A note on bi-derivations
of \(\ast\)-prime rings

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Abstract

A classical result of Matej Brešar asserts that any bi-derivation on a noncommutative prime ring \(R\) is the form \(B(x, y) = \lambda [x, y]\) where \(\lambda \in C\) (the extended centroid of \(R\)). The main purpose of this work is to study this result in the case of \(\ast\)-prime rings. Namely, we determine the bi-derivations in \(\ast\)-prime rings. First, we present some results on the left Martindale quotient ring and the extended centroid of a \(\ast\)-prime ring with involution \(\ast\). Afterwards, we describe the bi-derivations in noncommutative \(\ast\)-prime rings. This enable us to study the commuting maps in \(\ast\)-prime rings with involution.

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1 Introduction

This research has been motivated by the work of Matej Brešar, V.S. Martindale and C. Robert Miers [3]. Throughout this article, \(R\) will represent an associative ring with center \(Z(R)\). Recall that \(R\) is prime if \(aRb = (0)\) implies that either \(a = 0\) or \(b = 0\). An additive mapping \(x \mapsto x^\ast\) of \(R\) into itself is called involution if \((x^\ast)^\ast = x\) and \((xy)^\ast = y^\ast x^\ast\) holds for all \(x, y \in R\). A ring
equipped with an involution is known as ring with involution or $\star$-ring. A ring $R$ with involution $\star$ is called a $\star$-prime ring if $aRb = (0)$ and $aRb^\ast = (0)$ implies that either $a = 0$ or $b = 0$. It is clear that any prime ring with involution $\star$ is $\star$-prime but the converse is not true (see [1] for details). This is unifying notion in that it includes ordinary prime rings with involution and also prime rings $R$ without involution. It suffices to map $R$ into $R \times R^\circ$ via the embedding map $r \mapsto (r, r)$, where $R^\circ$ is the opposite ring. We note that if $R$ is a prime ring then $R \times R^\circ$ is $\star$-prime ring where $\star$ is the exchange involution defined by $(x, y)^\ast = (y, x)$. A ring $R$ is called a semiprime ring if $aRa = (0)$ implies $a = 0$. We can see that every $\star$-prime ring is a semiprime ring. We denote by $Q_r, Q_l, Q_s$, and $C$, right, left, symmetric Martindale ring of quotients and extended centroid of $\star$-prime ring $R$, respectively (see [2]). We write $[x, y]$ for $xy - yx$. It follows that for all $x, y, z \in R$

$$[xy, z] = x[y, z] + [x, z]y,$$

and

$$[x, yz] = [x, y]z + y[x, z].$$

An additive mapping $D : R \mapsto R$ is called a derivation if

$$D(xy) = D(x)y + xD(y),$$

for all $x, y \in R$.

An additive mapping $T : R \mapsto R$ is called a left (right) centralizer in case $T(xy) = xT(y)$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. An additive mapping $T : R \mapsto R$ is called a two-sided centralizer if $T$ is both a left and right centralizer. A bi-additive mapping $B : R \times R \mapsto R$ is called a $\star$-derivation if for every $x \in R$ the map $y \mapsto B(x, y)$ is a derivation of $R$ and for every $y \in R$ the map $x \mapsto B(x, y)$ is a derivation of $R$. For example, the mappings of the form $(x, y) \mapsto \lambda [x, y]$ where $\lambda$ is an element of the center of $R$ are $\beta$-derivations. In [3], it is shown that in the case of noncommutative prime ring $R$, the only $\beta$-derivation in $R$ are $B(x, y) = c[x, y]$ where $c \in C$ where $C$ is the extended centroid of $R$. Brešar [4] has extended this result to 2-torsion free semiprime ring by proving that if $R$ is a semiprime ring and $B : R \times R \mapsto R$ a $\star$-derivation, then there exists an idempotent $e \in C$ and an element $\nu \in C$ such that the algebra $(1 - e)R$ is commutative and $\epsilon B(x, y) = \epsilon [x, y]$, for all $x, y \in R$. Particularly, if $R$ is prime, then the only $\beta$-derivations of $R$ are the forms $B(x, y) = c[x, y]$. Our goal is to generalize this result in the case of $\star$-prime rings. The notion of $\beta$-derivation arise naturally in the study of additive commuting maps. Namely, if $f$ is an additive commuting map, then the map $B : R \times R \mapsto R$ given by $B(x, y) = [f(x), y]$ is a $\beta$-derivation. It is shown in [3] that if $f$ is an additive commuting map on a prime ring $R$, then $f(x) = \lambda x + \nu(x)$, where $\lambda \in C$ (the extended centroid of $R$) and $\nu : R \mapsto C$ an additive map. In this work, we describe the additive commuting maps on $\star$-prime rings.
2 The $\star$-extended centroid of a $\star$-prime ring

