Int. J. Open Problems Compt. Math., Vol. 11, No. 3, September 2018
ISSN 1998-6262; Copyright ©ICSRS Publication, 2018
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Existence and Uniqueness Results for Fractional Volterra-Fredholm Integro-Differential Equations

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Received 20 January 2018; Accepted 26 March 2018

Abstract

In this article, modified variational iteration technique has been successfully applied to find the approximate solution of Caputo fractional Volterra-Fredholm integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Moreover, we proved the existence and uniqueness results. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

Keywords: *Modified variational iteration method, Caputo fractional derivative, Volterra-Fredholm integro-differential equation, existence and uniqueness results, approximate solution.*
Mathematics subject classification (2010): 35A15, 26A33, 45J05.

1 Introduction

In recent years, numerous papers have been concentrating on the development of analytical and numerical methods for fractional integro-differential equations. In this paper, we consider Caputo fractional Volterra-Fredholm integro-differential equation of the form:

$$^{c}D^{\alpha}u(x) = a(x)u(x) + g(x) + \int_0^x K_1(x,t)F_1(u(t))dt + \int_0^1 K_2(x,t)F_2(u(t))dt, \tag{1}$$

with the initial condition

$$u(0) = u_0, \tag{2}$$

where $^{c}D^{\alpha}$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$ and $u : J \longrightarrow \mathbb{R}$, where $J = [0,1]$ is the continuous function which has to be determined, $g : J \longrightarrow \mathbb{R}$ and $K_i : J \times J \longrightarrow \mathbb{R}, i = 1,2$ are continuous functions. $F_i : \mathbb{R} \longrightarrow \mathbb{R}, i = 1,2$ are Lipschitz continuous functions. An application of fractional derivatives was first given in 1823 by Abel [1] who applied the fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [2, 4, 6, 7, 8, 9, 12, 13, 15]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. Al-Samadi and Gumah [3] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [18] applied HAM for system of fractional integro-differential equations. Yang and Hou [15] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [13] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [12] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [3, 16, 18].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated method as the modified variational iteration technique. Moreover, we proved the existence, uniqueness results of the Caputo fractional Volterra-Fredholm integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, modified variational iteration technique is constructed for solving Caputo fractional Volterra-Fredholm integro-differential equations. In Section 4, the existence and uniqueness results have been proved. In Section 5, the analytical example is presented to illustrate the accuracy of this method. In Section 6, we will give a report on our paper and a brief conclusion. Finally, Open Problem is given in Section 7.

2 Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [11, 14, 17].

Definition 2.1. ***(Riemann-Liouville fractional integral).*** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. ***(Caputo fractional derivative).*** *The fractional derivative of $f(x)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD_x^{\alpha} f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1<\alpha<m, \\ \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{4}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

1. $J^{\alpha} J^{v} f = J^{\alpha+v} f, \quad \alpha, v > 0.$
2. $J^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
3. $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x>0, \quad m-1<\alpha \leq m.$

Definition 2.3. ***(Riemann-Liouville fractional derivative).*** *The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha} f(x) = D^m J^{m-\alpha} f(x), \qquad m-1<\alpha \leq m, \quad m \in \mathbb{N}. \tag{5}$$

Theorem 2.4. *[17]* ***(Banach contraction principle).*** *Let (X,d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X i.e. $Tx = x$.*

Theorem 2.5. *[10]* ***(Schauder's fixed point theorem).*** *Let X be a Banach space and let A a convex, closed subset of X. If $T : A \longrightarrow A$ be the map such that the set $\{Tu : u \in A\}$ is relatively compact in X (or T is continuous and completely continouous). Then T has at least one fixed point $u^* \in A : Tu^* = u^*$.*

3 Modified Variational Iteration Method

We consider the following fractional equation

$$^{c}D^{\alpha}u + Mu + Nu = g(x), \tag{6}$$

where $^{c}D^{\alpha}$ is the Caputo fractional order derivative, M is a linear differential operator, N represents the nonlinear terms, and g is the source term. The basic character of He's method is the construction of a correction functional for (6), which reads

