A generalization of the Nielsen’s $\beta$-function

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Abstract

In this paper, we introduce a $p$-generalization of the Nielsen’s $\beta$-function. We further study among other things, some properties such as convexity, monotonicity and inequalities of the new function. In the end, we pose an open problem.

Keywords: Nielsen’s $\beta$-function, $p$-generalization, $p$-Gamma function, convolution theorem for Laplace transforms, completely monotonic.

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1 Introduction

The Nielsen’s $\beta$-function may be defined by any of the following equivalent forms (see [2], [3], [8], [11]).

\[ \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} \, dt, \quad x > 0, \]  
\[ = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} \, dt, \quad x > 0, \]  
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0, \]  
\[ = \frac{1}{2} \left\{ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right\}, \quad x > 0, \]
where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler’s Gamma function. It is known to satisfy the properties:

$$\beta(x + 1) = \frac{1}{x} - \beta(x),$$  \quad \text{(5)}

$$\beta(x) + \beta(1 - x) = \frac{\pi}{\sin \pi x}. \quad \text{(6)}$$

Lately, this special function has been studied in diverse ways. For instance, in [8], the author investigated some properties and inequalities of the function. Also, in [9], the function was applied to study some monotonicity and convexity properties and some inequalities involving a generalized form of the Wallis’ cosine formula. Then in [10], the author proved some monotonicity and convexity properties of the function. In this paper, we continue the investigation by establishing a $p$-generalization of this special function. In the meantime, we recall the following definitions concerning the $p$-analogue of the Gamma function. We shall use the notations $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The $p$-analogue (also known as $p$-extension or $p$-deformation) of the Gamma function is defined for $p \in \mathbb{N}$ and $x > 0$ as [1], [12]

$$\Gamma_p(x) = \frac{p! p^x}{x(x + 1) \ldots (x + p)} = \frac{p^x}{x(1 + \frac{x}{1}) \ldots (1 + \frac{x}{p})} \quad \text{(7)}$$

$$= \int_0^p \left(1 - \frac{t}{p}\right)^p t^{x-1} dt \quad \text{(8)}$$

where $\lim_{p \to \infty} \Gamma_p(x) = \Gamma(x)$. It satisfies the identities [5]

$$\Gamma_p(x + 1) = \frac{px}{x + p + 1} \Gamma_p(x),$$

$$\Gamma_p(1) = \frac{p}{p + 1}. \quad \text{(10)}$$

The $p$-analogue of the digamma functions is defined for $x > 0$ as [6]

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \ln p - \sum_{n=0}^{p} \frac{1}{n + x}, \quad \text{(9)}$$

$$= \ln p - \int_0^\infty \frac{1 - e^{-(p+1)t}}{1 - e^{-t}} e^{-xt} dt, \quad \text{(10)}$$

and satisfies the relation [5]

$$\psi_p(x + 1) = \frac{1}{x} - \frac{1}{x + p + 1} + \psi_p(x). \quad \text{(11)}$$

Also, it is well known in the literature that the integral

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt} dt$$

holds for $x > 0$ and $m \in \mathbb{N}_0$. 

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2 A $p$-Generalization of Nielsen’s $\beta$-function

In this section, we introduce a $p$-generalization of the Nielsen’s $\beta$-function and further study some of its properties.

**Definition 2.1.** The $p$-generalization of the Nielsen’s $\beta$-function is defined for $p \in \mathbb{N}$ as

\[
\beta_p(x) = \frac{1}{2} \left\{ \psi_p \left( \frac{x + 1}{2} \right) - \psi_p \left( \frac{x}{2} \right) \right\}, \quad x > 0,
\]

\[
= \sum_{n=0}^{p} \left( \frac{1}{2n + x} - \frac{1}{2n + x + 1} \right) x > 0,
\]

\[
= \int_0^{\infty} \frac{1 - e^{-2(p+1)t}}{1 + e^{-t}} e^{-xt} \, dt, \quad x > 0,
\]

\[
= \int_0^1 \frac{1 - t^{2(p+1)}}{1 + t} t^{x-1} \, dt, \quad x > 0,
\]

where $\beta_p(x) \to \beta(x)$ as $p \to \infty$.

