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c-Maximal submodules of finite modules

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Abstract

In this paper we introduce the basic definition of c-maximal submodule of finite modules, and to studying some properties of c-maximal submodule of finite modules.

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1 Introduction

In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Thus, a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication. Modules are very closely related to the representation theory of groups. They are also one of the central notions of commutative algebra and homological algebra, and are used widely in algebraic geometry and algebraic topology.
In a vector space, the set of scalars is a field and acts on the vectors by scalar multiplication, subject to certain axioms such as the distributive law. In a module, the scalars need only be a ring, so the module concept represents a significant generalization. In commutative algebra, both ideals and quotient rings are modules, so that many arguments about ideals or quotient rings

can be combined into a single argument about modules. In non-commutative algebra the distinction between left ideals, ideals, and modules becomes more pronounced, though some ring-theoretic conditions can be expressed either about left ideals or left modules. Much of the theory of modules consists of extending as many as possible of the desirable properties of vector spaces to the realm of modules over a "well-behaved" ring, such as a principal ideal domain. However, modules can be quite a bit more complicated than vector spaces; for instance, not all modules have a basis, and even those that do, free modules, need not have a unique rank if the underlying ring does not satisfy the invariant basis number condition, unlike vector spaces, which always have a (possibly infinite) basis whose cardinality is then unique. (These last two assertions require the axiom of choice in general, but not in the case of finite-dimensional spaces, or certain well-behaved infinite-dimensional spaces such as Lp spaces).
The normal index of a maximal subgroup of a finite group G, which was defined by Deskins [5], often yields awealth of information about the group G itself. In the past, it has been studied by many scholars (such as [2, 3, 4]). In [10], Wang defined c-normality of a subgroup and obtained some results. It is interesting to use some information on the submodules of a finite modules M to determine the structure of the module M. The maximality of submodules of a finite modules plays an important role in the study of finite modules. In this work, we introduce the concept of c-maximal submodule of finite modules and obtain some new results about them.

2 preliminaries

In this section we recall some of the fundamental concepts and definition, which are necessary for this paper. For details we refer to [8, 1, 6, 9, 7].

Definition 2.1 *A ring* $< R, +, . >$ *consists of a nonempty set* R *and two binary operations* $+$ *and* $.$ *that satisfy the axioms:*
(1) $< R, +, . >$ *is an abelian group;*
(2) $(ab)c = a(bc)$ *(associative multiplication) for all* $a, b, c \in R$*;*
(3) $a(b + c) = ab + ac, (b + c)a = ba + ca$ *(distributive laws) for all* $a, b, c \in R$
Moreover, the ring R *is a commutative ring if* $ab = ba$ *and ring with identity if* R *contains an element* 1_R *such that* $1_R a = a1_R = a$ *for all* $a \in R$*.*

Example 2.2 *(1) The ring* $\mathbb{Z}$ *of integers is a commutative ring with identity. So are* $\mathbb{Q}$*,* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{Z}_n$*,* $\mathbb{R}[x]$*, etc.*
(2) $3\mathbb{Z}$ *is a commutative ring with no identity.*
(3) The ring $\mathbb{Z}^{2\times 2}$ *of* 2×2 *matrices with integer coecients is anoncommutative*

ring with identity.
(4) $(3\mathbb{Z})^{2\times 2}$ *is a noncommutative ring with no identity.*

Definition 2.3 *Let R be a ring. A commutative group* $(M, +)$ *is called a left R-module or a left module over R with respect to a mapping*

$$. : R \times M \to M$$

if for all $r, s \in R$ *and* $m, n \in M$,
(1) $r.(m + n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,
(3) $(r + s).m = r.m + s.m$.
If R has an identity 1 and if $1.m = m$ *for all* $m \in M$, *then M is called a unitary or unital left R-module.*
A right R-module can be defined in a similar fashion.

