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Characterization of Regular Ordered Ternary Semigroups by Bi-ideals

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Abstract

In this paper we characterize regular ordered semigroups by its ordered ordered bi-ideals,ordered quasi-ideal, ordered left, right and lateral ideals.

Keywords: *ordered ternary semigroup, regular ordered ternary semigroup, ordered left ideal, ordered right ideal, ordered lateral ideal,ordered quasi-ideal, ordered bi-ideal.*
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1 Introduction

The literature of ternary algebraic system was introduced by Lehmer [10] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroup was known to S.Banach. He showed by an example that ternary semigroup does not necessarily reduce to an ordinary semigroup. In [11] Santiago developed the theory of ternary semigroups. He focused his attention mainly to the study of regular ternary semigroups, bi-ideals and ideals in ternary semigroups. Consider the semigroup $\mathbb{Z}$ of all integers under multiplication which plays a vital role in the literature of semigroup.The subset $\mathbb{Z}^+$of all positive integers of $\mathbb{Z}$ is a semigroup under multiplication. Now if we consider the subset $\mathbb{Z}^-$ of all negative integers of $\mathbb{Z}$, then it is not a semigroup under multiplication. Taking these facts in mind Lehmer [10] introduced the notion of ternary semigroup. $\mathbb{Z}^-$is a natural example of a ternary semigroup under the ternary multiplication. Kehayopulu in [8] developed the theory of po-semigroups. He mainly studied regular po-semigroups, ideals and bi-ideals in po-semigroups.In 1995, Dixit and Deewan [2]studied the properties of quasi-ideals and bi-ideals in ternary semigroups. In 1998, the notion of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu [7]. In 1999, Lee and Kang [5] gave characterizations of the regular po-semigroups in terms of some special types of ideals.

In 2009, Iampan [4] introduced an ordered ternary semigroups and characterized the minimality and maximality of ordered lateral ideals in ordered ternary semigroups. He also introduced the concept of ordered ideal extensions in ordered ternary semigroups[4]. In 2012, Daddi and Pawar[1] introduced the concepts of ordered quasi-ideal and ordered bi-ideal in ordered ternary semigroups. They also introduced regular ordered ternary semigroup and studied their properties. In Jailoka and Iampan [6] characterized minimality and maximality of ordered quasi-ideal in ordered ternary semigroups. In 2017, Hansda [3]characterized minimal bi-ideals in regular and completely regular ordered semigroups.

In this paper an attempt is made to characterize regular ordered ternary semigroups by their different types of ideals.

2 Preliminaries

Definition 2.1 *A nonempty set T together with a ternary operation [ ] defined on T is called a ternary semigroup if [ ] satisfies the following associative law:* $[a\ b\ [c\ d\ e]] = [a\ \ [b\ c\ d]\ \ e\ ] = [[a\ b\ c]\ \ d\ e]$, *for all* $a, b, c, d, e \in T$.

Every ordinary binary semigroup is a ternary semigroup under the same multiplication but not conversely. For this consider the following example. Let $T = \{\ldots\ldots$,-2i, -i, 0, i, 2i, $\ldots\ldots\}$. T is a ternary semigroup under the multiplication over complex number. Note that this ternary semigroup does not reduce to an ordinary semigroup under the same multiplication.

By $[ABC]$, we mean the set $\{[a\ b\ c]\ /a \in A, b\ \in B, c\ \in C\ \}$ for nonempty subsets A, B, C of T. If $A = \{a\}$, then we write $[\{a\}\, BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$.

Definition 2.2 *A nonempty subset S of a ternary semigroup T is called a ternary subsemigroup of T if* $[SSS] \subseteq S$.

Definition 2.3 *A ternary semigroup T is called an ordered ternary semigroup if there exits a partially ordered relation $\leq$ such that for $a, b\ \in T$ if $a\ \leq b$ implies* $[axy] \leq [bxy]\,, [xay] \leq [xby]\,, [xya] \leq [xyb]$, *for all* $x, y \in T$.

Example 2.4 *The set $\mathbb{Z}^-$ of all negative integers is an ordered ternary semigroup with respect to triple multiplication and a partial ordering relation $\leq$.*

Now onwards $T = < T, ., \leq >$ denotes an ordered ternary semigroup. For $H \subseteq S$ we denote $(H]$ the subset of T defined by
$(H] = \{t \in T / \ t \leq h$, for some $h \in T\}$.

