Int. J. Open Problems Compt. Math., Vol. 11, No. 1, March 2018
ISSN 1998-6262; Copyright © ICSRS Publication, 2018
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Quantum Logic Fuzzy Co-implication

(Some Properties and Applications)

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Abstract: ***Recently, many works have appeared of Fuzzy implications over triangular norms and triangular conorms. In this paper, we attempt a systematic study of Quantum logic Co-implications generated from a t-norm, t-conorm and strong negation. Also, some examples as well as application are discussed as well.***

Keywords: ***Fuzzy implications, Fuzzy Co-implications, QL-implication, QL-Coimplication.***

1. Introduction

A Quantum logic (in short, QL) connective plays an important role in computer programming [3]. In designing multi-valued logic circuits [7], the concepts of multi-valued fuzzy functions have been explored [1]. Fuzzy logic and fuzzy sets are basic framework when working with vague notions. In fuzzy logic, the classical binary negation, conjunction, disjunction and implication are extended to mappings that take values in the unit interval respectively. A fuzzy negation operator is normally modeled as a fuzzy negation. A fuzzy conjunction operator is normally modeled as a conjunction on the unit interval or (more usually) as a triangular norm (in short, t-norm)[16]. A fuzzy disjunction operator is normally modeled as a triangular co norm (t-conorm for short)[20]. There are many approaches to model a fuzzy implication operator. It can be constructed from the other three fuzzy logic operators, or it can be constructed from some parameterized generating functions.

2. Preliminaries

We suppose that the reader is customary with the classical results of fuzzy logic implications [14],[15], and [18].

2.1 Triangular Norm and Triangular Conorm

The conjunction $\wedge$ in fuzzy logic, it is often modeled as follows:

Definition 2.1. [9] A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a triangular norm (in short, t-norm), if for all $r,q,z,w\in[0,1]$ the following requirements are satisfied

$(T.1)\ T(r,q)=T(q,r)$, (T is commutative)

$(T.2)\ T(r,q)\le T(z,w)$ whenever $r\le z$ and $q\le w$, (T is increasing)

$(T.3)\ T(r,T(q,z))=T(T(r,q),z)$, ($T$ is associative)

$(T.4)\ T(r,1)=r$. (T has1 as identity)

Definition 2.2. [13] A mapping $S:[0,1]\times[0,1]\to[0,1]$ is a triangular conorm (in short, t-conorm), if for all $r,q,z,w\in[0,1]$ the following conditions are satisfied

$(S.1)\ S(r,q)=S(q,r)$. (S is commutative)

$(S.2)\ S(r,q)\le S(z,w)$ whenever $r\le z$ and $q\le w$. (S is increasing)

$(S.3)\ S(r,S(q,z))=S(S(r,q),z)$. ($S$ is associative)

$(S.4)\ S(r,0)=0$. (S has 0 as identity)

Proposition 2.1. [19] A mapping S is a triangular conorm iff there exists a triangular norm T such that $S(z,w)=1-T(1-z,1-w)$, $\forall z,w\in[0,1]$. In this case S is called the dual t-conorm of T.

The standard examples of t-norm and dual t-conorms are stated in the following:

t--norm (T)	**Dual t-conorm (*S*)**
$M(p,q)=\min(p,q)$. (Minimum t-norm)	$S_M(p,q)=\max(p,q)$. (Maximum t-conorm)
$\Pi(p,q)=pq$, (Product t-norm)	$S_M(p,q)=p+q-pq$, (Probabilistic sum t-conorm)
$W(p,q)=\begin{cases}p & \text{if } q=1,\\ q & \text{if } p=1,\\ 0 & \text{if } p,q\in[0,1).\end{cases}$ (Drastic or weak t-norm)	$S_W(p,q)=\begin{cases}p & \text{if } q=1,\\ q & \text{if } p=1,\\ 1 & \text{otherwise.}\end{cases}$ (Drastic or largest t-conorm)