Example 2.4 *(1) If K is a field, then the concepts "K-vector space" (a vector space over K) and K-module are identical.*
(2) If K is a field, and $K[x]$ *a univariate polynomial ring, then a* $K[x]$*-module M is a K-module with an additional action of x on M that commutes with the action of K on M.*
(3) The concept of a $\mathbb{Z}$*-module agrees with the notion of an abelian group. That is, every abelian group is a module over the ring of integers* $\mathbb{Z}$ *in a unique way. For* $n > 0$, *let* $nx = x+x+...+x$ *(n summands),* $0.x = 0$, *and* $(-n).x = -(nx)$.

Definition 2.5 *Let M be an R-module and N be a nonempty subset of M. Then N is called a submodule of M if N is a subgroup of M and for all* $r \in R, a \in N$, *we have* $ra \in N$.

Definition 2.6 *In algebra, a module homomorphism is a function between modules that preserves module structures. Explicitly, if M and N are left modules over a ring R, then a function* $f : M \to N$ *is called a module homomorphism or an R-linear map if for any x, y in M and r in R,* $f(x + y) = f(x) + f(y), f(rx) = rf(x)$. *If M, N are right modules, then the second condition is replaced with* $f(xr) = f(x)r$.

Example 2.7 *(1) The zero map* $M \to N$ *that maps every element to zero.*
(2) A linear transformation between vector spaces.

3 c-Maximal submodules of finite modules

Definition 3.1 *An R -module H is called c -maximal submodule of R -module M if there exists an R -submodule N of M such that* $M = HN$ *and* $H \cap N \leq H_M$ *(*$H \cap N$ *is a submodule of* H_M*), where* H_M *is the maximal submodules of M which is contained in H.*

Theorem 3.2 *Let M be an R -module with submodules A, B and C such that $B \leq A$ (B is a submodule of A). Then $A \cap BC = B(A \cap C)$.*

Proof 3.3 *Certainly, $B(A \cap C) \subseteq A \cap BC$ (since B is a submodule of A). Let $a \in A \cap BC$. Then $a = bc$ for some $b \in B$ and $c \in C$. As $B \subseteq A$ so $a \in B(A \cap C)$.*
Thus $A \cap BC = B(A \cap C)$.

Theorem 3.4 *Let M be an R -module with submodules A, B.*
(1) $A \cap B$ is a submodule of M.
(2) AB is a submodule of M.

Proof 3.5 *(1) It is clear that $A \cap B$ is subgroup of M. Let $r \in R$ and $x \in A \cap B$, then $rx \in A \cap B$(since A, B are submodules of M). Then $A \cap B$ is a submodule of M.*
(2) It is clear that AB is subgroup of M. If $r \in R$ and $x \in AB$, then we have $a \in A, b \in B$ such that $rx = rab = (ra)b \in AB$ (since A is submodule of M). Then from Definition 2.5, we get AB is a submodule of M.

Theorem 3.6 *Let M be an R -module with submodules H and an K such that $K \leq H \leq M$ and $K \leq N \leq M$. Then $M = HN$ if and only if $\frac{M}{K} = \frac{H}{K}\frac{N}{K}$.*

Proof 3.7 *Suppose that $M = HN$ and $m + K \in \frac{M}{K}$ for all $m \in M$. Then $m + K = (hn) + K = (h + K)(n + K) \in \frac{H}{K}\frac{N}{K}$ for all $h \in H$ and $n \in N$. Thus $\frac{M}{K} \subseteq \frac{H}{K}\frac{N}{K}$. Now if $(h+K)(n+K) \in \frac{H}{K}\frac{N}{K}$, then $(h+K)(n+K) = hn+k = m + K \in \frac{M}{K}$ and so $\frac{H}{K}\frac{N}{K} \subseteq \frac{M}{K}$. Therefore $\frac{M}{K} = \frac{H}{K}\frac{N}{K}$.*
Conversely, Suppose that $\frac{M}{K} = \frac{H}{K}\frac{N}{K}$. It is easy to show that $HN \subseteq M$. Let $m \in M$ and so $m + K \in \frac{M}{K} = \frac{H}{K}\frac{N}{K}$. Then $m + K = (h + k)(n + K) = hn + K \in \frac{HN}{K}$ for all $h \in H$ and $n \in N$. So $m \in HN$ and $M \subseteq HN$. Thus $M = HN$.