Definition 2.5 *A nonempty subset* I *of T is called an ordered left (respectively an ordered right, an ordered lateral) ideal of T if*
(1) $[TTI] \subseteq I$ (respectively $[ITT] \subseteq I$, $[TIT] \cup [TTITT] \subseteq I$),
(2) For $a \in$ I , $b \in T$ such that $b \leq a$ implies $b \in I$, that is $(I] =$ I .

Definition 2.6 *(a)A nonempty subset I of T is called a two-sided ordered ideal of T if I is both an ordered left and an ordered right ideal of T.*
(b)A nonempty subset I of T is called an ordered ideal if I is an ordered left, an ordered right and an ordered lateral ideal of T.

We denote by $L(A), R(A), M(A)$ and $I(A)$ the ordered left ideal, ordered right ideal, ordered lateral ideal and ordered ideal of T, respectively generated by a nonempty subset A of T , that is the least, with respect to the inclusion relation, ordered left ideal, ordered right ideal, ordered lateral ideal and ordered ideal of T, respectively containing A. As usual $L(A)$ (respectively $R(A), M(A)$) coinsides with the intersection of all ordered left (respectively ordered right, ordered lateral) ideals of T containing A, I(A) is the intersection of all ordered ideals of T containing A.

We observe the following properties:
$L(A) = (A \cup [TTA]]$,
$R(A) = (A \cup [ATT]]$,
$M(A) = (A \cup [TAT] \cup [TTATT]]$
$I(A) = (A \cup [TTA] \cup [ATT] \cup [TAT] \cup [TTATT]], \ A \subseteq T$.

For $\{a\}$, we write $L(a), R(a), M(a)$ and $I(a)$ simply for $L(\{a\}), R(\{a\}), M(\{a\})$ and $I(\{a\})$ respectively,and we call them the ordered principal left ideal,the ordered principal right ideal, the ordered principal lateral ideal and principal ordered ideal of T respectively generated by $a \in T$. Further we have,
$L(a) = \{t \in T/t \leq a$ or $t \leq [xya]$, for some $x, y \in T\} = (a \cup [TTa]]$

$$= (a] \cup ([TTa]] .$$

$R(a) = \{t \in T/t \leq a$ or $t \leq [xya]$, for some $x, y \in T\} = (a \cup [aTT]]$

$$= (a] \cup ([aTT]] .$$

$$M(a) = \{t \in T / t \leq a \text{ or } t \leq [xya], \text{for some } x, y \in T\} = (a \cup [TaT] \cup [TTaTT]]$$

$$= (a] \cup ([TaT]] \cup ([TTaTT]].$$

$I(a) = \{t \in T/t \leq a$ or $t \leq [xya]$, or $t \leq [yxa]$ *or* $t \leq [xay]$ *or* $t \leq [xyapq]$, for some $x, y, p, q \in T\} = (a] \cup ([TTa]] \cup ([aTT]] \cup ([TaT]] \cup ([TTaTT]]$.

Definition 2.7 *An ordered ternary semigroup T is called an ordered left(right or lateral) simple if it has no proper ordered left (right or lateral) ideal of T.*

Theorem 2.8 *For subsets A, B and C of an ordered ternary semigroup T, then the following statements hold.*
1) $A \subseteq (A]$, for all $A \subseteq T$.
2) If $A \subseteq B \subseteq T$ then $(A] \subseteq (B]$.
3) $((A]] = (A]$, for all $A \subseteq T$.
4) $(A](B](C] \subseteq ([ABC]]$.
5) $(A \cup B \cup C] = (A] \cup (B] \cup (C]$.
6) $(A \cap B \cap C] \subseteq (A] \cap (B] \cap (C]$.
7) For any $a \in T$, $([TTa]], ([TaT]] \cup ([TTaTT]]$ and $([aTT]]$ are ordered left, ordered lateral and ordered right ideal of T respectively.
8) For any non-empty subset A of T, $([TTA]], ([TAT]] \cup ([TTATT]]$ and $([ATT]]$ are ordered left, ordered lateral and ordered right ideal of T respectively.
9) For any $a \in T$, the set $([TTaTT]]$ is an ordered ideal of T .

Definition 2.9 *A nonempty subset Q of T is called an ordered quasi-ideal of T if*
(i)$[QTT] \cap [TQT] \cap [TTQ] \subseteq$ Q,
(ii) $[QTT] \cap [TTQTT] \cap [TTQ] \subseteq$ Q ,
(iii) For $a \in Q$ and $b \in T$ such that $b \leq a$ implies $b \in Q$. i.e. $(Q] = Q$.