The Martindale ring of quotients of a prime ring was introduced in [5] as a tool for studying rings satisfying a polynomial identity. The concept was extended to $\star$-prime rings in [2]. In this section we recall the construction of the Martindale ring of quotient and the $\star$-extended centroid of a $\star$-prime ring and we provide some essential results which will be needed in the next.

Throughout this section $R$ will denote a $\star$-prime ring with involution $\star$. Let $\mathfrak{F}_*$ be the set of all nonzero $\star$-ideals of $R$. We consider the set

$$\mathcal{L}_* = \{(I, f) / I \in \mathfrak{F}_*, f : I \to R \text{ a right centralizer}\}$$

Two elements $(I, f)$ and $(J, g)$ are equivalent if there exists $K \in \mathfrak{F}_*$ such that $K \subseteq I \cap J$ and $f = g$ on $K$. This is easily seen to yield an equivalence relation on $\mathcal{L}_*$. Let $[I, f]$ denotes the equivalence class of $(I, f)$ and $\mathcal{Q}_r$ the set of all equivalence classes. Let us define the addition and the multiplication on $\mathcal{Q}_r$ as follows

$$[I, f] + [J, g] = [I \cap J, f + g],$$

$$[I, f] \times [J, g] = [(I \cap J)^2, f \circ g].$$

Under this operations, $\mathcal{Q}_r$ is a ring with unit $[R, Id_R]$ which contains $R$ as a subring via the injection $r \mapsto [R, r]$ where $r(x) = rx$.

Definition 2.1. The ring $\mathcal{Q}_r$ is called right Martindale ring of quotients of $R$ and the centre $C$ of $\mathcal{Q}_r$ is called the extended centroid of $R$.

In the following proposition, we provide some results which will be needed in the next.

Proposition 2.2 ([2], Lemma 3.1). Let $R$ be a $\star$-prime ring. Then

1. For $q \in \mathcal{Q}_r$, there exists $u \in \mathfrak{F}_*$ such that $qu \in R$.

2. Let $f : I \to R$ be a right centralizer on $I \in \mathfrak{F}_*$, then there exists $q \in \mathcal{Q}_r$ such that $f(x) = qx$.

3. An element $[I, f]$ of $\mathcal{Q}_r$ belongs to $C$ if and only if $f$ is a two sided centralizer.

4. Let $f : I \to R$ be a two sided centralizer on $I \in \mathfrak{F}_*$, then there exists $q \in C$ such that $f(x) = qx$.

For $\lambda = [I, f] \in C$ we put $\lambda^* = [I, g]$ where $g(x) = (f(x^*))^*$ for all $x \in I$. It is clear that $\star$ is an involution on $C$.
Definition 2.3. The set of symmetric elements $C_\ast$ of $C$ under the above involution is called the $\ast$-centroid of $R$.

The structure of $C_\ast$ is given in [2] as follows:

Proposition 2.4. Let $R$ be a $\ast$-prime ring. Then

1. $C_\ast$ is the set of $[I,f] \in C$ such that $f$ commutes with $\ast$.

2. $C_\ast$ is a field.

Remark 2.5. Let $f : I \to R$ be a right and left centralizer on $I \in \mathfrak{F}_\ast$ such that $f$ commutes with $\ast$. Then there exists $\lambda \in C_\ast$ such that $f(x) = \lambda x$.

It is known that if $R$ is a prime ring and $a, b \in R$ verify $axb = bxa$ for all $x \in R$, then either $a = 0$ or there exists $\lambda \in C$ such that $b = \lambda a$ (see [6], Theorem 1). Now we provide the corresponding result for $\ast$-prime rings.

Theorem 2.6. Let $R$ be a $\ast$-prime ring.

Suppose that $axb = bxa$, $axb^\ast = bxa^\ast$, and $a^\ast xb = b^\ast xa$, for all $x \in R$, then either $a = 0$ or $b = \lambda a$ where $\lambda \in C_\ast$.

