of (2.4).

**Theorem 3.** If $f(z) \in THP_{\lambda}\left(\alpha,\beta,n,m,k\right)$, $\lambda,\alpha \geq 0, \lambda+\alpha \geq 1,\ 0 \leq \beta < 1,\ |z| = r < 1$, then

$$|f(z)| \leq \left(1+|b_1|\right)r + \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(1-|b_1|-\beta\right)r^2$$

$$|f(z)| \geq \left(1-|b_1|\right)r - \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(1-|b_1|-\beta\right)r^2 \qquad (2.5)$$

**Proof:** Let $f(z) \in THP_{\lambda}\left(\alpha,\beta,n,m,k\right)$. Taking the absolute value of $f(z)$ we obtain

$$|f(z)| \leq \left(1+|b_1|\right)r + \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right)r^k ,$$

$$\leq \left(1+|b_1|\right)r + \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right)r^2 , \qquad |z| = r < 1$$

$$\leq \left(1+|b_1|\right)r + \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(\sum_{k=2}^{\infty}(1+\lambda)^n\,(1+\alpha)(m+1)\left(|a_k|+|b_k|\right)\right)r^2$$

$$\leq \left(1+|b_1|\right)r + \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right)\right)r^2$$

$$\leq \left(1+|b_1|\right)r + \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(1-|b_1|-\beta\right)r^2$$

and

$$|f(z)| \geq \left(1-|b_1|\right)r - \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right)r^k ,$$

$$\geq \left(1-|b_1|\right)r - \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right)r^2 , \qquad |z| = r < 1$$

$$\geq \left(1-|b_1|\right)r - \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(\sum_{k=2}^{\infty}(1+\lambda)^n\,(1+\alpha)(m+1)\left(|a_k|+|b_k|\right)\right)r^2$$

$$\geq \left(1-|b_1|\right)r - \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right)\right)r^2$$

$$\geq \left(1-|b_1|\right)r - \frac{1}{(1+\lambda)^n\,(1+\alpha)(m+1)}\left(1-|b_1|-\beta\right)r^2.$$

The bounds given in Theorem 3. For the functions $f(z)=h(z)+\overline{g(z)}$ of the form (1.4) also hold for functions of the form (1.1) if the coefficient condition (2.1) satisfied the functions

$$f(z)=z+|b_1|\bar{z}-\frac{1}{(1+\lambda)^n(1+\alpha)(m+1)}(1-|b_1|-\beta)\bar{z}^2$$

*and*

$$f(z)=z-|b_1|z-\frac{1}{(1+\lambda)^n(1+\alpha)(m+1)}(1-|b_1|-\beta)z^2.$$

For $|b_1|\le 1-\beta$ shows that the bounds given in Theorem 3 are sharp.

The following result follows from the left hand inequality in Theorem 3.

**Corollary 1.** If $f(z)\in THP_\lambda(\alpha,\beta,n,m,k)$. Then

$$\left\{w:|w|<\frac{(1+\lambda)^n(1+\alpha)(m+1)+\beta-1}{(1+\lambda)^n(1+\alpha)(m+1)}+\frac{1-(1+\lambda)^n(1+\alpha)(m+1)}{(1+\lambda)^n(1+\alpha)(m+1)}|b_1|\right\}\subset f(U).$$

Next, we determine the extreme points of the closed convex hulls of $THP_\lambda(\alpha,\beta,n,m,k)$, denoted by $clcoTHP_\lambda(\alpha,\beta,n,m,k)$.

**Theorem 4.** A function $f(z)\in clcoTHP_\lambda(\alpha,\beta,n,m,k)$, if and only if

$$f(z)=\sum_{k=1}^{\infty}(\mu_k h_k+\eta_k g_k) \tag{2.6}$$

where

$$h_1(z)=z$$

$$h_k(z)=z-\frac{1-\beta}{[1+(k-1)\lambda]^n(1-\alpha+k\alpha)C(m,k)}z^k,\quad k=2,3,...$$

$$g_k(z)=z-\frac{1-\beta}{[1+(k-1)\lambda]^n(1-\alpha+k\alpha)C(m,k)}\bar{z}^k,\quad k=1,2,3,...$$

$$\sum_{k=1}^{\infty}(\mu_k+\eta_k)=1,\quad \mu_k\ge 0 \ \text{and}\ \eta_k\ge 0.$$

In particular, the extreme points of $THP_\lambda(\alpha,\beta,n,m,k)$ are $\{h_k\}$ *and* $\{g_k\}$.

**Proof:** For the functions $f(z)$ of the form (2.6), we have

$$f(z)=\sum_{k=1}^{\infty}\left(\mu_k h_k+\eta_k g_k\right)$$
$$=\sum_{k=1}^{\infty}(\mu_k+\eta_k)z-\sum_{k=2}^{\infty}\frac{1-\beta}{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}\mu_k z^k$$
$$+(-1)^n\sum_{k=1}^{\infty}\frac{1-\beta}{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}\eta_k \overline{z}^{k}$$

then

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}{1-\beta}\left(\frac{1-\beta}{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}\mu_k\right)+$$
$$\sum_{k=1}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}{1-\beta}\left(\frac{1-\beta}{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}\eta_k\right)$$
$$=\sum_{k=2}^{\infty}\mu_k+\sum_{k=1}^{\infty}\eta_k=1-\mu_1\le 1$$

and so $f(z)\in clcoTHP_{\lambda}(\alpha,\beta,n,m,k)$.