For $a \in T$, an ordered quasi-ideal of T generated by a is denoted by $Q(a)$, It is the smallest ordered quasi-ideal of T containing a.
That is $Q(a) = (a \cup (\{[aTT] \cap ([TaT] \cup [TTaTT]) \cap [TTa]\}]]$

Definition 2.10 *A nonempty subset B of T is called an ordered bi-ideal of T if*
(i)$[BTBTB] \subseteq B$,
(ii) for $a \in B$, $b \in T$ such that $b \leq a$ implies $b \in B$ i.e. $(B] = B$.

For $a \in T$, an ordered bi-ideal of T generated by a is denoted by $B(a)$, It is the smallest ordered bi-ideal of T containing a.

That is $B(a) = (a \cup [aTaTa]]$

Definition 2.11 *An ordered bi-ideal* B *of* T *is called sub-idempotent ordered bi-ideal if* $B^3 = [BBB] \subseteq B$.

Theorem 2.12 *In an ordered ternary semigroup* T,
1)Intersection of any number of ordered quasi-ideals of T *is an ordered quasi-ideal of* T*, provided it is nonempty.*
2)If S *is an ordered ternary subsemigroup and* Q *is an ordered quasi-ideal of* T *then* $Q \cap S$ *is an ordered quasi-ideal of* S.
3)If A *be an ordered ideal and* Q *be an ordered quasi-ideal of* T*, then* $A \cap Q$ *is an ordered quasi-ideal of* T.
4) Every ordered ideal of T *is an ordered bi-ideal of* T.
5) Every ordered quasi-ideal of T *is an ordered bi-ideal of* T.
6) Intersection of any number of ordered bi-ideals of T *is an ordered bi-ideal of* T*, provided it is nonempty.*
7) If S *is an ordered ternary subsemigroup and* B *is an ordered bi-ideal of* T*, then* $B \cap S$ *is an ordered bi-ideal of* S.
8) If S *is an ordered ternary subsemigroup and* B *is an ordered quasi-ideal of* T*, then* $Q \cap S$ *is an ordered quasi-ideal of* S.

Definition 2.13 *An element* $a \in T$ *is called regular if there exists* $x \in T$ *such that* $a \leq [axa]$*. If every element of* T *is regular then* T *is called regular ordered ternary semigroup.*

Theorem 2.14 *Let* S *be an ordered ternary subsemigroup of* T*. Then* S *is regular if and only if* $a \in ([aSa]]$ *, for all* $a \in S$.

3 Main results

Theorem 3.1 *An ordered ternary semigroup* T *is regular if and only if for every ordered right ideal R, ordered lateral ideal M and ordered left ideal L of T,* $([RML]] = R \cap M \cap L$

Proof. First suppose that T is a regular ordered ternary semigroup. Let R, M and L be an ordered right, ordered lateral and ordered left ideal of T respectively. Since $([RML]] \subseteq ([RTT]] \subseteq (R] = R$.
Similarly $([RML]] \subseteq ([TMT]] \subseteq ([TMT]] \cup ([TTMTT]] \subseteq (M] = M$ and

$([RML]] \subseteq ([TTL]] \subseteq (L] = L$.
Therefore $([RML]] \subseteq R \cap M \cap L$. ——————(1)
Now let $a \in R \cap M \cap L \subseteq T$, then there exists $x \in T$ such that $a \leq [axa]$.
As $a \leq [axa] \leq [axaxa] \leq [axaxaxa] = [[axa][xax]a] \in [RML]$.
Thus $a \in ([RML]]$.
So that $R \cap M \cap L \subseteq ([RML]]$. ——————-(2)
Therefore from (1) and (2), $([RML]] = R \cap M \cap L$.
Conversely suppose that $([RML]] = R \cap M \cap L$.
For any $a \in T$, $R(a), M(a)$ and $L(a)$ are ordered right, ordered lateral and ordered left ideals of T respectively.
Hence by assumption$([R(a)M(a)L(a)]] = R(a) \cap M(a) \cap L(a)$.
As $a \in R(a), a \in M(a)$ and $a \in L(a)$,
we have $a \in R(a) \cap M(a) \cap L(a) = ([R(a)M(a)L(a)]]$.
Hence $a \leq [[at_1t_2][t_3at_4][t_5t_6a]] = [a[[t_1t_2t_3]a[t_4t_5t_6]]a] = [axa]$, where
$x = [[t_1t_2t_3]a[t_4t_5t_6]] \in T$.Therefore T is regular.