Proof. Suppose that $a \neq 0$. we define the map

$$f : RaR + Ra^\ast R \longrightarrow R$$

$$\sum_i x_iay_i + \sum_j x_ja^\ast y_j \longmapsto \sum_i x_iby_i + \sum_j x_jb^\ast y_j$$

The mapping $f$ is well defined. Indeed, suppose that

$$\sum_i x_iay_i + \sum_j x_ja^\ast y_j = 0.$$ 

Then, for all $r \in R$,

$$0 = br(\sum_i x_iay_i + \sum_j x_ja^\ast y_j)$$

$$= \sum_i brx_iay_i + \sum_j brx_ja^\ast y_j$$

$$= \sum_i arx_i by_i + \sum_j arx_jb^\ast y_j$$

$$= ar(\sum_i x_iby_i + \sum_j x_jb^\ast y_j)$$

It follows that

$$aR(\sum_i x_iby_i + \sum_j x_jb^\ast y_j) = (0) \quad (1)$$
 Likewise, for all \( r \in R \),
\[
0 = b^* r (\sum_i x_i a y_i + \sum_j x_j a^* y_j)
\]
\[
= \sum_i b^* r x_i a y_i + \sum_j b^* r x_j a^* y_j
\]
\[
= \sum_i a^* r x_i b y_i + \sum_j a^* r x_j b^* y_j
\]
\[
= a^* r (\sum_i x_i b y_i + \sum_j x_j b^* y_j)
\]

Hence
\[
a^* R (\sum_i x_i b y_i + \sum_j x_j b^* y_j) = (0) \tag{2}
\]

From (1) and (2) and the fact that \( R \) is \( \ast \)-prime ring, we deduce that
\[
\sum_i x_i b y_i + \sum_j x_j b^* y_j = 0,
\]
so that \( f \) is well defined.

In other hand , we can see that \( f \) commutes with \( \ast \). Indeed,
\[
f((\sum_i x_i a y_i + \sum_j x_j a^* y_j)^*) = f((\sum_i y_i^* a^* x_i^* + \sum_j y_j^* a x_j^*)
\]
\[
= \sum_i y_i^* b^* x_i^* + \sum_j y_j^* b x_j^*
\]
\[
= (\sum_i x_i b y_i + \sum_j x_j b^* y_j)^*
\]

Furthermore, \( f \) is a two-sided centralizer of \( R \), then there exists \( \lambda \in C_\ast \) such that \( f(x) = \lambda x \). As \( f(a) = b \), we are done. \( \square \)

3 Bi-derivations in \( \ast \)-prime rings

We start this section with the following result which provide the description of bi-derivations of \( \ast \)-prime rings.

**Theorem 3.1.** Let \( R \) be a \( \ast \)-prime ring with involution \( \ast \) and \( B : R \times R \rightarrow R \) a bi-derivation such that \( B^*(x, y) = B(y^*, x^*) \) for all \( x, y \in R \). Then there exists \( \lambda \in C_\ast \) such that \( B(x, y) = \lambda [x, y] \), for all \( x, y \in R \).

For the proof of the theorem 3.1 we need the following lemmas.
Lemma 3.2. Let $R$ be a ring and $B : R \rightarrow R$ a bi-derivation. Then, for all $x, y, z, u, v \in R$,
\[
B(x, y)z[u, v] = [x, y]zB(u, v)
\]

Proof. As $B$ is a bi-derivation then, for all $x, y, u, v$ in $R$

\[
B(xu, yv) = B(x, yv)u + xB(u, yv).  
\]

Then
\[
B(xu, yv) = B(x, yv)u + yB(x, u)v + xyB(u, v).  
\]

As $B(xu, yv) = B(xu, yv) + yB(xu, v)$, we deduce that

\[
B(xu, yv) = B(x, yv)u + yB(x, v)u + yxB(u, v).  
\]

From (3) and (4), we obtain

\[
B(x, y)[u, v] = [x, y]B(u, v), \forall x, y, u, v \in R.  
\]

Replacing $u$ by $zu$ in (5), we get

\[
B(x, y)([z, v]u + z[u, v]) = [x, y](B(z, v)u + zB(u, v)),  
\]

so that

\[
B(x, y)[z, v]u + B(x, y)z[u, v] = [x, y]B(z, v)u + [x, y]zB(u, v).  
\]