**Remark 2.2.** The relations (14) and (15) are respectively derived from (9) and (10), and by a change of variable, (16) is obtained from (15).

**Proposition 2.3.** The function $\beta_p(x)$ satisfies the functional equation

\[
\beta_p(x+1) = \frac{1}{x} - \frac{1}{x + 2(p+1)} - \beta_p(x).
\]

**Proof.** By using representation (16), we obtain

\[
\beta_p(x+1) + \beta_p(x) = \int_0^1 \frac{1 - t^{2(p+1)}}{1 + t} t^x \, dt + \int_0^1 \frac{1 - t^{2(p+1)}}{1 + t} t^{x-1} \, dt
\]

\[
= \int_0^1 \frac{1 - t^{2(p+1)}}{1 + t} t^x \left( \frac{t + 1}{t} \right) \, dt
\]

\[
= \int_0^1 (1 - t^{2(p+1)}) t^{x-1} \, dt
\]

\[
= \frac{1}{x} - \frac{1}{x + 2(p+1)},
\]

which completes the proof. \(\square\)

As an immediate consequence of (17), we obtain the upper bound

\[
\beta_p(x) \leq \frac{1}{x} - \frac{1}{x + 2(p+1)}.
\]
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Also, successive applications of (17) yields the generalized form

$$\beta_p(x+n) = \sum_{s=0}^{n-1} \frac{(-1)^{s+n+1}}{x+s} + \sum_{s=0}^{n-1} \frac{(-1)^{s+n}}{x+s+2(p+1)} + (-1)^n \beta_p(x), \ n \in \mathbb{N}. \quad (19)$$

Also, successive differentiations of (13), (15), (16) and (17) yields respectively

$$\beta_p^{(n)}(x) = \frac{1}{2n+1} \left\{ \psi_p^{(n)} \left( \frac{x+1}{2} \right) - \psi_p^{(n)} \left( \frac{x}{2} \right) \right\}, \quad (20)$$

$$= (-1)^n \int_0^\infty \frac{t^n - t^n e^{-2(p+1)t}}{1 + e^{-t}} dt, \quad (21)$$

$$= \int_0^1 \frac{(\ln t)^n - (\ln t)^n t^{2(p+1)}}{1 + t} t^{x-1} dt, \quad (22)$$

$$\beta_p^{(n)}(x+1) = \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{(x+2(p+1))^{n+1}} - \beta_p^{(n)}(x), \quad (23)$$

where $n \in \mathbb{N}_0$ and $\beta_p^{(n)}(x) \to \beta^{(n)}(x)$ as $p \to \infty$.

**Remark 2.4.** It follows easily from (20)-(22) that:

(a) $\beta_p(x)$ is positive and decreasing,

(b) $\beta_p^{(n)}(x)$ is positive and decreasing if $n \in \mathbb{N}_0$ is even,

(c) $\beta_p^{(n)}(x)$ is negative and increasing if $n \in \mathbb{N}_0$ is odd.

**Theorem 2.5.** The function $\beta_p(x)$ satisfies the inequality

$$\beta_p \left( \frac{x+y}{u+v} \right) \leq [\beta_p(x)]^{\frac{1}{u}} [\beta_p(y)]^{\frac{1}{v}}, \ x, y \in (0, \infty), \quad (24)$$

where $u > 1, v > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$. Put in another way, the function $\beta_p(x)$ is logarithmically convex on $(0, \infty)$.

**Proof.** Let $u > 1, v > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$ and $x, y \in (0, \infty)$. Then Hölder’s inequality implies

$$\beta_p \left( \frac{x+y}{u+v} \right) = \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{x+y-1} dt$$

$$= \int_0^1 \left( \frac{1-t^{2(p+1)}}{1+t} t^{x-1} \right)^{\frac{1}{u}} \left( \frac{1-t^{2(p+1)}}{1+t} t^{y-1} \right)^{\frac{1}{v}} dt$$

$$\leq \left( \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{x-1} dt \right)^{\frac{1}{u}} \left( \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{y-1} dt \right)^{\frac{1}{v}}$$

$$= [\beta_p(x)]^{\frac{1}{u}} [\beta_p(y)]^{\frac{1}{v}},$$

which gives the desired result. \qed
Remark 2.6. As a by-product of Theorem 2.5, we obtain immediately the following results.