Theorem 3.2 *Let T be a regular ordered ternary semigroup and B be an ordered bi-ideal of T, then $B = ([BTBTB]]$.*

Proof. Since B is an ordered bi-ideal of T, therefore $[BTBTB] \subseteq B$ and $(B] = B$.
Hence $([BTBTB]] \subseteq (B] = B$.——————(1)
Let $b \in B \subseteq T$, since T is regular, then there exists $x \in T$ such that $b \leq [bxb] \leq [bx[bxb]] \in [BTBTB]$.
Therefore $b \in ([BTBTB]]$. Hence $B \subseteq ([BTBTB]]$.——————(2)
Therefore from (1) and (2), $B = ([BTBTB]]$.

Theorem 3.3 *An ordered ternary semigroup T in which all ordered bi-ideals are idempotent is regular.*

Proof. Let R, M and L be an ordered right, ordered lateral and ordered left ideal of T respectively.
Then by Theorem 2.12(4),$R \cap M \cap L$ is an ordered bi-ideal of T.
But by the hypothesis,
$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L)(R \cap M \cap L)(R \cap M \cap L) \subseteq [RML]$.
As $[RML] \subseteq R \cap M \cap L$.
Hence $[RML] = R \cap M \cap L$. Thus by Theorem 3.1 T is regular.

Theorem 3.4 *An ordered ternary semigroup T is simple if and only if it has no proper bi-ideal.*

Proof. First assume that T is a simple ordered ternary semigroup. Let B be an ordered bi-ideal of T.
Since B is non-empty set, then $([BTT]], ([TBT] \cup [TTBTT]]$ and $([TTB]]$ are ordered right, ordered lateral and ordered left ideals of T respectively, by Theorem 2.8(8). But T is simple.
Therefore $([BTT]] = T, ([TBT] \cup [TTBTT]] = T$ and $([TTB]] = T$.
Now $T = ([BTT]] = ([BT[TTB]] = ([BT[BTT]TB]] = ([BTB[TTT]B]] \subseteq ([BTBTB]] \subseteq (B] = B$ (Since B is an ordered bi-ideal of T).
Thus $T \subseteq B$. Hence $T = B$. Therefore B is not a proper bi-ideal of T. Hence T has no proper bi-ideal.
Conversely assume that T has no proper bi-ideal. Let L be an ordered left ideal of T. Then By Theorem 2.12(4), L is an ordered bi-ideal of T.
Hence $L = T$. Therefore T is a left simple. Similarly we can prove that T is right as well as lateral simple. Hence T is simple.

For a ternary semigroup T (without order), the set P(T) of all non-empty finite subsets of T is a semilattice ordered ternary semigroup with ternary operation [ ] and $\leq$ defined as follows:
For $A, B, C \in P(T)$,
$[ABC] = \{[a\ b\ c]/a \in A, b \in B, c \in C\}$
and
$A \leq B$ if and only if $A \subseteq B$.

Theorem 3.5 *If T is regular, then P(T) is regular.*

Proof. By Theorem 2.14, as T is regular, $a \in ([aTa]]$ for all $a \in T$.
Therefore $A \subseteq ([ATA]]$, for all $A \subseteq T$. Hence A is regular in P(T).
Therefore P(T) is regular.

Theorem 3.6 *Let T be a regular ordered ternary semigroup, then the following statements hold in T:*
(i) For every $a \in T$, $B(a) = ([R(a)M(a)L(a)]]$.
(ii)$([ATT]] \cap ([TAT]] \cap ([TTA]] = ([ATT] \cap [TAT] \cap [TTA]]$,*for any non-empty subset A of T.*

Proof. (i) Let $b \in B(a)$. As T is a regular ordered ternary semigroup, there exits $x \in T$ such that $a \leq [axa]$.
Now $b \in B(a)$ implies $b \leq a \leq [axa] \leq [ax[axa]] \leq [[axa][xax]a] \in [R(a)M(a)L(a)]$.
Therefore $b \in ([R(a)M(a)L(a)]]$.
Hence $B(a) \subseteq ([R(a)M(a)L(a)]]$.——————(1)
Again for $y \in ([R(a)M(a)L(a)]]$, there exist $t_1, t_2, t_3, t_4, t_5, t_6 \in T$ such that

