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Some Results on a Class of Analytic Functions Defined By Fractional Integral

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Abstract

In the present paper, we introduce a new class namely £ (A, B, f, p, δ) of analytic functions, in terms of fractional integral operator and prove various sharp results. In our first result we obtain a necessary and sufficient condition for G (z) subsequently a containment relation is proved. We also find a class preserving integral operator.

Keywords: Univalent function, Analytic function, Complex order, Starlike function, Fractional Integral Operator, class **£** (A, B, f, p, δ).

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1 Introduction and Definitions

Let $\mathcal{T}$ denote the class of the functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_1 > 0; a_n \geq 0) \tag{1.1}$$

Which are univalent in the open unit disk $u = \{z : |z| < 1\}$.

A function f (z) belongs to the class $\mathcal{T}$* (A, B, p) if and only if

$$\left| \frac{\frac{zf'(z)}{pf(z)} - 1}{\frac{Bzf'(z)}{pf(z)} - A} \right| < 1, \qquad z \in u \tag{1.2}$$

Where $-1 \le A < B \le 1$. The class $\mathcal{T}$* (A, B, p) is studied by Goel and Sohi [5].

Now, we investigate a new class **£** (A, B, f, p, δ), of analytic starlike functions, in terms of fractional integral operator, over the elements of $\mathcal{T}$* (A, B, p) having negative coefficients.

A function G belongs to the class **£** (A, B, f, p, δ), if it satisfies

$$G(z) = \frac{\Gamma(1+p+\delta)}{\Gamma(1+p)} z^{-\delta} D_z^{-\delta} f(z), \quad z \in u \tag{1.3}$$

for some f (z) belonging to the class $\mathcal{T}$* (A, B, p).

Here $D_z^{-\delta} f(z)$ denotes the fractional integral of f (z) of order δ, defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\delta}}, \qquad (\delta > 0) \tag{1.4}$$

Where f is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

By giving the specific values of parameters δ and p, we obtain the following subclasses of univalent functions studied by Kumar [6] and Libra [3].

(i) Taking p = 1 in (1.3), the class **£** (A, B, f, p, δ) reduces to the class **£** (δ, k, ρ, f) for some f belongs to the class $\mathcal{T}$ (ρ, k), studied by Kumar [6].

(ii) Taking δ = 1 in (1.3), the class **£** (A, B, f, p, δ) reduces to the integral operator

$$G(z) = \frac{1+p}{z} \int_0^z f(\zeta)d\zeta .$$

(iii) Taking p = 1 and δ = 1 in (1.3), the class **£** (A, B, f, p, δ) reduces to

$$G(z) = \frac{2}{z} \int_0^z f(\zeta)d\zeta .$$

The integral operator studied by Libera [3]. Where the operator defined by (1.3) may be treated as generalization of the Libera integral operator.

In a paper Sharma and Singh [4] introduced the class G (λ, μ, A, B, b) of analytic functions f (z) of complex order b, using convolution technique. They estimated the coefficient a_n and generalized Ahuja [2] further obtained sufficient conditions for the function f (z) belonging to the class G (λ, μ, A, B, b) extending Chaudhary [1].
This motivates our main results in the setting of newly introduced class **£** (A, B, f, p, δ) of analytic functions.

2 Main Results

To prove our main results, we state a lemma due to Goel and Sohi [5].
Lemma (1) [5]: A function f (z) defined by (1.1) belongs to the class $\mathcal{T}^*$ (A, B, p) if and only if

$$\sum_{n=1}^{\infty}\{(1+B)n+(B-A)p\}\left|a_{p+n}\right| \leq (B-A)p \tag{2.1}$$

The result is sharp with the external function

$$f(z)=z^p-\frac{(B-A)p}{\{(1+B)n+(B-A)p\}}z^{p+n}\quad ,\qquad (n\in N) \tag{2.2}$$

Now we obtain necessary and sufficient Condition:

Theorem (1): A function $G(z)=z^p-\sum_{n=1}^{\infty}\left|c_{p+n}\right|z^{p+n}$ belongs to the class **£** (A, B, f, p, δ) if and only if

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}\left|c_{p+n}\right| \leq 1 \tag{3.1}$$

Proof: By definition G belongs to the class **£** (A, B, f, p, δ) if it satisfies the relation (1.3) for some f (z) belongs to $\mathcal{T}^*$ (A, B, p). Let f(z) defined by (1.1), then after a simple computation, we obtain

$$G(z)=\frac{\Gamma(1+p+\delta)}{\Gamma(1+p)}z^{-\delta}D_z^{-\delta}f(z)$$

$$=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}\left|a_{p+n}\right|z^{p+n}$$

clearly $$\left|c_{p+n}\right|=\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}\left|a_{p+n}\right|$$

or $$\left|a_{p+n}\right|=\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}\left|c_{p+n}\right|,\qquad (n\in N) \tag{3.2}$$

The required result follows now by using (3.2) in Lemma (1).

Let G belongs to the class **£** (A, B, f, p, δ), where f (z) defined by (1.1), then

$$G(z) = z^p - \sum_{n=1}^{\infty} |c_{p+n}| z^{p+n} \tag{3.3}$$

where $|c_{p+n}| = \dfrac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)} |a_{p+n}| = \dfrac{|a_{p+n}|}{\gamma(n,p,\delta)}$

and $\gamma(n,p,\delta) = \dfrac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p-\delta)}$.