$N(p,q)=\begin{cases}\min(p,q) \text{ if } p+q\geq 1,\\ 0 \quad\quad\quad \text{if } p+q<1.\end{cases}$ (Nilpotent t-norm)	$\mathrm{S}_N(p,q)=\begin{cases}\max(p,q) \text{ if } p+q<1,\\ 0 \quad\quad \text{if } p+q\geq 1.\end{cases}$ (Nilpotent t-conorm)
$L(p,q)=\max(p+q-1,0),$ (Lukasiewicz t-norm)	$\mathrm{S}_L(p,q)=\min(p+q,1).$ (Bounded Sum t-conorm)
$H(p,q)=\begin{cases}0 \quad\quad\quad\quad \text{if } p=q=0,\\ \dfrac{pq}{p+q-pq} \quad \text{otherwise.}\end{cases}$ (Hamacher t-norm)	$S_H(p,q)=\begin{cases}0 \quad\quad \text{if } p=q=0,\\ \dfrac{p+q-2pq}{1-pq} \quad \text{otherwise.}\end{cases}$ (Hamacher t-conorm)
$D_\alpha(p,q)=\dfrac{pq}{\max(p,q,\alpha)}, \quad \alpha\in(0,1).$ (Dubois-Prade t-norm)	$\mathrm{S}_{D_\alpha}(p,q)=1-\dfrac{(1-p)(1-q)}{\max(1-p,1-q,\alpha)},$ $\alpha\in(0,1).$ (Dubois − Prade t-conorm)

2.2 Negation Function

Definition 2.3. [4] A mapping N from $[0,1]$ into $[0,1]$ is a negation function, iff:

1. $N(0)=1, N(1)=0;$
2. $N(p)\leq N(q),$ if $p\geq q.\ \forall p,q\in[0,1].$ (Monototonicity)

A negation function is strict, iff:

1. $N(p)$ is continuous;
2. $N(p)<N(q),$ if $p>q.\ \forall p,q\in[0,1].$

A strict negation function is strong or volutive, iff:

1. $N(N(p))=p, \forall p\in[0,1].$

A negation function is weak, iff N is not strong.

Example 2.1.[15] The strong negation $(N_C(p)=1-p)$, strict negation but not strong $(N_k(p)=1-p^2)$, weaker negations $\left(N_{D_1}(p)=\begin{cases}1 \text{ if } p=0,\\ 0 \text{ if } p>0.\end{cases}\right)$, and strongest negations $\left(N_{D_2}(p)=\begin{cases}1 \text{ if } p<1,\\ 0 \text{ if } p=1.\end{cases}\right)$.

Definition 2.4. [9] Let T be a t-norm and S be a t-conorm. A mapping N_T, N_S from $[0,1]$ into $[0,1]$ defined by:

$$N_T(p) = \sup\{r \in [0,1] : T(p,r) = 0\}, \text{ for every } p \in [0,1],$$
$$N_S(p) = \inf\{r \in [0,1] : S(p,r) = 1\}, \text{ for every } p \in [0,1],$$

are called the natural negation of T and S, respectively.

3. Fuzzy Implications

In classical logic all assertions are either true or false (i.e have truth values 1 or 0 respectively). In case of fuzzy logic the truth value may be any value in the interval [0 , 1]. Connected with fuzzy logic is the notion of fuzzy sets. Classical set is given by it's characteristic function with values 0 and 1. Likewise, a fuzzy set is given by it's membership function with values from interval $[0.1]$.

In the following they are four ways to define an implication in the Boolean lattice (L,∧,∨,¬):

(1) $p \Rightarrow q \equiv \neg p \vee q$. (S-Implication, see [13] and [20])

(2) $p \Rightarrow q \equiv max\{t \in L \mid r \wedge t \leq q\}$. (R-Implication, see [12] and [14])

(3) $p \Rightarrow q \equiv \neg p \vee (p \wedge q)$. (Quantum logic, see [14] and [20])

(4) $p \Rightarrow q \equiv q \vee (\neg p \wedge \neg q)$. (D-Implication, see [14] and [20])

where $p.q \in L$.

A fuzzy implication I is an extension of the implication operator in the classical binary logic. So I must satisfy at least the boundary conditions

$$I1:\ I(0,0) = I(0,1) = I(1,1) = 1 \text{ and } I(1,0) = 0.$$

Besides $I1$, there are several other potential axioms for I to satisfy in different theories and applications, among which the most important ones are (notice that $FI5$ is a part of $I1$): [14]

$(FI.1)$ $(\forall x_1, x_2, y \in [0,1])(x_1 < x_2) \rightarrow I(x_1, y) \geq I(x_2, y))$.