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[^{c}D^{\alpha}u_n(s) + M\tilde{u}_n(s) + N\tilde{u}_n(s) - g(s)]ds, \tag{7}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [7], u_n is the n^{th} approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To solve (6) by VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximations $u_n(x), n \geq 0$, of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With determined λ, then several approximations $u_n(x), n \geq 0$, follow immediately. We have the following variational iteration formula

$$\begin{aligned} u_0(x) & \quad \text{is an arbitrary initial guess,} \\ u_{n+1}(x) &= u_n(x) + \int_0^x \lambda(s)[^{c}D^{\alpha}u_n(s) + Mu_n(s) + Nu_n(s) - g(s)]ds. \end{aligned} \tag{8}$$

Now we are applying the operator J^{α} to both sides of (6), and using the given conditions, we obtain

$$u = R - J^{\alpha}[Mu] - J^{\alpha}[Nu], \tag{9}$$

where the function R represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed. we have the following variational iteration formula for (9)

$$\begin{aligned} & u_0(x) \quad \text{is an arbitrary initial guess,} \\ & u_{n+1}(x) = R(x) - J^{\alpha}[Mu_n(x)] - J^{\alpha}[Nu_n(x)]. \end{aligned} \tag{10}$$

In this paper, we are using modified variational iteration method, that was introduced by Ghorbani et al [5], can be settled based on the supposition that the function $R(x)$ of the iterative relation (10) can be split into two parts, namely $R_0(x)$ and $R_1(x)$. Under this an assumption, we set

$$R(x) = R_0(x) + R_1(x). \tag{11}$$

Now, we structure the following modified variational iteration formula

$$\begin{aligned} &u_0(x) = R_0(x), \\ &u_1(x) = R(x) - J^\alpha[Mu_0(x)] - J^\alpha[Nu_0(x)], \\ &. \\ &. \\ &. \\ &u_{n+1}(x) = R(x) - J^\alpha[Mu_n(x)] - J^\alpha[Nu_n(x)]. \end{aligned} \tag{12}$$

Consequently, the approximate solution is given by

$$\lim_{n \longrightarrow \infty} u_n(x) = u(x). \tag{13}$$

4 The Main Results

In this section, we shall give an existence and uniqueness results of Eq.(1) with the initial condition (2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exist two constants $L_{F_1}, L_{F_2} > 0$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|F_1(u_1(x)) - F_1(u_2(x))| \leq L_{F_1} |u_1 - u_2|$$

and

$$|F_2(u_1(x)) - F_2(u_2(x))| \leq L_{F_2} |u_1 - u_2|$$

(A2) There exist two functions $K_1^*, K_2^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$K_1^* = \sup_{x,t \in [0,1]} \int_0^x |K_1(x,t)|\, dt < \infty, \; K_2^* = \sup_{x,t \in [0,1]} \int_0^x |K_2(x,t)|\, dt < \infty.$$

(A3) The two functions $a, g : J \to \mathbb{R}$ are continuous.

Lemma 4.1. *If $u_0(x) \in C(J,\mathbb{R})$, then $u(x) \in C(J,\mathbb{R}^+)$ is a solution of the problem* $(1)-(2)$ *iff u satisfying*

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau\right)ds, \end{aligned}$$

for $x \in J$.

Our first result is based on the Banach contraction principle.

Theorem 4.2. *Assume that (A1), (A2) and (A3) hold. If*

$$\left(\frac{\|a\|_\infty + K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right) < 1. \tag{14}$$

Then there exists a unique solution $u(x) \in C(J)$ to $(1)-(2)$.

Proof. By Lemma 4.1. we know that a function u is a solution to $(1)-(2)$ iff u satisfies

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau\right)ds. \end{aligned}$$

Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$\begin{aligned} (Tu)(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau\right)ds, \end{aligned}$$

we can see that, If $u \in C(J,\mathbb{R})$ is a fixed point of T, then u is a solution of $(1)-(2)$.