By virtue of relation (5) we have

\[
B(x, y)[z, v]u = [x, y]B(z, v)u, \forall x, y, z, u, v \in R,
\]

and we obtain the assertion of the lemma. \qed

Lemma 3.3. Let $S$ be any set and $R$ be a $\ast$-prime ring with involution $\ast$. Suppose that functions $f : S \rightarrow R$ and $g : S \rightarrow R$ satisfy

\[
F(s)xG(t) = G(s)xF(t), \forall s, t \in S, \forall x \in R,
\]

\[
F^\ast(s)xG(t) = G^\ast(s)xF(t), \forall s, t \in S, \forall x \in R,
\]

and

\[
F(s)xG^\ast(t) = G(s)xF^\ast(t), \forall s, t \in S, \forall x \in R.
\]

Then there exists $\lambda \in C_\ast$, the $\ast$-extended centroid of $R$ such that

\[
G(s) = \lambda F(s), \forall s \in S.
\]
Proof. Let us consider $s \in S$. Then, for all $x \in R$,

$$F(s)xG(s) = G(s)xF(s),$$

$$F^*(s)xG(s) = G^*(s)xF(s),$$

and

$$F(s)xG^*(s) = G(s)xF^*(s)$$

If $F(s) \neq 0$, then from Theorem 2.6 there exists $\lambda(s) \in C_*$ such that

$$G(s) = \lambda(s)F(s).$$

Let $t \in S$ such that $F(t) \neq 0$. For all $x \in R$,

$$\lambda(t)F(s)xF(t) = F(s)x\lambda(t)F(t) = F(s)xG(t) = G(s)xG(t) = \lambda(t)F(s)xF(t).$$

Hence

$$(\lambda(t) - \lambda(s))F(s)xF(t) = F(s)xF(t), \forall x \in R. \quad (8)$$

In the other hand, the relation $F(s)xG^*(t) = G(s)xF^*(t)$ implies

$$\lambda(t)F(s)xF^*(t) = F(s)x\lambda(t)F^*(t) = F(s)(\lambda(t)F(t))^* = F(s)xG^*(t) = G(s)xF^*(t) = \lambda(s)F(s)xF^*(t).$$

It follows that

$$(\lambda(t) - \lambda(s))F(s)xF^*(t) = 0, \forall x \in R. \quad (9)$$

From (8) and (9), the $*$-primeness of $R$ yields $\lambda(t) = \lambda(s)$, for all $t, s \in S$. Hence, there exists $\lambda \in C_*$ such that $G(s) = \lambda F(s)$, for all $s \in S$ where $F(s) \neq 0$. However, if $F(s) = 0$, then $G(s)xF(t) = 0$ and $G(s)xF^*(t) = 0$ so that $G(s) = 0$. Finally, there exists $\lambda \in C_*$ such that $G(s) = \lambda F(s)$, for all $s \in S$.

Now, we are ready to prove the theorem 3.1.
Proof. Let $S = R \times R$, and define $F: S \to R$ such that

$$F(x, y) = B(x, y),$$

and define $G: S \to R$ such that

$$G(x, y) = [x, y].$$

From Lemma 3.2, for all $x, y, z, u, v \in R$

$$F(x, y)zG(u, v) = B(x, y)z[u, v]$$

$$= [x, y]zB(u, v)$$

$$= G(x, y)zF(u, v),$$

and

$$F(x, y)zG^*(u, v) = B(x, y)z[u, v]^*$$

$$= [x, y]z[v^*, u^*]$$

$$= [x, y]zB(v^*, u^*)$$

$$= [x, y]zF^*(u, v)$$

and

$$F^*(x, y)zG(u, v) = B^*(x, y)z[u, v]$$

$$= B(y^*, x^*)z[u, v]$$

$$= [y^*, x^*]zB(u, v)$$

$$= G^*(x, y)zF(u, v)$$

By virtue of Lemma 3.3 there exists $\lambda \in C_*$ such that

$$F(x, y) = \lambda G(x, y), \forall x, y \in R,$$

so that

$$B(x, y) = \lambda [x, y], \forall x, y \in R.$$

\[\square\]

Corollary 3.4. Let $R$ a $*$-prime ring with involution $*$ and $B : R \times R \to R$ a bi-derivation. Then there exists $\lambda \in C_*$ such that

$$B(x, y) + B^*(y^*, x^*) = \lambda [x, y], \forall x, y \in R$$

Proof. Let $B$ be a bi-derivation on $R$. We consider the map $B_1 : R \times R \to R$ defined by

$$B_1(x, y) = B(x, y) + B^*(y^*, x^*), \forall x, y \in R.$$
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For all elements \(x, y, z\) in \(R\)

\[
B_1(xy, z) = B(xy, z) + B^*(z^*, y^*x^*) \\
= B(x, z)y + xB(y, z) + [B(z^*, y^*)x^* + y^*B(z^*, x^*)]^* \\
= B(x, z)y + xB(y, z) + xB^*(z^*, y^*) + B^*(z^*, x^*)y \\
= B_1(x, z)y + xB_1(y, z).
\]