$y \leq [[at_1t_2][t_3at_4][t_5t_6a]] = [a[t_1t_2t_3]a[t_4t_5t_6]a] \subseteq [aTaTa]$
Therefore $y \in ([aTaTa]] = B(a)$.
Hence $([R(a)M(a)L(a)]] \subseteq B(a)$.——————(2)
Therefore from (1) and (2), For every $a \in T$, $B(a) = ([R(a)M(a)L(a)]]$.
(ii) Let A be a non-empty subset of T and let $x \in ([ATT]] \cap ([TAT]] \cap ([TTA]$.
Then there exist $t_1, t_2, t_3, t_4, t_5, t_6 \in T$ and $a, b, c \in A$ such that $x \leq [at_1t_2]$, $x \leq [t_3bt_4]$ and $x \leq [t_5t_6c]$.
As T is a regular ordered ternary semigroup. Therefore for $x \in T$ there exits $z \in T$ such that $x \leq [xzx]$.
$x \leq [xzx] \leq [xzxzx] \leq [[at_1t_2]z[t_3bt_4]z[t_5t_6c]]$.
Now $[a[t_1t_2zt_3b][t_4zt_5t_6c]] \in [ATT]$,
$[[at_1t_2zt_3]b[t_4zt_5t_6c]] \in [TAT]$,
$[[at_1t_2zt_3][bt_4zt_5t_6]c] \in [TTA]$.
Therefore $[[at_1t_2]z[t_3bt_4]z[t_5t_6c] \in [ATT] \cap [TAT] \cap [TTA]$.
Hence $x \in ((([ATT]] \cap ([TAT]] \cap ([TTA]]]$.
Thus $([ATT]] \cap ([TAT]] \cap ([TTA]] \subseteq ((([ATT]] \cap ([TAT]] \cap ([TTA]]]$.
$((([ATT]] \cap ([TAT]] \cap ([TTA]]] \subseteq ([ATT]] \cap ([TAT]] \cap ([TTA]]$ is obviously true, by Theorem 2.8 (6).
Therefore $((([ATT]] \cap ([TAT]] \cap ([TTA]]] = ([ATT]] \cap ([TAT]] \cap ([TTA]]$.

Theorem 3.7 *Let X, Y, Z be three non-empty subsets of an ordered ternary semigroup T and $N = ([XYZ]]$. Then N is an ordered bi-ideal of T if one of the following conditions holds*
(1)X is an ordered bi-ideal of T.
(2)Y is an ordered bi-ideal of T.
(3)Z is an ordered bi-ideal of T.
(4)At least one of X, Y, Z is an ordered left or an ordered right or ordered lateral ideal of T.

Proof.(1) Since $N = ([XYZ]]$ and X is an ordered bi-ideal of T, we have $[NTNTN] = [([XYZ]]T([XYZ]]T([XYZ]]] \subseteq ([[X[TTT]X[TTT]X]YZ]] \subseteq ([[XTXTX]YZ]] \subseteq ([XYZ]] = N$ and $(N] = N$.
Hence N is an ordered bi-ideal of T.
(2) and (3) can be proved similarly as (1).
(4)Assume that X is an ordered right ideal of T.Now we have,

$$\begin{aligned}[NTNTN] &= [([XYZ]]T([XYZ]]T([XYZ]]] \\ &\subseteq ([[X[TTT]T[TTT]T]YZ]] \\ &\subseteq ([[X[TTT]T]YZ]] \\ &\subseteq ([[XTT]YZ]] \\ &\subseteq ([XYZ]] = N.\end{aligned}$$

and $(N] = N$.
Therefore N is an ordered bi-ideal of T.

Similar proofs can be given when either X or Y or Z is an ordered left or an ordered right or ordered lateral ideal of T.

Theorem 3.8 *Let X, Y, Z be three non-empty subsets of an ordered ternary semigroup T and* $N = ([XYZ]]$*. Then N is an ordered sub-idempotent bi-ideal of T if one of the following conditions holds*
(1)$Y, Z \subseteq X$ *and X is an ordered sub-idempotent bi-ideal of T.*
(2)$X, Z \subseteq Y$ *Y is an ordered sub-idempotent bi-ideal of T.*
(3)$X, Y \subseteq Z$ *Z is an ordered sub-idempotent bi-ideal of T.*
(4)At least one of X, Y, Z is an ordered left or an ordered right or ordered lateral ideal of T.