Now we prove T has a fixed point u in $C(J,\mathbb{R})$. For that, let $u_1, u_2 \in C(J,\mathbb{R})$ and for any $x \in [0,1]$ such that

$$\begin{aligned} u_1(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u_1(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u_1(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u_1(\tau))d\tau\right)ds, \end{aligned}$$

and,

$$\begin{aligned}u_2(x) &= u_0+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u_2(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds+\frac{1}{\Gamma(\alpha)}\\ &\quad \times\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u_2(\tau))d\tau+\int_0^1 K_2(s,\tau)F_2(u_2(\tau))d\tau\right)ds.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&|(Tu_1)(x)-(Tu_2)(x)|\\ \le\ & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)|\,|u_1(s)-u_2(s)|\,ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\begin{array}{c}\int_0^s |K_1(s,\tau)|\,|F_1(u_1(\tau))-F_1(u_2(\tau))|\,d\tau\\ +\int_0^1 |K_2(s,\tau)|\,|F_2(u_1(\tau))-F_2(u_2(\tau))|\,d\tau\end{array}\right)ds\\ \le\ & \frac{\|a\|_\infty}{\Gamma(\alpha+1)}|u_1(x)-u_2(x)|+\frac{K_1^*L_{F_1}}{\Gamma(\alpha+1)}|u_1(x)-u_2(x)|+\frac{K_2^*L_{F_2}}{\Gamma(\alpha+1)}|u_1(x)-u_2(x)|\\ =\ & \left(\frac{\|a\|_\infty+K_1^*L_{F_1}+K_2^*L_{F_2}}{\Gamma(\alpha+1)}\right)|u_1(x)-u_2(x)|\,.\end{aligned}$$

From the inequality (14) we have

$$\|Tu_1-Tu_2\|_\infty\le\|u_1-u_2\|_\infty\,.$$

This means that T is contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point u in $C(J,\mathbb{R})$. □

Now, we will study the existence result by means of Schauder's fixed point Theorem.

Theorem 4.3. *Assume that F_1, F_2 are continuous functions and (A2), (A3) hold, If*

$$\frac{\|a\|_\infty}{\Gamma(\alpha+1)}<1. \tag{15}$$

Then there exists at least a solution $u(x)\in C(J,\mathbb{R})$ to problem (1) – (2).

Proof. Let the operator $T: C(J,\mathbb{R})\to C(J,\mathbb{R})$, be defined as in Theorem 4.2. Firstly, we prove that the operator T is completely continuous.

(1) We show that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $u_n, u \in C(J,\mathbb{R})$ and for any $x \in J$ we have

$$\begin{aligned}
& |(Tu_n)(x) - (Tu)(x)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)| \, |u_n(s) - u(s)| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \begin{pmatrix} \int_0^s |K_1(s,\tau)| \, |F_1(u_n(\tau)) - F_1(u(\tau))| \, d\tau \\ + \int_0^1 |K_2(s,\tau)| \, |F_2(u_n(\tau)) - F_2(u(\tau))| \, d\tau \end{pmatrix} ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \sup_{s\in J} |a(s)| \sup_{s\in J} |u_n(s) - u(s)| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \begin{pmatrix} \sup_{s,\tau\in J} \int_0^\tau |K_1(s,\tau)| \sup_{\tau\in J} |F_1(u_n(\tau)) - F_1(u(\tau))| \, d\tau \\ + \sup_{s,\tau\in J} \int_0^1 |K_2(s,\tau)| \sup_{\tau\in J} |F_2(u_n(\tau)) - F_2(u(\tau))| \, d\tau \end{pmatrix} ds \\
\leq\ & \|a\|_\infty \|u_n(.) - u(.)\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds \\
& + K_1^* \|F_1(u_n(.)) - F_1(u(.))\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds \\
& + K_2^* \|F_2(u_n(.)) - F_2(u(.))\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds.
\end{aligned}$$

Since $\int_0^x (x-s)^{\alpha-1} ds$ is bounded, $\lim_{n\to\infty} u_n(x) = u(x)$ and F_1, F_2 are continuous functions, we conclude that $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$, thus, T is continuous on $C(J,\mathbb{R})$.