$(FI.2)$ $(\forall (x, y_1, y_2) \in [0,1])(y_1 < y_2) \rightarrow I(x_1, y) \leq I(x_2, y))$,

$(FI.3)$ $(\forall x \in [0,1])(I(0, x) = 1)$,

$(FI.4)$ $(\forall x \in [0,1])(I(x, 1) = 1)$,

$(FI.5)$ $I(1,0) = 0$,

$(FI.6)$ $(\forall x \in [0,1])(I(1, x) = x)$,

$(FI.7)$ $(\forall(x,y,z)\in[0,1])\left(I\big(x,I(y,z)\big)=I\big(y,I(x,z)\big)\right)$,

$(FI.8)$ $(\forall(x,y)\in[0,1])(I(x,y)=1\rightarrow x\leq y)$,

$(FI.9)$ $(\forall x\in[0,1])\big(N(x)=I(x,0)\big)$, is a strong fuzzy negation(SN).

$(FI.10)$. $(\forall(x,y)\in[0,1]^2)(I(x,y)\geq y)$,

$(FI.11)$. $(\forall x\in[0,1])(I(x,x)=1)$,

$(FI.12)$ $(\forall x,y\in[0,1])(I(x,y)=I(N(y),N(x)))$, where$N$ is a strong fuzzy negation.

Definition 3.1. [9] A function $I:[0,1]\times[0,1]\rightarrow[0,1]$ is called an (S,N) implication if there exists a *t −conorm* S and a fuzzy negation N such that :

$$I(x,y)=S\left(N(x),y\right),x,y\in[0,1]$$

Definition 3.2. [9] Let T a left-continuous t-norm. Then, the residual implication or R-implication derived from T is given by:

$$I_T(x,y)=sup\{z\in[0,1]\backslash T(z,x)\leq y\} \text{ for every } x,y\in[0,1].$$

4. **QL-Implication**

A QL-implication is generated from a strong fuzzy negation, a t-conorm and a t-norm, getting idea from the equivalency in classical binary logic:

$$r\Rightarrow q\equiv\big(\neg r\vee(r\wedge q)\big),\forall\, r,q\in[0,1].$$

Let *S* be a t-conorm, *N* be a strong fuzzy negation and *T* be a t-norm. A QL-implication is defined by:

$$I(p.q)=S\big(N(p),T(p,q)\big),\forall p,q\in[0,1].$$

Table 4.1.

Name	Formula of $I(x.y)$	S-implication	R-implication	QL-implication
Kleen-Dienes I_b	$I_b=\max(1-x,y)$	1	0	1
Lukasiewicz I_L	$I_L(x,y)=\min(1-x+y,1)$	1	1	1
Least strickt I_{LR}	$\mathrm{I}_{LR(x.y)}=\begin{cases}y & \text{if } x=1.\\ 1 & \text{otherwise.}\end{cases}$	1	1	1
Reichenbrach I_r	$I_r(x,y)=1-x+xy$	1	0	1
Early Zadeh I_m	$I_m(x.y)=Max(1-x,min(x.y))$	0	0	1
Klir and Yuan I_p	$I_p(x,y)=1-x+x^2y$	0	0	1

Table 4.1 lists the popular fuzzy implications in the literature, which belong to the classes S- implications, R- implications or QL-implications. For the intersections of S-implications, R- implications and QL-implications, we refer to [13], [14], and [20].

QL-implications satisfy less axioms than S-implications and R-implications generated from t-norms. First we see which axioms a QL-implication satisfies [15].

A QL-implication I generated from a t-conorm S, a strong fuzzy negation N and a t-norm T satisfies:

$(FI.2)$ $(\forall x, y_1, y_2 \in [0,1])(y_1 < y_2) \rightarrow I(x_1, y) \leq I(x_2, y))$;

$(FI.3)$ $(\forall x \in [0,1])(I(0, x) = 1)$;

$(FI.5)$ $I(1,0) = 0$;

$(FI.6)$ $(\forall x \in [0,1])(I(1, x) = x)$;

$(FI.9)$ $(\forall x \in [0,1])\big(N(x) = I(x, 0)\big)$, is a strong fuzzy negation(S_N);

The most strict implication I_M given in Table 4.1 also satisfies $(FI.2), (FI.3), (FI.5), (FI.6)$ and $(FI.9)$ (because it is an S-implication). We prove that I_M is not a QL-implication, Assume I_M to be a QL-implication. Then there exist a t-conorm S, a strong fuzzy negation N and a t-norm T such that:

$$I_M(p, q) = S\big(N(p), T(p, q)\big) = S_M(N_{1b}(p), q), \forall p, q \in [0,1].$$

Take $q = 0$, we have $N(p) = N_{1b}(p)$. So

$$I_M(p, q) = S\big(N_{1b}(p), T(p, q)\big) = \begin{cases} 1, & \text{if } \ p = 0. \\ T(p, q), & \text{otherwise.} \end{cases}$$

So we obtain $T(p, q) = q$, for all $p > 0$. If we take $x_0 = 0.1$ and $y_0 = 0.2$, then $T(x_0, y_0) = 0.2$ while $T(x_0, y_0) = 0.1$, which means that T does not satisfy $(T.3)$. So T is not a t-norm, which is a contradiction with the assumption. Thus I_M is not a QL-implication.

