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δ - Dynamic Chromatic number of Double Wheel Graph Families

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Abstract

An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the δ-dynamic chromatic number of middle, total, central and line graph of double wheel graph ($r = \delta$).

Keywords: *δ-dynamic coloring, middle graph, total graph, central graph.*
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1 Introduction

Throughout this paper all graphs are finite and simple. The r-dynamic chromatic number was first introduced by Montgomery [14]. An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i)if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v and $d(v)$ its degree. The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in [1, 2, 3, 4, 8]. There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example,

For a graph G with $\Delta(G) \geq 3$, Lai et al. [8] proved that $\chi_d(G) \leq \Delta(G)+1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [6]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2log_2\alpha(G) + 3$. Taherkhani gave in [15] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e., $\chi_2(G) - \chi(G) \leq \left\lceil (\Delta e)/\delta log\left(2e\left(\Delta^2+1\right)\right)\right\rceil$.

Li et al. proved in [10] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Liu and Zhou [9] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

N.Mohanapriya et al. [11, 12] studied the dynamic chromatic number for various graph families. Also, it was proven in [13] that the r-dynamic chromatic number of line graph of a helm graph H_n.

In this paper, we study $\chi_r(G)$ when r is δ, the minimum degree of the graph. We find the δ-dynamic chromatic number for middle, total, central and line graph of double wheel graph.

2 Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [5] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [5] of G, denoted by $T(G)$ is defined in the following way. The vertex set of

$T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

The central graph [16] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [7] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

For any integer $n \geq 3$, the double wheel graph DW_n is a graph of size n can be composed of $2C_n + K_1$ i.e. it consists of two cycle of size n , where the vertices of two cycles are all connected to a common hub.Let $V\left(DW_n\right) = \{v, v_1, v_2, \ldots v_n, w_1, w_2, \ldots w_n\}$ and
$E(DW_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1, w_1w_2, w_2w_3, \ldots, w_{n-1}w_n, w_nw_1\}$
$\cup\{vv_1, \ldots, vv_n, ww_1, \ldots, ww_n\}$.

3 Main results

Theorem 3.1 *Let $n \geq 3$. The δ-dynamic chromatic number of the line graph of a double wheel of order n is $\chi_\delta(L(DW_n)) = 2n$.*

Proof: Let $V\left(L\left(DW_n\right)\right) = \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{f_1, f_2, \ldots, f_n\} \cup \{x_1, x_2, \ldots, x_n\}$, where u_i is the vertex corresponding to the edge v_iv_{i+1} of $DW_n\,(1 \leq i \leq n-1)$, $u_n = v_nv_1$, e_i is the vertex corresponding to the edge vv_i of $DW_n\,(1 \leq i \leq n)$, x_i is the vertex corresponding to the edge w_iw_{i+1} of $DW_n\,(1 \leq i \leq n-1)$, $x_n = w_nw_1$ and f_i is the vertex corresponding to the edge vw_i of $DW_n(1 \leq i \leq n)$. By definition of the line graph, the vertices $\{e_i : (1 \leq i \leq n), f_i : (1 \leq i \leq n)\}$ induce a clique of order K_{2n} in $L(DW_n)$. Thus, $\chi_\delta(L(DW_n)) \geq 2n$.
Consider the following $2n$-coloring of $L\left(DW_n\right)$:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{i+n} to f_i . Complete the coloring by assigning colors $\{c_1, c_2, \ldots, c_n, c_{n+1}, \ldots, c_{2n}\}$ to the remaining vertices consecutively as follows: colour c_{i+n} to u_i and c_i to x_i.An easy check shows that this coloring is a 4- dynamic coloring. Hence, $\chi_\delta(L(DW_n)) \leq 2n$. Therefore, $\chi_\delta(L(DW_n)) = 2n$, $\forall\ n \geq 3$.

Theorem 3.2 *Let $n \geq 3$. The δ- dynamic chromatic number of the middle graph of a double wheel of order n is $\chi_\delta(M(DW_n)) = 2n + 1$.*

Proof: Let $V\left(M\left(DW_n\right)\right) = \{v, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\} \cup \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\} \cup \{u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n\}$, where u_i and x_i is

the vertex corresponding to the edge v_iv_{i+1} and w_iw_{i+1} of $DW_n\,(1 \leq i \leq n-1)$, $u_n = v_nv_1$ and $x_n = w_nw_1$. e_i and f_i is the vertex corresponding to the edge vv_i and $vw_i$of $DW_n\,(1 \leq i \leq n)$. By definition of the middle graph, the vertices $\{v, e_i : (1 \leq i \leq n), f_i : (1 \leq i \leq n)\}$ induce a clique of order K_{2n+1} in $M(DW_n)$. Thus, $\chi_\delta(M(DW_n)) \geq 2n+1$.
Consider the following $2n+1$-coloring of $M\,(DW_n)$:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{i+n} to f_i and color c_{2n+1} to v. For $1 \leq i \leq n$, assign to vertex u_i and x_i one of the allowed colors - such color exists, because $deg(u_i) = deg(x_i) = 6$. For $1 \leq i \leq n$, if any, assign to vertex v_i and w_i one of the allowed colors - such color exists, because $deg(v_i) = deg(w_i) = 3$. Also, $\delta(G) = 3$. An easy check shows that $N(u)$ and $N(x)$ contains an induced clique of order atleast 3, for every $u, x \in V\,(M\,(DW_n))$. Thus, this coloring is a 3-dynamic coloring. Hence, $\chi_\delta(M(DW_n)) \leq 2n+1$. Therefore, $\chi_\delta(M(DW_n)) = 2n+1$, $\forall\ n \geq 3$.