Theorem 3.8 *Let M be an R -module. If H is a submodule of M, then H is a c -maximal submodule of M.*

Proof 3.9 *Let that H is a submodule of M. Since M is a submodule of itself, then $M = HM$ and $H \cap M = H \leq H_M$. So H is c -maximal submodule of M.*

Theorem 3.10 *Let M be an R -module with submodules H and K. If H is c-maximal submodule of M with $H \leq K \leq M$, then H is c-maximal submodule of K.*

Proof 3.11 *Suppose that H is c -maximal submodule of M. Then there exists a submodule N in M such that $M = HN$ and $H \cap N \leq H_M$. By Theorem 3.2 we have that $K = K \cap M = K \cap HN = H(K \cap N)$. Theorem 3.4(1) give that $K \cap N$ is submodule of K. Then $H \cap (N \cap K) = (H \cap N) \cap K \leq H_M \cap K \leq H_K$ and so H is c-maximal submodule of K.*

Theorem 3.12 *Let K be a submodule of an R -module M with $K \leq H$. Then H is c -maximal submodule of M if and only if $\frac{H}{K}$ is c -maximal submodule of $\frac{M}{K}$.*

Proof 3.13 *Let $\frac{H}{K}$ is c -maximal submodule of $\frac{M}{K}$. Then there exists a submodule $\frac{N}{K}$ of $\frac{M}{K}$ such that $\frac{M}{K} = (\frac{H}{K})(\frac{N}{K})$ and $(\frac{H}{K}) \cap (\frac{N}{K}) \leq (\frac{H}{K})_{(\frac{M}{K})}$. Now Theorem 3.6 give that $M = HK$ and $H \cap N \leq H_M$. Thus H is c -maximal submodule of M.*
Conversely, let H is c -maximal submodule of M. Then there exists a submodule N of M such that $M = HN$ and $H \cap N \leq H_M$. From Theorem 3.6 we obtain that $\frac{M}{K} = (\frac{H}{K})(\frac{N}{K}) = (\frac{H}{K})(\frac{NK}{K})$ (since $K \leq N$). By Theorem 3.4(2) we know that KN is submodule of M. Now

$$(\frac{H}{K}) \cap (\frac{NK}{K}) = \frac{(H \cap NK)}{K} = \frac{K(H \cap N)}{K} \leq \frac{KH_M}{K} = (\frac{H}{K})_{(\frac{M}{K})}.$$

Then $\frac{H}{K}$ is c -maximal submodule of $\frac{M}{K}$.

4 Open Problem

In abstract algebra, a bimodule is an abelian group that is both a left and a right module, such that the left and right multiplications are compatible. Besides appearing naturally in many parts of mathematics, bimodules play a clarifying role, in the sense that many of the relationships between left and right modules become simpler when they are expressed in terms of bimodules. If R and S are two rings, then an $R-S-$bimodule is an abelian group M such that:

(1) M is a left $R-$module and a right $S-$module.
(2) For all r in R and s in S and m in M: $(rm)s = r(ms)$.
An $R-R-$bimodule is also known as an $R-$bimodule.
(1) For positive integers n and m, the set $M_{n,m}(R)$ of $n \times m$ matrices of real numbers is an $R-S-$bimodule, where R is the ring $M_n(R)$ of $n \times n$ matrices, and S is the ring $M_m(R)$ of $m \times m$ matrices. Addition and multiplication are carried out using the usual rules of matrix addition and matrix multiplication; the heights and widths of the matrices have been chosen so that multiplication is defined.
(2) If R is a ring, then R itself is an $R-R-$bimodule, and so is R^n (the n-fold direct product of R).
(3) If R is a subring of S, then S is an $R-R-$bimodule. It is also an $R-S-$ and an $S-R-$bimodule.
(4) Any two-sided ideal of a ring R is an $R-R-$bimodule. The open problem here is to investigate c-maximal submodules of bimodules and to give some new results about them as we did.

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