(a) The inequality \( \beta_p(x)\beta_p''(x) \geq (\beta_p'(x))^2 \) holds for \( x \in (0, \infty) \).

(b) The function \( \frac{\beta_p(x)}{\beta_p'(x)} \) is increasing on \( (0, \infty) \).

Corollary 2.7. The inequalities

\[
\begin{align*}
[\beta_p(x + y)]^2 &< \beta_p(x)\beta_p(y), \\
\beta_p(x + y) &< \beta_p(x) + \beta_p(y),
\end{align*}
\]  

hold for \( x, y \in (0, \infty) \).

Proof. Let \( u = v = 2 \) in Theorem 2.5. Then by the decreasing property of \( \beta_p(x) \), it follows easily that

\[
\beta_p(x + y) < \beta_p\left(\frac{x + y}{2}\right) \leq \sqrt{\beta_p(x)\beta_p(y)},
\]

which gives (25). Next, by (27) and the basic AM-GM inequality, we obtain

\[
\beta_p(x + y) < \sqrt{\beta_p(x)\beta_p(y)} \leq \frac{\beta_p(x)}{2} + \frac{\beta_p(y)}{2} \leq \beta_p(x) + \beta_p(y),
\]

which gives (26). \( \square \)

Corollary 2.8. The inequality

\[
1 < \frac{\beta_p(z)}{\beta_p(z + 1)} < \frac{\beta_p(z - 1)}{\beta_p(z)}
\]  

holds for \( z > 1 \).

Proof. Let \( z > 1 \). Then the left-hand side of (28) follows directly from the decreasing property of \( \beta_p(x) \). Next, by letting \( x = z - 1 \) and \( y = z + 1 \) in right-hand side of (27), we obtain

\[
\beta_p^2(z) < \beta_p(z - 1)\beta_p(z + 1),
\]

which when rearranged, gives the right-hand side of (28). Alternatively, we could proceed as follows. Let \( f(x) = \frac{\beta_p(x)}{\beta_p(x + 1)} \) for \( x > 0 \). Then

\[
f'(x) = f(x) \left[ \frac{\beta_p'(x)}{\beta_p(x)} - \frac{\beta_p'(x + 1)}{\beta_p(x + 1)} \right] < 0,
\]

which implies that \( f(x) \) is decreasing. Hence \( f(z) < f(z - 1) \) which also gives the right-hand side of (28). \( \square \)
Theorem 2.9. The function

\[ \phi(x) = u^x \beta_p(x), \quad u > 0, \]  

is convex on \((0, \infty)\).

Proof. Let \(a > 1, b > 1, \frac{1}{a} + \frac{1}{b} = 1\) and \(x, y \in (0, \infty)\). Then the log-convexity of \(\beta_p(x)\) implies

\[ \phi \left( \frac{x}{a} + \frac{y}{b} \right) = u^{x+b} \beta_p \left( \frac{x}{a} + \frac{y}{b} \right) \leq [u^x \beta_p(x)]^\frac{1}{a} [u^y \beta_p(y)]^\frac{1}{b}, \]

and by the classical Young's inequality, we obtain

\[ [u^x \beta_p(x)]^\frac{1}{a} [u^y \beta_p(y)]^\frac{1}{b} \leq \frac{u^x \beta_p(x)}{a} + \frac{u^y \beta_p(y)}{b} = \frac{\phi(x)}{a} + \frac{\phi(y)}{b}. \]

Hence, \(\phi(x)\) is convex on \((0, \infty)\). \(\square\)

Theorem 2.10. The inequality

\[ \exp \left\{ \beta_p \left( x + \frac{1}{2} \right) \right\} \leq \frac{\Gamma_p \left( \frac{x}{2} + 1 \right) \Gamma_p \left( \frac{x}{2} + \frac{1}{2} \right)}{\Gamma_p ^2 \left( \frac{x}{2} + \frac{1}{2} \right)} \leq \exp \left\{ \frac{1}{2x - \frac{1}{2x + 4(p+1)}} \right\} \]  

holds for \(x > 0\).