(2) We verify that T maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, just we show that for any $\lambda > 0$ there exists a positive constant ℓ such that for each $u \in \mathbb{B}_\lambda = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \lambda\}$, one has $\|Tu\|_\infty \leq \ell$.

Let $\mu_1 = \sup_{(u)\in J\times[0,\lambda]} F_1(u(x)) + 1$, and $\mu_2 = \sup_{(u)\in J\times[0,\lambda]} F_2(u(x)) + 1$.

and for any $u \in \mathbb{B}_r$ and for each $x \in J$, we have

$$\begin{aligned}
& |(Tu)(x)| \\
=\ & |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)| \, |u(s)| \, ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |g(s)| \, ds + \\
& \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |K_1(s,\tau)| \, |F_1(u(\tau))| \, d\tau + \int_0^1 |K_2(s,\tau)| \, |F_2(u(\tau))| \, d\tau \right) ds \\
\leq\ & |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^* \mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{K_2^* \mu_2 x^\alpha}{\Gamma(\alpha+1)} \\
\leq\ & \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K_1^* \mu_1 + K_2^* \mu_2}{\Gamma(\alpha+1)} \right) \\
:\ & = \ell.
\end{aligned}$$

Therefore, $\|Tu\| \leq \ell$ for every $u \in \mathbb{B}_r$, which implies that $T\mathbb{B}_r \subset \mathbb{B}_\ell$.

(3) We examine that T maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $\mathbb{B}_\lambda$ is defined as in (**2**) and for each $u \in \mathbb{B}_\lambda$, $x_1, x_2 \in [0,1]$, with $x_1 < x_2$ we have

$$
\begin{aligned}
& |(Tu)(x_2) - (Tu)(x_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \\ - \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \end{array} \right| \\
=\ & \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds \\ + \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \end{array} \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds \\ + \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \end{array} \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \\ - \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \\ + \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \\ - \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds. \end{array} \right|
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& |(Tu)(x_2) - (Tu)(x_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left(\begin{array}{c} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |a(s)| |u(s)| ds \\ + \int_0^{x_1} (x_1 - s)^{\alpha-1} - (x_2 - s)^{\alpha-1} |a(s)| |u(s)| ds \end{array} \right) \\
& + \frac{1}{\Gamma(\alpha)} \left(\begin{array}{c} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |g(s)| ds \\ + \int_0^{x_1} (x_1 - s)^{\alpha-1} - (x_2 - s)^{\alpha-1} |g(s)| ds \end{array} \right) \\
& + \frac{1}{\Gamma(\alpha)} \left(\begin{array}{c} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \left(\begin{array}{c} \int_0^s |K_1(s,\tau)| |F_1(u(\tau))| d\tau \\ + \int_0^1 |K_2(s,\tau)| |F_2(u(\tau))| d\tau \end{array} \right) ds \\ + \int_0^{x_1} (x_1 - s)^{\alpha-1} - (x_2 - s)^{\alpha-1} \left(\begin{array}{c} \int_0^s |K_1(s,\tau)| |F_1(u(\tau))| d\tau \\ + \int_0^1 |K_2(s,\tau)| |F_2(u(\tau))| d\tau \end{array} \right) ds \end{array} \right) \\
=\ & I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)}(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|a(s)||u(s)|\,ds \\
&\quad+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|a(s)||u(s)|\,ds) \\
&\le \frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{x_1^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda-\frac{x_2^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda \\
&= \frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+(x_1^\alpha-x_2^\alpha)\right) \\
&\le \frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|g(s)|\,ds+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|g(s)|\,ds\right) \\
&\le \frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{x_1^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty-\frac{x_2^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty \\
&= \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+(x_1^\alpha-x_2^\alpha)\right) \\
&\le \frac{\|g\|_\infty}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha, \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \frac{1}{\Gamma(\alpha)}\begin{array}{c} \int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\begin{array}{c}\int_0^s|K_1(s,\tau)||F_1(u(\tau))|\,d\tau \\ +\int_0^1|K_2(s,\tau)||F_2(u(\tau))|\,d\tau\end{array}\right)ds \\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\begin{array}{c}\int_0^s|K_1(s,\tau)||F_1(u(\tau))|\,d\tau \\ +\int_0^1|K_2(s,\tau)||F_2(u(\tau))|\,d\tau\end{array}\right)ds \end{array} \\
&\le \frac{(K_1^*\mu_1+K_2^*\mu_2)}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+(x_1^\alpha-x_2^\alpha)\right) \\
&\le \frac{(K_1^*\mu_1+K_2^*\mu_2)}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha, \qquad (18)
\end{aligned}$$