Hence a: [0.1]×[0.1] → [0.1] mapping satisfying $(FI.2), (FI.3), (FI.5), (FI.6)$ and$(FI.9)$ is not always a QL-implication.

Proposition 4.1. [17] Let S be a t-conorm, T be a t-norm and N be a strong negation, then $\mathrm{S}\big(\mathrm{p}.\mathrm{N}(\mathrm{p})\big) = 1$ is satisfied $\forall \mathrm{p} \in [0.1]$ for the QL-implication $I(p, q) = \mathrm{S}\big(\mathrm{N}(\mathrm{p}), \mathrm{T}(\mathrm{p}, \mathrm{q})\big)$ and also satisfy $(\mathrm{FI}.1), (\mathrm{FI}.7), (\mathrm{FI}.8)$ or $(\mathrm{FI}.12)$.

5 QL-Coimplication

This section will be devoted to introduce the concept of QL-Coimplication. The relation between classical logic and fuzzy logic as well as some examples are also discussed.

Definition 5.1.[17] A mapping $J: [0,1]\times[0,1] \to [0,1]$ is a fuzzy co-implication if, for all $p, q, r \in [0,1]$, the following conditions are satisfied:

$(J.1): J(1,1) = J(1,0) = J(0,0) = 0$ and $J(0,1) = 1$.

$(J.2): J(p,q) \geq J(r,q)\ if\ p \leq r$.

$(J.3): J(p,q) \leq J(p,r)\ if\ q \leq r$.

The set of all fuzzy co-implications is denoted by $Co - FI$.

From the previous definition we can deduce that for each fuzzy co-implication $J(1,q) = J(p,0) = 0$ for each $p.q \in [0,1]$. Moreover, J satisfies also the normality condition $J(p,p) = 0$.

Lemma 5.1. [6] If a function $J: [0,1]^2 \to [0,1]$ satisfies $(I.1)$ and $(I.2)$, then the map $N_J: [0,1] \to [0,1]$ defined by:

$$N_J(p) = J(p,1).\forall p \in [0,1]$$

is a fuzzy negation.

If J is a fuzzy co-implication, then the function N_J is called the natural negation of J. In the following we list a few of the most important properties of fuzzy co-implication. They are generalizations of the corresponding properties of the classical implication.

Definition 5.2.[11]A fuzzy implication *J* is said to satisfy

(i) The left neutrality property if

$$J(0,b) = b.\ \forall\, b \in [0,1]; \qquad \text{(Co-NP)}$$

(ii) The exchange principle, if

$$J\big(a,J(b,c)\big) = J\big(b,J(a,c)\big), \forall a,b,c \in [0,1]\,; \qquad \text{(Co-EP)}$$

(iii) The identity principle, if

$$J(a,a) = 0,\ \forall a \in [0,1]\,; \qquad \text{(Co-IP)}$$

(iv) The ordering property, if

$$J(a,b) = 0 \Leftrightarrow a \geq b,\ \forall a,b \in [0,1]. \qquad \text{(Co-OP)}$$

In 2016 Jebril introduce the definition of (T,N) co-implication and residual co-implication (R^*-implication) in dual Heting algebra. [9]

Definition 5.3. [9] A mapping J from $[0,1]^2$ into $[0,1]$ is called an (T,N) co_implication if there exists a t-norm T and a fuzzy negation N such that

$$J_{T,N}(p,q) = T(q, N(p)),\ \forall p,q \in [0,1].$$

Definition 5.4. [9] Let S is the t-conorm of right continuous T. Then, the residual co-implication (R^*-implication) derived from S, is

$$J_S(p,q) = \inf\{r \in [0,1] \setminus S(r,p) \geq q\},\ \forall p,q \in [0,1].\ (R^*)$$

Fuzzy co-implications are extensions of the Boolean co-implication ($p \not\Leftarrow q$) meaning that p is not necessary for q. In classical logic, the operator `$\not\Rightarrow$' (material co-implication) is generated by Boolean negation `$\neg$' and conjunction `$\wedge$'

$$q \not\Rightarrow p \equiv ((p \vee q) \wedge \neg p).$$

In the following table 5.5, we can see the truth table for the classical co-implication.