Proof. We make use of the Hermite-Hadamard's inequality

\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(s) \, ds \leq \frac{f(a) + f(b)}{2}, \]  

for a convex function \(f : (a,b) \subset \mathbb{R} \to \mathbb{R}\). Since every logarithmically convex function is also convex, it follows that \(\beta_p(x)\) is convex. Now, letting \(f(s) = \beta_p(s) = \frac{1}{2} \left\{ \psi_p \left( \frac{s+1}{2} \right) - \psi_p \left( \frac{s}{2} \right) \right\}, a = x > 0\) and \(b = x + 1\) in (32) gives

\[ \beta_p \left( x + \frac{1}{2} \right) \leq \left| \ln \Gamma_p \left( \frac{x}{2} + \frac{1}{2} \right) - \ln \Gamma_p \left( \frac{x}{2} \right) \right| \leq \beta_p \left( x + \frac{1}{2} \right) + \beta_p(x), \]

which by (17) implies

\[ \beta_p \left( x + \frac{1}{2} \right) \leq \ln \frac{\Gamma_p \left( \frac{x}{2} + 1 \right) \Gamma_p \left( \frac{x}{2} + \frac{1}{2} \right)}{\Gamma_p ^2 \left( \frac{x}{2} + \frac{1}{2} \right)} \leq \frac{1}{2} \left( \frac{1}{x} - \frac{1}{x + 2(p+1)} \right). \]

Then by exponentiation, we obtain the required result (31). \(\square\)
Remark 2.11. The function \( \frac{\Gamma_p(x + \frac{1}{2})\Gamma_p(x)}{\Gamma_p(x + \frac{1}{2})} \) is a special case of 
\[
T_p(x, y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x + y/2)}, \quad x, y > 0,
\]
which is a \( p \)-analogue of Gurland’s ratio \([4]\) for the Gamma function. For more information concerning the Gurland’s ratio, one may refer to \([7]\) and the related references therein.

Lemma 2.12. Let \( f(t) \) and \( g(t) \) be any two functions with convolution \( f \ast g = \int_0^t f(s)g(t - s)\,ds \). Then the Laplace transform of the convolution is given as 
\[
\mathcal{L}\{f \ast g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.
\]
That is 
\[
\int_0^\infty \left[ \int_0^t f(s)g(t - s)\,ds \right] e^{-xt}\,dt = \int_0^\infty f(t)e^{-xt}\,dt \int_0^\infty g(t)e^{-xt}\,dt. \tag{33}
\]

The above lemma is well-known in the literature as the convolution theorem for Laplace transforms. We shall rely on it in proving some of the results that follow.

Theorem 2.13. The function \( Q(x) = x\beta_p(x) \) is completely monotonic on \((0, \infty)\).

Proof. Recall that a function \( f : (0, \infty) \to \mathbb{R} \) is said to be completely monotonic on \((0, \infty)\) if \( f \) has derivatives of all order and \((-1)^n f^{(n)}(x) \geq 0 \) for all \( x \in (0, \infty) \) and \( n \in \mathbb{N} \). By repeated differentiation, we obtain 
\[
Q^{(n)}(x) = n\beta_p^{(n-1)}(x) + x\beta_p^{(n)}(x). \tag{34}
\]
Then by (12), (15) and (33), we obtain
\[
\frac{(-1)^nQ^{(n)}(x)}{x} = (-1)^n \left[ \frac{n}{x}\beta_p^{(n-1)}(x) + \beta_p^{(n)}(x) \right]
\]
\[
= -n \int_0^\infty e^{-xt}\,dt \int_0^\infty \frac{t^{n-1}(1 - e^{-2(p+1)t})}{1 + e^{-t}} e^{-xt}\,dt
\]
\[
+ \int_0^\infty \frac{t^n(1 - e^{-2(p+1)t})}{1 + e^{-t}} e^{-xt}\,dt.
\]
\[
= -n \int_0^\infty \left[ \int_0^t \frac{s^{n-1}(1 - e^{-2(p+1)s})}{1 + e^{-s}} ds \right] e^{-xt}\,dt
\]
\[
+ \int_0^\infty \frac{t^n(1 - e^{-2(p+1)t})}{1 + e^{-t}} e^{-xt}\,dt.
\]
\[
= \int_0^\infty W(t)e^{-xt}\,dt,
\]
where 
\[
W(t) = \frac{1}{1 + e^{-t}}.
\]
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where