In the other hand

\[
B_1(x, yz) = B(x, yz) + B^*(z^*y^*, x^*) \\
= B(x, y)z + yB(x, z) + [B(z^*, x^*)y^* + z^*B(y^*, x^*)]^* \\
= B_1(x, y)z + yB_1(x, z).
\]

It follows that the mapping \(B_1\) is a bidrivation on \(R\). Furthermore,

\[
B_1^*(x, y) = B_1(y^*, x^*), \forall x, y \in R.
\]

By virtue of Theorem 3.1, there exists \(\lambda \in C_*\) such that \(B_1(x, y) = \lambda [x, y]\) for all \(x, y \in R\).

\[\square\]

**Corollary 3.5.** Let \(R\) a \(\star\)-prime ring with involution \(\star\) and \(f\) be an additive commuting mapping on \(R\) which commutes with \(\star\). Then there exists an element \(\lambda \in C_*\) and an additive mapping \(\nu : R \rightarrow C\) such \(f(x) = \lambda x + \nu(x)\).

**Proof.** Let \(f\) be an additive commuting mapping in \(R\) which commutes with \(\star\). We define the mapping \(B\) on \(R \times R\) as follows

\[
B(x, y) = [f(x), y], \forall x, y \in R.
\]

As \(f\) is commuting, then \([f(x) + f(y), x + y] = 0\) for all \(x, y \in R\).

Hence

\[
[f(x), y] = [x, f(y)], \forall x, y \in R.
\]

We can easily verify that \(B\) is a bi-derivation and

\[
B^*(x, y) = [f(x), y]^* \\
= [y^*, f(x^*)] \\
= -[f(x^*), y^*] \\
= -[x^*, f(y^*)] \\
= [f(y^*), x^*] \\
= B(y^*, x^*).
\]
From Theorem 3.1, there exists an element $\lambda$ in $C_*$ such that

$$[f(x), y] = \lambda[x, y] \text{ for all } x, y \in R.$$  

Then

$$[f(x) - \lambda x, y] = 0 \text{ for all } x, y \in R.$$  

It follows that for any $x \in R$ $\nu(x) = f(x) - \lambda x \in C$, and the proof is complete.

\[\square\]

**Corollary 3.6.** Let $R$ a $\ast$-prime ring with involution $\ast$ and $f$ be an additive commuting mapping in $R$. Then there exists an element $\lambda \in C_*$ and an additive mapping $\nu : R \rightarrow C$ such $f(x) + f^*(x^*) = \lambda x + \nu(x)$

**Proof.** It suffices to remark that the mapping $g(x) = f(x) + f^*(x^*)$ is an additive commuting map which commutes with $\ast$.  

\[\square\]

**Corollary 3.7.** Let $R$ be a noncommutative $\ast$-prime ring with involution $\ast$ such that $\text{char}(R) \neq 2$. Then $R$ can not admits a nonzero symmetric bi-derivation.

**Proof.** Suppose that $R$ is a noncommutative $\ast$-prime ring and $B$ a nonzero symmetric bi-derivation on $R$. By virtue of theorem 3.4, there exists $\lambda \in C_*$ such that $B(x, y) + B^*(y^*, x^*) = \lambda[x, y]$ for all $x, y \in R$. As $B$ is symmetric then $\lambda[y, x] = B(y, x) + B^*(x^*, y^*) = \lambda[x, y]$ for all $x, y \in R$. It follows that $2\lambda[x, y] = 0$ so that $[x, y] = 0$ for all $x, y \in R$. This result is impossible because $R$ is not commutative.

\[\square\]

### 4 Open Problem

In this work we have described the bi-derivations $B : R \times R \rightarrow R$ where $R$ is a $\ast$-prime ring under the condition (c): $B^*(x, y) = B(y^*, x^*)$, $\forall x, y \in R$. This result enabled us to deduce the additive commuting maps on $R$ which commute with $\ast$.

The open problems here are:

P1: Does the theorem 3.1 remain true without the condition (c)?

P2: What is the forms of the additive commuting mapping on $\ast$-prime rings?
References