Table 5.5

p	q	$p \Rightarrow q$	$(p \vee q)$	$\neg p$	$(p \vee q) \wedge \neg p$	$q \not\Rightarrow p$
0	0	1	0	1	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	0
1	1	1	1	0	0	0

Definition5.5. A function J: [0.1]×[0.1] → [0.1] is called an QL-Coimplication if there exists a t-norm T and a fuzzy negation N such that :

$$J(p,q) = T\big(S(p,q), N(p)\big),\ \forall p,q \in [0.1].$$

If N is a strong negation, then J is called a strong QL-Coimplication, If N is not strong negation then J is called non-strong QL-Coimplication. Moreover, if an QL-Coimplication is generated from T,S and N, then we will denote this by $J_{T.S.N}$. A relation between fuzzy negations and (S,N) implication is given in the next proposition.

Proposition 5.1. Let $J_{T.S.N}$ be an QL implication, then $N_{J_{T.S.N}} = N$.

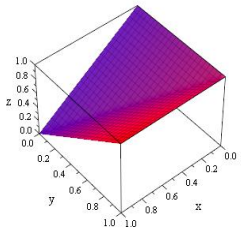
Proof: For any $x \in [0.1]$ we have

$$J_{T,S,N}(x) = J_{T,S,N}(x,1) = T\big(1, N(x)\big) = N(x).$$

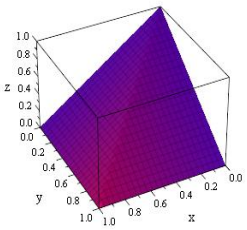
Remark 5.1. In the above, *N* is assumed to be a strong negation. However, *N* is not necessary supposed strong, even not necessary to be continuous.

Example 5.1. For any t-norm *T* and t-conorm *S* and a fuzzy negation *N* by graphing Kleene –Dienes Ql Implication

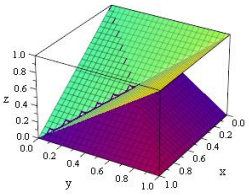
$$I_b = \max(1 - x, y).$$



Kleene -Dienes QL-implication $I_b = max(1 - x, y)$



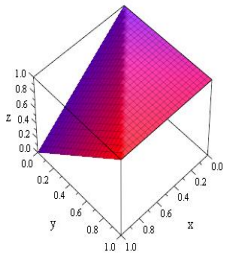
Kleene -Dienes QL-Coimplication $J_b = min(1 - y, 1 - x)$



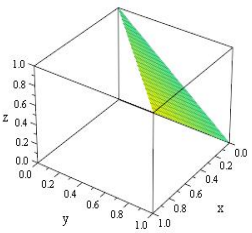
Kleene -Dienes QL-implication and QL-Coimplication I_b and J_b

Example 5.2. For any t-norm *T* and t-conorm S and a fuzzy negation *N* by graphing Łukasiewicz *QL* Implication

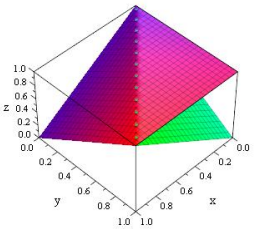
$$I_L = min(1 - x + y, 1).$$



Łukasiewicz QL-implication $I_L = min(1 - x + y, 1)$

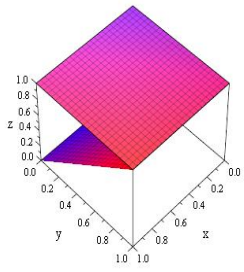


Łukasiewicz QL-Coimplication $J_L = max(0,1 - y + x)$

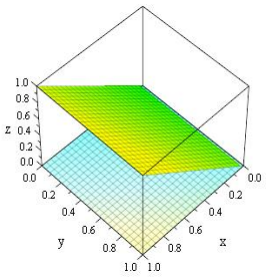


Łukasiewicz QL-implication and QL-Coimplication $I_L$and J_L

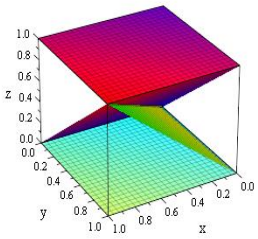
Example 5.3. For any t-norm and t-conorm *S* and a fuzzy negation *N* by graphing least strict I_{LR}.