$$W(t) = -n \int_0^t s^{n-1} \frac{(1 - e^{-2(p+1)s})}{1 + e^{-s}} ds + \frac{t^n(1 - e^{-2(p+1)t})}{1 + e^{-t}}.$$ 

Then $W(0) = \lim_{t \to 0} W(t) = 0$. In addition,

$$W'(t) = 2(p + 1) t^n e^{-2(p+1)t} + t^n e^{-t} \frac{1 - e^{-2(p+1)t}}{(1 + e^{-t})^2} > 0,$$

which implies that $W(t)$ increasing. Hence for $t > 0$, we have $W(t) > W(0) = 0$. Therefore,

$$(-1)^n Q^{(n)}(x) \geq 0 \quad (35)$$

which concludes the proof.

\[\Box\]

**Remark 2.14.** Theorem 2.13 implies that $Q(x) = x\beta_p(x)$ is decreasing and convex. These further imply that

$$\beta_p(x) + x\beta'_p(x) < 0 \quad (36)$$

and

$$2\beta'_p(x) + x\beta''_p(x) > 0 \quad (37)$$

respectively.

**Corollary 2.15.** The function $H(x) = x\beta_p(x)$ is increasing and concave on $(0, \infty)$.

**Proof.** By (34), (35) and (37), we obtain

$$H'(x) = \beta'_p(x) + x\beta''_p(x) > 2\beta'_p(x) + x\beta''_p(x) > 0,$$

$$H''(x) = 2\beta''_p(x) + x\beta'''_p(x) < 3\beta''_p(x) + x\beta'''_p(x) < 0,$$

which conclude the proof. \[\Box\]

**Theorem 2.16.** The inequality

$$\beta_p(xy) \leq \beta_p(x) + \beta_p(y), \quad (38)$$

holds for $x > 0$ and $y \geq 1$.

**Proof.** Let $\phi(x, y) = \beta_p(xy) - \beta_p(x) - \beta_p(y)$ for $x > 0$ and $y \geq 1$. By fixing $y$, we obtain

$$\frac{\partial}{\partial x} \phi(x, y) = y\beta'_p(xy) - \beta'_p(x)$$

$$= \frac{1}{x} [xy\beta'_p(xy) - x\beta'(x)]$$

$$\geq 0.$$
since \( x\beta'_p(x) \) is increasing. Hence, \( \phi(x, y) \) is increasing. Then for \( 0 < x < \infty \), we obtain
\[
\phi(x, y) \leq \lim_{x \to \infty} \phi(x, y) = -\beta_p(y) < 0,
\]
which gives the result (38).

\[\square\]

**Remark 2.17.** Note that \( \left| \beta^{(n)}_p(x) \right| = (-1)^n \beta^{(n)}_p(x) \) for all \( n \in \mathbb{N}_0 \). In respect of this, the recurrence relation (23) yields
\[
\left| \beta^{(n)}_p(x + 1) \right| = \frac{n!}{x^{n+1}} - \frac{n!}{(x + 2(p + 1))^{n+1}} - \left| \beta^{(n)}_p(x) \right|.
\]

It is also worth noting that, if \( F(x) = \left| \beta^{(n)}_p(x) \right| \), then \( F'(x) = -\left| \beta^{(n+1)}_p(x) \right| \).

This implies that the \( \left| \beta^{(n)}_p(x) \right| \) is decreasing for all \( n \in \mathbb{N}_0 \). Furthermore, it follows readily from (39) that
\[
\left| \beta^{(n)}_p(x) \right| \leq \frac{n!}{x^{n+1}} - \frac{n!}{(x + 2(p + 1))^{n+1}}.
\]

This is a generalization of (18).