Clearly, $\dfrac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)} = \dfrac{(1+p)(2+p)......................(n+p)}{(1+p+\delta)(2+p+\delta)......(n+p+\delta)}$

$$\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)} = \prod_{j=1}^{n} \frac{(j+p)}{(j+p+\delta)} < 1 \quad \text{for all} \quad \delta > 0.$$

Thus $|c_{p+n}| < |a_{p+n}|$, for all n ≥ 1 and therefore

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n + (B-A)p\}}{(B-A)p} |c_{p+n}| < \sum_{n=1}^{\infty} \frac{\{(1+B)n + (B-A)p\}}{(B-A)p} |a_{p+n}| \le 1,$$

Since f (z) belongs to the class 𝒯* (A, B, p). Hence G belongs to the class **£** (A, B, f, p, δ) and thus we get containment relation

$$\text{£}(A, B, f, p, \delta) \subset \mathcal{T}^*(A, B, p). \tag{3.4}$$

Since $\lim_{\beta \to 0}$ **£** (A, B, f, p, δ) ≡ 𝒯* (A, B, p).

The relation (3.4) can also be written as **£** (A, B, f, p, δ) ⊂ $\lim_{\beta \to 0}$ **£** (A, B, f, p, β).

In our next result we obtain containment Relation:

Theorem (2): If $0 < \beta \le \delta$, then **£** (A, B, f, p, δ) ⊂ **£** (A, B, f, p, β).

Proof: Let the function G defined by (3.3) belongs to the class **£** (A, B, f, p, δ). Then from (3.1), we have

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n + (B-A)p\}}{(B-A)p} \frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)} |c_{p+n}| \le 1. \tag{4.1}$$

Next, since β ≤ δ, we have

$$\frac{\Gamma(1+p)\Gamma(n+1+p+\beta)}{\Gamma(n+1+p)\Gamma(1+p+\beta)} = \prod_{j=1}^{n} \left\{ \frac{j+p+\beta}{j+p} \right\} \le \prod_{j=1}^{n} \left\{ \frac{j+p+\beta}{j+p} \right\}, \text{ since } \beta \le \delta$$

$$\frac{\Gamma(1+p)\Gamma(n+1+p+\beta)}{\Gamma(n+1+p)\Gamma(1+p+\beta)} = \frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}|c_{p+n}|$$

$$\leq\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}|c_{p+n}|$$

(4.2)

Using (4.1) in (4.2), we get

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\frac{\Gamma(1+p)\Gamma(n+1+p+\beta)}{\Gamma(n+1+p)\Gamma(1+p+\beta)}|c_{p+n}|\leq 1.$$

Hence G belongs to the class **£** (A, B, f, p, δ) and this complete the proof of the theorem.

With the aid of Theorem (1), we immediately obtain the following theorems.

Theorem (3): Let $0\leq\delta\leq 1$, $-1\leq A_1\leq A_2<1$ and $0\leq B\leq 1$. Then

£ (A_1, B, f, p, δ) ⊂ **£** (A_2, B, f, p, δ).

Theorem (4): Let $0\leq\delta\leq 1$, $-1\leq A<1$ and $0\leq B_1\leq B_2\leq 1$. Then

£ (A, B_1, f, p, δ) ⊂ **£** (A, B_2, f, p, δ)

Corollary (4.1): Let $0\leq\delta\leq 1$, $-1\leq A_1\leq A_2<1$ and $0\leq B_1\leq B_2\leq 1$. Then

£ (A_1, B_1, f, p, δ) ⊂ **£** (A_1, B_1, f, p, δ) ⊂ **£** (A_2, B_1, f, p, δ).

Theorem (5): Let $0\leq\delta\leq 1$, $-1\leq A<1$ and $0\leq B\leq 1$. Then

£ (A, B, f, p, δ) = **£** **{**(1-B+2A/1+B) f, p, δ). More generally, -1≤A'<1 and 0≤B'≤1, then

£ (A, B, f, p, δ) = **£** (A', B', f, p, δ) if and only if

{(1+B) + (B-A) p}/ (B-A)p = {(1+B') + (B'-A')p}/(B'-A')p

Finally we obtain a result for integral Operator:

Theorem (6): Let c be a real number such that c > -p. If G belongs to the class **£** (A, B, f, p, δ), then the function F defined by $F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}G(t)dt$ is also an element of **£** (A, B, f, p, δ).

Proof: Let the function G defined by $G(z)=z^p-\sum_{n=1}^{\infty}|c_{p+n}|z^{p+n}$. Then by definition of F (z) it is clear that $F(z)=z^p-\sum_{n=1}^{\infty}|d_{p+n}|z^{p+n}$,

Where $|d_{p+n}|=\left(\frac{c+p}{c+p+n}\right)|c_{p+n}|<|c_{p+n}|$. Therefore

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\gamma(n,p,\delta)\left|d_{p+n}\right|$$
$$\leq\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\gamma(n,p,\delta)\left|c_{p+n}\right|$$
≤ 1, by theorem (1). Hence, F (z) is also an element of **£** (A, B, f, p, δ).

3 Open Problem

In our last section, we suggest an open problem as follows:

Let c be a real number such that c > -p. If F belongs to the class **£** (A, B, f, p, δ) and if

$$F(z)=\frac{c+p}{z^{c}}\int_{0}^{z}t^{c-1}G(t)dt$$

Then
i.) Whether G (z) is starlike?
ii.)Whether the result is sharp?
or
In other words does the converse problem of the above **Theorem 6** exist?

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