Least strictI_{LR}
QL-implication
$I_{LR} = \begin{cases} y & \text{for } x = 1. \\ 1 & \text{otherwise.} \end{cases}$

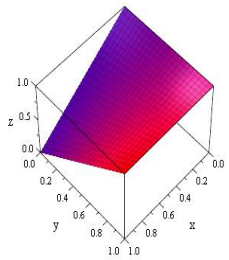


Least strickt I_{LR}
QL-Coimplication
$J_{LR} = \begin{cases} 0 & \text{otherwise.} \\ x & \text{if } y = 1. \end{cases}$

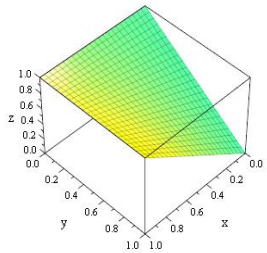


Least strict
QL-implication
and QL-Coimplication
I_{LR} and J_{LR}

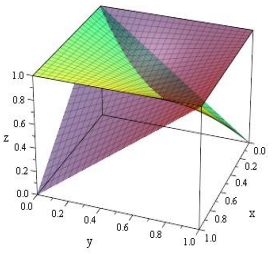
Example 5.4. For any t-norm *T* and t-conorm *S* and a fuzzy negation *N* by graphing Reichenbach $I_r(x, y) = 1 - x + xy$.



Reichenbach
QL-implication
$I_r(x, y) = 1 - x + xy$



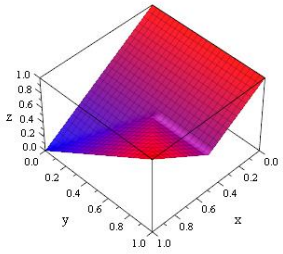
Reichenbach
QL-coimplication
$J_r(x, y) = xy - y = 1$



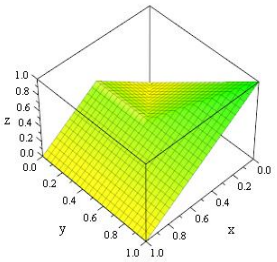
Reichenbach
QL-implication
and QL-coimplication
I_r and J_r

Example 5.5. For any t-norm *T* and t-conorm *S* and a fuzzy negation *N* by graphing Early Zadeh

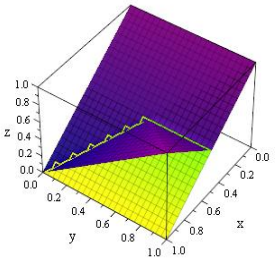
$$I_m(x, y) = max\big(1 - x, min(x, y)\big).$$



Early Zadeh
QL-implication
$I_m = max(1 - x, min(x, y))$



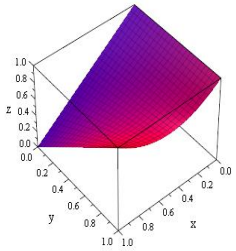
Early Zadeh
QL-Coimplication
$J_m = min(\max(x, y), 1 - x)$



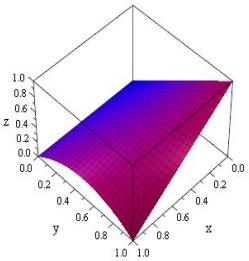
Early Zadeh
QL-implication
and QL-Coimplication
I_m and J_m

Example 5.6. For any t-norm *T* and t-conorm *S* and a fuzzy negation N by graphing Klir and Yuan QL implication

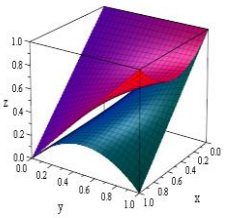
$$I_p(x, y) = 1 - x + x^2 y.$$



Klir and Yuan
QL-implication
$I_p = 1 - x + x^2 y$



Klir and Yuan
QL-Coimplication
$I_p = yx^2 + 1 - x$



Klir and Yuan
QL-implication
and QL-Coimplication
I_p and j_p

Conclusion

In this paper we have studied the class of *QL*-co-implications under certain conditions In crisp logic and Fuzzy logic as a generalization of the implication defined in Quantum logic $\Rightarrow q \equiv \neg p \vee (p \wedge q)$. However, *QL*-co-implications have not received as more attention as (*S*,*N*) and *R*-co-implications within Fuzzy logic. We wish that our new topic can be used in Fuzzy decision making or in multivariate Statistical Analysis.

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