**Theorem 2.18.** Let \( \Delta_n \) be defined for \( x > 0 \) and \( n \in \mathbb{N}_0 \) as
\[
\Delta_n(x) = \frac{x^{n+1}}{n!} \left| \beta^{(n)}_p(x) \right|.
\]

Then,

(a) \( \lim_{x \to 0} \Delta_n(x) = 1 \) and \( \lim_{x \to 0} \Delta_n'(x) = 0 \).

(b) \( \Delta_n(x) \) is decreasing.

**Proof.** (a) By virtue of (39), we obtain
\[
\lim_{x \to 0} \Delta_n(x) = \lim_{x \to 0} \left\{ 1 - \left( \frac{x}{x + 2(p + 1)} \right)^{n+1} - \frac{x^{n+1}}{n!} \left| \beta^{(n)}_p(x) \right| \right\} = 1.
\]

Also,
\[
\lim_{x \to 0} \Delta_n'(x) = \lim_{x \to 0} \left\{ \left( \frac{n + 1}{n!} \right) x^n \left| \beta^{(n)}_p(x) \right| - \frac{x^{n+1}}{n!} \left| \beta^{(n+1)}_p(x) \right| \right\}
\]
\[
= \lim_{x \to 0} \left\{ \frac{(n + 1)x^{n+1}}{(x + 2(p + 1))^{n+2}} - \frac{(n + 1)x^n}{(x + 2(p + 1))^{n+1}} \right. \\
\left. + \frac{x^{n+1}}{n!} \left| \beta^{(n+1)}_p(x + 1) \right| - (n + 1) \frac{x^{n+1}}{n!} \left| \beta^{(n)}_p(x + 1) \right| \right\}
\]
\[
= 0.
\]
(b) By using (21) and (33), we obtain
\[ \frac{n!}{x^{n+1}} \Delta_n'(x) = \frac{n+1}{x} |\beta_p^{(n)}(x)| - |\beta_p^{(n+1)}(x)| \]
\[ = (n + 1) \int_0^\infty e^{-xt} dt \int_0^\infty \frac{t^n(1 - e^{-2(p+1)t})}{1 + e^{-t}} e^{-xt} dt \]
\[ - \int_0^\infty e^{-t} t^{n+1} \left(1 - e^{-2(p+1)t}\right) e^{-xt} dt \]
\[ = (n + 1) \int_0^\infty \left[ \int_0^t \frac{s^n(1 - e^{-2(p+1)s})}{1 + e^{-s}} ds \right] e^{-xt} dt \]
\[ - \int_0^\infty e^{-t} t^{n+1} \left(1 - e^{-2(p+1)t}\right) e^{-xt} dt \]
\[ = \int_0^\infty K(t) e^{-xt} dt, \]
where
\[ K(t) = (n + 1) \int_0^t \frac{s^n(1 - e^{-2(p+1)s})}{1 + e^{-s}} ds - \frac{t^{n+1}(1 - e^{-2(p+1)t})}{1 + e^{-t}}. \]

Then \( K(0) = \lim_{t \to 0} K(t) = 0. \) Furthermore,
\[ K'(t) = -2(p + 1) \frac{t^{n+1} e^{-2(p+1)t}}{1 + e^{-t}} - t^{n+1} e^{-t} \frac{1 - e^{-2(p+1)t}}{(1 + e^{-t})^2} < 0, \]
which implies that \( K(t) \) decreasing. Hence for \( t > 0 \), we have \( K(t) < K(0) = 0. \) Therefore \( \Delta_n'(x) < 0 \) which gives the desired result.

\[ \begin{array}{c}
3 \quad \text{Open Problem}
\end{array} \]

The function \( x\beta_p(x) \) has been shown to be completely monotonic in Theorem 2.13. Show that the generalized form \( \frac{x^{n+1}}{n!} |\beta_p^{(n)}(x)|, \ x > 0, \ n \in \mathbb{N}_0 \) is completely monotonic.

\[ \text{References} \]