we can conclude the right-hand side of (16), (17) and (18) is independently of $u \in \mathbb{B}_\lambda$ and tends to zero as $x_2 - x_1 \to 0$. This leads to $|(Tu)(x_2) - (Tu)(x_1)| \to 0$ as $x_2 \to x_1$. i.e. the set $\{T\mathbb{B}_\lambda\}$ is equicontinuous.

From I_1 to I_3 together with the Arzela–Ascoli theorem, we can conclude that $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.

Finally, we need to investigate that there exists a closed convex bounded subset $\mathbb{B}_{\widetilde{\lambda}} = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \le \widetilde{\lambda}\}$ such that $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$. For each positive integer $\widetilde{\lambda}$, then $\mathbb{B}_{\widetilde{\lambda}}$ is clearly closed, convex and bounded of $C(J,\mathbb{R})$.

We claim that there exists a positive integer ϵ such that $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$. If this property is false, then for every positive integer $\widetilde{\lambda}$, there exists $u_{\widetilde{\lambda}} \in \mathbb{B}_{\widetilde{\lambda}}$ such that $(Tu_{\widetilde{\lambda}}) \notin T\mathbb{B}_{\widetilde{\lambda}}$, i.e. $\left\|Tu_{\widetilde{\lambda}}(t)\right\|_\infty > \widetilde{\lambda}$ for some $x_{\widetilde{\lambda}} \in J$ where $x_{\widetilde{\lambda}}$ denotes x depending on $\widetilde{\lambda}$. But by using the previous hypotheses we have

$$\begin{aligned}
&\leq |u_0| + \|u\|_\infty \|a\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^*\mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{K_2^*\mu_2 x^\alpha}{\Gamma(\alpha+1)} \\
&\leq \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K_1^*\mu_1 + K_2^*\mu_2}{\Gamma(\alpha+1)}\right)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\lambda} &< \left\|Tu_{\widetilde{\lambda}}\right\|_\infty \\
&= \sup_{x\in J} \left|(Tu_{\widetilde{\lambda}})(x)\right| \\
&\leq \sup_{x\in J} \left\{ \begin{array}{c} |u_0| + \left|\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s) \left|u(s)\right| ds\right| + \left|\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds\right| + \frac{1}{\Gamma(\alpha)} \\ \times \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |K_1(s,\tau)| \left|F_1(u(\tau))\right| d\tau + \int_0^1 |K_2(s,\tau)| \left|F_2(u(\tau))\right| d\tau\right) ds \end{array} \right\} ds \\
&\leq \sup_{x\in J} \left\{ |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^*\mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{K_2^*\mu_2 x^\alpha}{\Gamma(\alpha+1)} \right\} \\
&\leq \sup_{x\in J} \left(|u_0| + \frac{\|a\|_\infty \widetilde{\lambda} + \|g\|_\infty + K_1^*\mu_1 + K_2^*\mu_2}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Dividing both sides by $\widetilde{\lambda}$ and taking the limit as $\widetilde{\lambda} \to +\infty$, we obtain

$$1 < \frac{\|a\|_\infty}{\Gamma(\alpha+1)},$$

which contradicts our assumption (15). Hence, for some positive integer $\widetilde{\lambda}$, we must have $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$.