Conversely, suppose that $f(z)\in clcoTHP_{\lambda}(\alpha,\beta,n,m,k)$. Set

$$\mu_k=\frac{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}{1-\beta}|a_k|,\ k=2,3,...$$

*and*

$$\eta_k=\frac{\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)}{1-\beta}|b_k|,\ k=1,2,3,...$$

Then note that by Theorem 2, $0\le\mu_k\le 1\ (k=2,3,...)$, and $0\le\eta_k\le 1\ (k=1,2,3,...)$.

We define $\mu_1=1-\sum_{k=2}^{\infty}\mu_k+\sum_{k=1}^{\infty}\eta_k$ and note that, by Theorem 2, $\mu_1\ge 0$.

Consequently, we obtain

$$f(z)=\sum_{k=1}^{\infty}\left(\mu_k h_k+\eta_k g_k\right)$$

as required. Using Theorem 2 it is easily seen that $THP_{\lambda}(\alpha,\beta,n,m,k)$, is convex and closed, so $clcoTHP_{\lambda}(\alpha,\beta,n,m,k)=THP_{\lambda}(\alpha,\beta,n,m,k)$. Then the statement of Theorem 4 is really for $f(z)\in THP_{\lambda}(\alpha,\beta,n,m,k)$.

**Theorem 5.** Each member of $THP_\lambda\left(\alpha,\beta,n,m,k\right)(\alpha \geq 0, \lambda+\alpha \geq 1, 0 \leq \beta < 1)$ maps $U$ on to a starlike domain.

**Proof:** We only need to show that if $f(z) \in THP_\lambda\left(\alpha,\beta,n,m,k\right)$, then

$$\operatorname{Re}\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\} > 0.$$

Using the fact that $\operatorname{Re} w > 0$ if and only if $|1+w| > |1-w|$, it suffices to show that

$$\begin{aligned}
&\left|h(z)+\overline{g(z)}+zh'(z)-\overline{zg'(z)}\right|-\left|h(z)+\overline{g(z)}+zh'(z)+\overline{zg'(z)}\right| \\
&=\left|2z-\sum_{k=2}^{\infty}(k+1)|a_k|z^k+\sum_{k=1}^{\infty}(k-1)|b_k|\overline{z}^k\right|-\left|\sum_{k=2}^{\infty}(k-1)|a_k|z^k-\sum_{k=1}^{\infty}(k+1)|b_k|\overline{z}^k\right| \\
&\geq 2|z|-\left|\sum_{k=2}^{\infty}(k+1)|a_k|z^k-\sum_{k=1}^{\infty}(k-1)|b_k|\overline{z}^k\right|-\left|\sum_{k=2}^{\infty}(k-1)|a_k|z^k-\sum_{k=1}^{\infty}(k+1)|b_k|\overline{z}^k\right| \\
&\geq 2|z|\left\{1-\left(\sum_{k=2}^{\infty}k|a_k||z|^{k-1}+\sum_{k=1}^{\infty}k|b_k||z|^{k-1}\right)\right\} \\
&> 2|z|\left\{1-\left(\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)|a_k|+\sum_{k=1}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)|b_k|\right)\right\} \\
&\geq 2|z|\left[1-(1-\beta)\right] \\
&= 2|z|\beta \geq 0
\end{aligned}$$

**Theorem 6.** If $f(z) \in THP_\lambda\left(\alpha,\beta,n,m,k\right)(\alpha \geq 0, \lambda+\alpha \geq 1, 0 \leq \beta < 1)$, then $f(z)$ is convex in the disc

$$|z| < \min_k \left[\frac{1-\beta-|b_1|}{k}\right]^{1/k-1}, \quad k = 2,3,\ldots, \quad 1-\beta > |b_1|$$

**Proof:** Let $f(z) \in THP_\lambda\left(\alpha,\beta,n,m,k\right)$ and let r be fixed such that $0 < r < 1$, then If $r^{-1}f(rz) \in THP_\lambda\left(\alpha,\beta,n,m,k\right)$ and have

$$\sum_{k=2}^{\infty}k^2\left(|a_k|+|b_k|\right)r^{k-1} = \sum_{k=2}^{\infty}k\left(|a_k|+|b_k|\right)\left(k\,r^{k-1}\right)$$

$$\leq \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right)\left(k\,r^{k-1}\right)$$
$$\leq 1-\beta-|b_1|.$$

Provided $k\,r^{k-1}\leq 1-\beta-|b_1|$, which is true if

$$r\leq \min_k\left[\frac{1-\beta-|b_1|}{k}\right]^{1/k-1},\quad k=2,3,\ldots,\ 1-\beta>|b_1|.$$