Theorem 3.3 *Let $n \geq 3$. The δ-dynamic chromatic number of the total graph of a double wheel of order n is $\chi_\delta(T(DW_n)) = 2n+1$.*

Proof: Let $V\,(T\,(DW_n)) = \{v, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\} \cup \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\} \cup \{u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n\}$, where u_i and x_i is the vertex corresponding to the edge v_iv_{i+1} and w_iw_{i+1} of $DW_n\,(1 \leq i \leq n-1)$, $u_n = v_nv_1$ and $x_n = w_nw_1$. e_i and f_i is the vertex corresponding to the edge vv_i and $vw_i$of $DW_n\,(1 \leq i \leq n)$. By the definition of the total graph, the vertices $\{v, e_i : (1 \leq i \leq n), f_i : (1 \leq i \leq n)\}$ induce a clique of order K_{2n+1} in $T(DW_n)$. Thus, $\chi_\delta(T(DW_n)) \geq 2n+1$.
Consider the following $2n+1$-coloring of $T\,(DW_n)$:
For $1 \leq i \leq n$, assign color c_i to e_i and c_{i+n} to f_i and assign color c_{2n+1} to v. We complete the coloring by assigning colors $\{c_1, c_2, \ldots, c_n, c_{n+1}, \ldots, c_{2n}\}$ to the remaining vertices consecutively as follows: color c_1 to w_1, c_2 to w_2, and color c_{n+1} to v_1, c_{n+2} to v_2, …and for $1 \leq i \leq n$, assign to vertex u_i and x_i one of the allowed colors - such color exists, because $deg(u_i) = deg(x_i) = 6$.

An easy check shows that this coloring is a 6-dynamic coloring. Hence, $\chi_\delta(T(DW_n)) \leq 2n+1$. Therefore, $\chi_\delta(T(DW_n)) = 2n+1$, $\forall\ n \geq 3$.

Theorem 3.4 *Let $n \geq 4$. The δ-dynamic chromatic number of the central graph of a double wheel of order n is $\chi_\delta(C(DW_n)) = 2n+1$.*

Proof: Let $V\,(C\,(DW_n)) = \{v, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\} \cup \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\} \cup \{u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n\}$, where u_i and x_i is the vertex corresponding to the edge v_iv_{i+1} and w_iw_{i+1} of $DW_n\,(1 \leq i \leq n-1)$, $u_n = v_nv_1$ and $x_n = w_nw_1$. e_i and f_i is the vertex corresponding to the edge vv_i and $vw_i$of $DW_n\,(1 \leq i \leq n)$.
Clearly, the graph induced by $\{v_{2i} :\ i = 1, 2, \ldots, \lfloor (n)/2 \rfloor\}$ is a complete graph.

Thus, a proper coloring assign at least $\lfloor (n)/2 \rfloor$ colors to them. The same happens with the subgraph induced by $\{v_{2i-1} : \ i = 1, 2, \ldots, \lfloor (n)/2 \rfloor\}$.The graph induced by $\{w_{2i} : \ i = 1, 2, \ldots, \lfloor (n)/2 \rfloor\}$ is a complete graph. Thus, a proper coloring assign at least $\lfloor (n)/2 \rfloor$ colors to them. The same happens with the subgraph induced by $\{w_{2i-1} : \ i = 1, 2, \ldots, \lfloor (n)/2 \rfloor\}$. Moreover, if we are considering a δ-dynamic coloring when n is odd, v_n should have a different color from v_{2i-1}, $\{i = 1, 2, \ldots, \lfloor (n)/2 \rfloor\}$, because v_n and v_1 are the only neighbors of u_n, and v_n is adjacent to v_{2i-1}, $\{i = 2, \ldots, \lfloor (n)/2 \rfloor\}$.The same procedure is followed in coloring the vertices w_i $(1 \leq i \leq n)$.
A similar reasoning also shows that in a δ-dynamic coloring, the colors assigned to odd vertices should be different to the colors assigned to even vertices and that all of them should be different from the color assigned to v. Thus, $\chi_\delta(C(DW_n)) \geq 2n + 1$.
Consider the following $2n + 1$-coloring of $C\,(DW_n)$:
For $1 \leq i \leq n$, assign the color c_{i+n} to v_i $1 \leq i \leq n$, assign the color c_i to w_i and assign the color c_{2n+1} to v. For $1 \leq i \leq n$, assign to vertex u_i , x_i , e_i,f_i one of the allowed colors - such color exists, because $\deg(u_i) = \deg(x_i) = \deg(e_i) = \deg(f_i) = 2$. An easy check shows that this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(C(DW_n)) \leq 2n + 1$. Therefore, $\chi_\delta(C(DW_n)) = 2n + 1$, $\forall\ n \geq 4$.

4 Open Problem

In this paper, we have studied the $\delta-$ Dynamic chromatic number of double wheel graph families. As a motivation of this work, it can be extended to find the r - Dynamic chromatic number of some general graphs.

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