An application of Schauder's fixed point Theorem shows that there exists at least a fixed point u of T in $C(J, \mathbb{R})$. Then u is the solution to (1) – (2) on J, and the proof is completed. □

5 Illustrative Example

In this section, we present the analytical technique based on modified variational iteration method to solve Caputo fractional Volterra-Fredholm integro-differential equations.

Example 5.1 Consider the following Caputo fractional Volterra-Fredholm integro-differential equation.

$${}^cD^{0.5}[u(x)] = \frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2} - \frac{x^2 e^x}{3} u(x) + \int_0^x e^x s u(s) ds + \int_0^1 x^2 u(s) ds, \quad (19)$$

with the initial condition

$$u(0) = 0, \tag{20}$$

and the the exact solution is $u(x) = x$. Applying the operator $J^{0.5}$ to both sides of Eq.(19)

$$u(x) = 0 + J^{0.5}[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}] + J^{0.5}[-\frac{x^2 e^x}{3}u(x)] + J^{0.5}[\int_0^x e^x s u(s) ds + \int_0^1 x^2 u(s) ds].$$

Then,

$$u(x) = J^{0.5}[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}] + J^{0.5}[-\frac{x^2 e^x}{3}u(x)] + J^{0.5}[\int_0^x e^x s u(s) ds + \int_0^1 x^2 u(s) ds].$$

Then,

$$\begin{aligned}
J^{0.5}g(x) &= J^{0.5}[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}], \\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^x (x-s)^{-0.5} s^{0.5} ds - \frac{1}{2\Gamma(0.5)} \int_0^x (x-s)^{-0.5} s^2 ds, \\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^x (1-\frac{s}{x})^{-0.5} x^{-0.5} s^{0.5} ds - \frac{1}{2\Gamma(0.5)} \int_0^x (1-\frac{s}{x})^{-0.5} x^{-0.5} s^2 ds, \\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^1 (1-\tau)^{-0.5} \tau^{0.5} x d\tau - \frac{1}{2\Gamma(0.5)} \int_0^1 (1-\tau)^{-0.5} x^{2.5} \tau^2 d\tau, \\
&= \frac{x}{\Gamma(1.5)\Gamma(0.5)} \beta(0.5, 1.5) - \frac{x^{2.5}}{2\Gamma(0.5)} \beta(0.5, 3), \\
&= x - \frac{x^{2.5}}{\Gamma(3.5)}.
\end{aligned}$$

Now, we apply the modified variational iteration,

$$J^{0.5}g(x) = J^{0.5}[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}],$$

$$R(x) = J^{0.5}g(x) = R_0(x) + R_1(x) = x - \frac{x^{2.5}}{\Gamma(3.5)}.$$

We get,

$$R_0(x) = x, \qquad R_1(x) = -\frac{x^{2.5}}{\Gamma(3.5)}.$$

The modified iterative relations

$$\begin{aligned}
u_0(x) &= R_0(x) = x. \\
u_1(x) &= R(x) + J^{0.5}\left(f(x)u_0(x)\right) + J^{0.5}\left(\int_0^x K_1(x,s)u_0 ds + \int_0^1 K_2(x,s)u_0 ds\right), \\
&= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2e^x}{3}u_0(x)\right) + J^{0.5}\left(\int_0^x e^x s u_0(s)ds + \int_0^1 x^2 u_0(s)ds\right), \\
&= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2e^x}{3}x\right) + J^{0.5}\left(\int_0^x e^x s^2 ds + \int_0^1 x^2 s ds\right), \\
&= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3} + \frac{x^2}{2}\right), \\
&= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3}\right) + J^{0.5}\left(\frac{x^2}{2}\right), \\
&= x. \\
u_2(x) &= x. \\
&\cdot \\
&\cdot \\
&\cdot \\
u_n(x) &= x.
\end{aligned}$$

Therefore, the obtained solution is

$$u(x) = x.$$

6 Conclusions

The modified variational iteration method has been successfully applied to find the approximate solution of Caputo fractional Volterra-Fredholm integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear fractional Volterra-Fredholm integro-differential equations. Moreover, we proved the existence and uniqueness of the solution. The illustrative example establish the precision and efficiency of the proposed technique.