Proof. (1) Since $N = ([XYZ]]$, $Y, Z \subseteq X$ and X is an ordered bi-ideal of T, we have
$[NNN] = [([XYZ]]([XYZ]]([XYZ]]]$
$\subseteq ([XXX][XXX][XYZ]]$
$\subseteq ([[XXX]YZ]]$
$\subseteq ([XYZ]] = N$ and $[NTNTN] = [([XYZ]]T([XYZ]]T([XYZ]]]$
$\subseteq ([[X[TTT]X[TTT]X]YZ]]$
$\subseteq ([[XTXTX]YZ]] \subseteq ([XYZ]] = N$
and $(N] = N$.
Hence N is an ordered sub-idempotent bi-ideal of T.
(2) and (3) can be proved similarly as (1).
(4)Assume that X is an ordered right ideal of T.Now we have,
$[NNN] = [([XYZ]]([XYZ]]([XYZ]]]$
$\subseteq ([XTT][TTT][TYZ]]$
$\subseteq ([[X[TTT]T]YZ]]$
$\subseteq ([[XTT]YZ]]$
$\subseteq ([XYZ]] = N$
and $[NTNTN] = [([XYZ]]T([XYZ]]T([XYZ]]]$
$\subseteq ([[X[TTT]T[TTT]T]YZ]]$
$\subseteq ([[X[TTT]T]YZ]]$
$\subseteq ([[XTT]YZ]]$
$\subseteq ([XYZ]] = N$.
and $(N] = N$.
Therefore N is an ordered sub-idempotent bi-ideal of T.
Similar proofs can be given when either X or Y or Z is an ordered left or an ordered right or ordered lateral ideal of T.

By Theorem 2.12(5), every ordered quasi-ideal of an ordered ternary semigroup T is an ordered bi-ideal of T, but converse need not be true.

Now we use Theorem 3.7 to give an example of an ordered ternary semigroup in which an ordered bi-ideal is not an ordered quasi-ideal of T.

Example 3.9 *Let T be an ordered ternary semigroup such that T is not regular and X, Y, Z be respectively a minimal ordered right, a minimal ordered lateral,a minimal ordered left ideal of T satisfying the condition of Theorem 3.8. Thus* $N = ([XYZ]]$ *is an ordered bi-ideal of T. We will show that N is not an ordered quasi-ideal of T.*

Proof. As $([XYZ]] \subseteq ([XTT]] \subseteq (X] \subseteq X$
$([XYZ]] \subseteq ([TYT]] \subseteq (Y] \subseteq Y$
$([XYZ]] \subseteq ([TTZ]] \subseteq (Z] \subseteq Z.$
Therefore $([XYZ]] \subseteq X \cap Y \cap Z$ which is a a minimal ordered quasi-ideal of T.

If we assume that $([XTT]]$ is an ordered quasi-ideal of T, then $([XYZ]] = X \cap Y \cap Z$ (By Theorem 3.1).

Thus T is regular. Hence it contradicts the hypothesis. So $([XYZ]]$ is not an ordered quasi-ideal, but an ordered bi-ideal of T (By Theorem 3.8).

Theorem 3.10 *In a regular ternary ordered semigroup every ordered bi-ideal is an ordered quasi-ideal.*

Proof. Let T be an ordered ternary semigroup.If B is an ordered quasi-ideal of T, then, from Theorem 2.12(5), it follows that B is an ordered bi-ideal of T. Conversely, let B be an ordered bi-ideal of T. From Theorem 3.1, An ordered ternary semigroup T is regular if and only if for every ordered right ideal R, ordered lateral ideal M and ordered left ideal L of T, $([RML]] = R \cap M \cap L$.
Now, $[BTT] \cap [TBT] \cap [TTB] = ([BTT][TBT][TTB]] = ([B[TTT]B[TTT]B] \subseteq ([BTBTB]] \subseteq (B] = B$.
and $[BTT] \cap [TTBTT] \cap [TTB] = ([BTT][TTBTT][TTB]] = ([B[TTT][TBT][TTT]B] \subseteq ([BTTBTTB]] \subseteq (B] = B$.
Hence B is an ordered quasi-ideal of T.

From Theorem 3.1 and Theorem 3.10, we get the following corollary

Corollary 3.11 *In a regular ordered ternary semigroup T, a non-empty subset B of T is a bi-ideal of T if and only if* $B = (RML] = R \cap M \cap L$ *for an ordered right ideal R, ordered lateral ideal M and ordered left ideal L of T.*

4 Open Problem

Here the open problem is to characterize ordered bi-ideals and ordered quasi-ideals in completely regular ordered ternary semigroups.

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