Following Ruscheweyh [6 ], we call the set

$$N_\delta f(z)=\left\{F:F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\bar{z}^k \text{ and } \sum_{k=1}^{\infty}k\left(|a_k-A_k|+|b_k-B_k|\right)\leq\delta\right\} \tag{2.7}$$

as the $\delta$-neighbourhood of $f(z)$. From (2.7) we obtain

$$\sum_{k=1}^{\infty}k\left(|a_k-A_k|+|b_k-B_k|\right)=|b_1-B_1|+\sum_{k=2}^{\infty}k\left(|a_k-A_k|+|b_k-B_k|\right)\leq\delta. \tag{2.8}$$

**Theorem 7.** Let $f(z)\in THP_\lambda(\alpha,\beta,n,m,k)(\alpha\geq 0,\lambda+\alpha\geq 1,0\leq\beta<1)$ and $\delta\leq\beta.$ If $F\in N_\delta(f)$, then $F$ is a harmonic starlike function.

**Proof:** Let $F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\bar{z}^k\in N_\delta f(z),$ we have

$$\sum_{k=2}^{\infty}k\left(|A_k|+|B_k|\right)+|B_1|\leq\sum_{k=2}^{\infty}k\left(|a_k-A_k|+|b_k-B_k|\right)+\sum_{k=2}^{\infty}k\left(|a_k|+|b_k|\right)+|B_1-b_1|+|b_1|$$

$$\leq\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)\left(|a_k-A_k|+|b_k-B_k|\right)+|B_1-b_1|+|b_1|$$
$$+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n(1-\alpha+k\alpha)C(m,k)\left(|a_k|+|b_k|\right)$$
$$\leq\delta+|b_1|+\left(1-\beta-|b_1|\right)\leq 1.$$

Hence, $F(z)$ is a harmonic starlike function.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

we define the convolution of two harmonic functions $f(z)$ and $F(z)$ as

$$\left(f*F\right)(z)=f(z)*F(z)=z-\sum_{k=2}^{\infty}|a_k||A_k|z^k-\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k. \tag{2.9}$$

Using this definition, we show that the class $THP_\lambda\left(\alpha,\beta,n,m,k\right)$ is closed under convolution.

**Theorem 8.** For $0\le\alpha_1\le\alpha_2$, $0\le\beta_1\le\beta_2<1$, $\lambda+\alpha\ge1$, let $f(z)\in THP_\lambda\left(\alpha_2,\beta_2,n,m,k\right)$ and $F(z)\in THP_\lambda\left(\alpha_1,\beta_1,n,m,k\right)$.
Then $\left(f*F\right)(z)\in THP_\lambda\left(\alpha_2,\beta_2,n,m,k\right)\subset THP_\lambda\left(\alpha_1,\beta_1,n,m,k\right)$.

**Proof:** Let $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $THP_\lambda\left(\alpha_2,\beta_2,n,m,k\right)$ and

$$F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\overline{z}^k \text{ be in } THP_\lambda\left(\alpha_1,\beta_1,n,m,k\right).$$

Then the convolution $\left(f*F\right)$ is given by (2.9). We wish to show that the coefficient of $\left(f*F\right)$ satisfies the required condition given in Theorem 2.
For $F(z)\in THP_\lambda\left(\alpha_1,\beta_1,n,m,k\right)$ we note that $|A_k|<1$ and $|B_k|<1$. Now, for the convolution function $f*F$, we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha_1+k\alpha_1\right)C(m,k)}{1-\beta_1}|a_k||A_k|+\sum_{k=1}^{\infty}\frac{\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha_1+k\alpha_1\right)C(m,k)}{1-\beta_1}|b_k||B_k|$$

$$\le\sum_{k=2}^{\infty}\frac{\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha_1+k\alpha_1\right)C(m,k)}{1-\beta_1}|a_k|+\sum_{k=1}^{\infty}\frac{\left[1+\left(k-1\right)\lambda\right]^n\left(1-\alpha_1+k\alpha_1\right)C(m,k)}{1-\beta_1}|b_k|$$

$$\le \sum_{k=2}^{\infty} \frac{\left[1+\left(k-1\right)\lambda\right]^n \left(1-\alpha_2+k\alpha_2\right)C(m,k)}{1-\beta_2}\left|a_k\right| + \sum_{k=1}^{\infty} \frac{\left[1+\left(k-1\right)\lambda\right]^n \left(1-\alpha_2+k\alpha_2\right)C(m,k)}{1-\beta_2}\left|b_k\right| \le 1.$$

Since $0 \le \alpha_1 \le \alpha_2$, $0 \le \beta_1 \le \beta_2 < 1$, $\lambda + \alpha \ge 1$ and $f(z)THP_\lambda\left(\alpha_2, \beta_2, n, m, k\right)$, thus $\left(f * F\right)(z) \in THP_\lambda\left(\alpha_2, \beta_2, n, m, k\right) \subset THP_\lambda\left(\alpha_1, \beta_1, n, m, k\right)$.

Note that some other related work using different types of operators can be found in ([14]-[16]).

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