7 Open Problem

Study the Homotopy perturbation method (HPM) and modified variational iteration method (MVIM) for solving Caputo fractional Volterra-Fredholm integro-differential equation of the form:

$$\sum_{j=0}^{k} p_j(x) D^{\alpha} u(x) = f(x) + \lambda_1 \int_a^x \sum_{m=0}^{r} A_m(x,t) F_m[u(t)]^p dt + \lambda_2 \int_a^b \sum_{i=0}^{s} B_i(x,t) G_i[u(t)]^q dt.$$

with the mixed conditions

$$\sum_{j=0}^{k-1} [a_{ij} u^{(j)}(a) + b_{ij} u^{(j)}(b) + c_{ij} u^{(j)}(\xi)] = \beta_i, \quad i = 0, 1, ..., k-1. \quad a \leq \xi \leq b,$$

where D^{α} is the Caputo's fractional derivative, $n - 1 < \alpha \leq n$ and $n \in \mathbb{N}$.

ACKNOWLEDGEMENTS.
The authors would like to thank the referees for their valuable suggestions and comments.

References

[1] N. Abel, *Solution de quelques problemes a laide dintegrales definites*, Christiania Grondahl, Norway, (1881), 16–18.

[2] S. Alkan and V. Hatipoglu, Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order, *Tbilisi Mathematical Journal*, **10**(2) (2017), 1–13.

[3] M. AL-Smadi and G. Gumah, On the homotopy analysis method for fractional SEIR epidemic model, *Research J. Appl. Sci. Engrg. Technol.*, **18**(7) (2014), 3809–3820.

[4] M. Bani Issa, A. Hamoud, K. Ghadle and Giniswamy, Hybrid method for solving nonlinear Volterra-Fredholm integro-differential equations, J. Math. Comput. Sci. **7**(4) (2017), 625–641.

[5] A. Ghorbani, J. Saberi-Nadjafi, An effective modification of He's variational iteration method, *Non-linear Anal.: Real World Appl.*, **10** (2009), 2828–2833.

[6] A. Hamoud and K. Ghadle, The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations, *Korean J. Math.*, **25**(3)(2017), 323–334.

[7] A. Hamoud and K. Ghadle, On the numerical solution of nonlinear Volterra-Fredholm integral equations by variational iteration method, *Int. J. Adv. Sci. Tech. Research*, **3** (2016), 45–51.

[8] A. Hamoud and K. Ghadle, The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations, *J. Korean Soc. Ind. Appl. Math.*, **21** (2017), 17–28.

[9] A. Hamoud and K. Ghadle, Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations, *J. Indian Math. Soc.*, **85**(1-2) (2018), 52–69.

[10] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud. Elsevier, Amsterdam, **204** [2006].

[11] V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Analysis: Theory, Methods and Appl.* **69**(10) (2008), 3337–3343.

[12] X. Ma and C. Huang, Numerical solution of fractional integro-differential equations by a hybrid collocation method, *Appl. Math. Comput.*, **219**(12) (2013), 6750–6760.

[13] R. Mittal and R. Nigam, Solution of fractional integro-differential equations by Adomian decomposition method, *Int. J. Appl. Math. Mech.*, **4**(2) (2008), 87–94.

[14] S. Samko, A. Kilbas and O. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Yverdon, [1993].

[15] C. Yang and J. Hou, Numerical solution of integro-differential equations of fractional order by Laplace decomposition method, *Wseas Trans. Math.*, **12**(12) (2013), 1173–1183.

[16] X. Zhang, B. Tang, and Y. He, Homotopy analysis method for higher-order fractional integro-differential equations, *Comput. Math. Appl.*, **62**(8) (2011), 3194–3203.

[17] Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, **6** [2014].

[18] M. Zurigat, S. Momani and A. Alawneh, Homotopy analysis method for systems of fractional integro-differential equations, *Neur. Parallel Sci. Comput.,* **17** (2009), 169–186.