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Some generalizations involving open problems of F. Qi

B. Halim, A. Senouci[1]

Department of Mathematics, Faculty of Mathematics and Computer Sciences,
Ibn-Khaldoun University, Tiaret, Algeria
e-mail: benalihalim19@yahoo.fr; kamer295@yahoo.fr

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Abstract

***In this paper we generalize some inequalities obtained in* [2] *by considering two parameters and weight functions. Moreover we give a partial answer to an open problem posed in* [2].**

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1 Introduction

In [3], F. Qi proved the following

Theorem 1.1 *For $x_i \geq 0$, $i = 1, 2, .., n$, $n \in N$, $n \geq 2$, the inequality*

$$\frac{e^2}{4} \sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1}$$

is valid. Equality in (1) holds if $x_i = 2$ for some $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1) is the best possible.

[1]Corresponding author

Theorem 1.2 *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then*

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{2}$$

Equality in (2) holds if $x_i = 2$ for some $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (2) is the best possible.

In the same paper F. Qi posed the following two open problems.

Problem 1.3 *For $x_i \geq 0$, $i = 1, 2, .., n$, $n \in N$, $n \geq 2$, determine the best possible constants α_n $\lambda_n \in \mathbb{R}$ and $\beta_n > 0$, $\mu_n < \infty$ such that*

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{3}$$

Problem 1.4 *What is the integral analogue of the two-sided inequality (3)?*

Huan-Nan Shi gave a partial answer in [5] to Problem 1.3. In [2] is given a complete answer to Problem 1.3 and partial answer to Problem 1.4. Namely the following statements were proved there.

Lemma 1.5 *Let $r > 0$, $x > 0$ real numbers, then the inequality*

$$x^r \leq \frac{r^r}{e^r} e^x \tag{4}$$

is valid. Equality in (4) holds if $x = r$. Thus, the constant $C = \frac{r^r}{e^r}$ in (4) is the best possible.

Theorem 1.6 *Let $0 < p \leq 1$ and $x_i > 0$, $i = 1, 2, .., n$, $n \in N$ and $n \geq 2$, then*

$$\sum_{i=1}^{n} x_i^p \leq n^{1-p}\frac{p^p}{e^p}\exp\left(\sum_{i=1}^{n} x_i\right). \tag{5}$$

is valid. Equality in (5) holds if $x_i = p\, n^{-1}$ for all $i = 1, .., n$. Thus the constant $n^{1-p}\frac{p^p}{e^p}$ is the best possible.

Theorem 1.7 *let $0 < p \leq 1$ and $x_i > 0$, $i = 1, 2, .., n$, $n \in N$ and $n \geq 2$ such that $0 < x_i \leq p$ for all $i = 1, ..., n$, then*

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p}. \tag{6}$$

is valid. Equality in (6) holds if $x_i = p$ for all $i = 1, .., n$. Thus the constant $\frac{p^p}{n} e^{np}$ is the best possible.

Theorem 1.8 *Let $0 < p \leq 1$, be a real number and let f be a non negative continuous function on $[a, b]$, then the inequality*

$$\int_a^b f^p \, dx \leq (b-a)^{1-p} \frac{p^p}{e^p} \exp\left(\int_a^b f \, dx\right). \tag{7}$$

is valid. Equality in (7) holds if $f(x) = p(b-a)^{-1}$. Thus, the constant $\frac{p^p}{e^p}(b-a)^{1-p}$ in (7) is the best possible.

In [2] was proposed the next open problem.
For $p \geq 1$ a real number and a function f, determine the best possible constant $\alpha \in \mathbb{R}$ such that

$$\int_a^b |f(x)|^p \, dx \leq \alpha \frac{p^p}{e^p} \exp\left(\int_a^b |f(x)| \, dx\right). \tag{8}$$

The aim of this paper is to generalize Theorem 1.6 and Theorem 1.8 for two parameters p, q with weighted function w. Moreover we give integral analogue of Theorem 1.7 and an answer to an open problem posed in [2] for some functions classes.

2 Preliminaries

In this section, we state and prove the following Lemmas which are useful in the proofs of our results.

Lemma 2.1 *Let $0 < p < \infty$ and p, p' are conjugate, $f \in L_p([a, b]), g \in L_{p'}([a, b])$ and w be a weight function (non negative measurable function).*

1) If $1 \leq p$, then

$$\int_a^b f \, g \, w \, dx \leq \left(\int_a^b f^p \, w \, dx\right)^{\frac{1}{p}} \left(\int_a^b g^{p'} \, w \, dx\right)^{\frac{1}{p'}}. \tag{9}$$

2) If $0 < p < 1$, then

$$\int_a^b f \, g \, w \, dx \geq \left(\int_a^b f^p \, w \, dx\right)^{\frac{1}{p}} \left(\int_a^b g^{p'} \, w \, dx\right)^{\frac{1}{p'}}. \tag{10}$$

Proof 2.2 *The following two inequalities are the well-known Hölder inequalities.*

1) If $p \geq 1$*, then*

$$\int_a^b |fg|dx \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^{p'} dx\right)^{\frac{1}{p'}}. \qquad (*)$$

2) If $0 < p < 1$*, then*

$$\int_a^b |fg|dx \geq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^{p'} dx\right)^{\frac{1}{p'}}. \qquad (**)$$

By replacing f *and* g *with* $f_1 = fw^{\frac{1}{p}}$*,* $g_1 = gw^{\frac{1}{p'}}$ *in (*) and (**), we obtain inequalities (9) and (10).*

Lemma 2.3 *Let* $0 < p < q < \infty$*, and* f, w *be non negative measurable functions on* $[a, b]$ *such that* $\int_a^b f^q\, w\, dx < \infty$*, then*

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^q w\, dx\right)^{\frac{1}{q}}. \qquad (11)$$

Proof 2.4 *Let* $0 < p < q < \infty$*. Applying (9) of Lemma 2.1 with* $g = 1$*, and using parameter* $\frac{q}{p} > 1$*, its conjugate* $(\frac{q}{p})' = \frac{q}{q-p}$*, we obtain*

$$\int_a^b f^p\, w\, dx \leq \left(\int w\, dx\right)^{\frac{q-p}{q}} \left(\int_a^b f^q\, w\, dx\right)^{\frac{p}{q}}$$

or

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}.$$

Remark 2.5

1) If $q = \infty$*, then (11) is satisfied*

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{1}{p}} \|f\|_{L_\infty(a,b)}. \qquad (12)$$

2) If $w(x) = 1$ *in (11), then*

$$\left(\int_a^b f^p\, dx\right)^{\frac{1}{p}} \leq (b-a)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^q\, dx\right)^{\frac{1}{q}}. \qquad (13)$$

Equality in (13) holds if $f = 1$*. Thus the constant* $(b-a)^{\frac{1}{p}-\frac{1}{q}}$ *is the best possible.*

3 Main results

We give some inequalities with proofs.

Lemma 3.1 *Let* (x_i) *be a sequence of non negative real numbers and let* $0 < p \leq q \leq \infty$, *then*

$$\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}. \tag{14}$$

Proof 3.2 *The inequality*

$$\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$$

is the well known Jensen's inequality and the right-hand side inequality follows by Hölder's inequality.

The following Theorem is the generalization of Theorem 1.6.

Theorem 3.3 *Let* $0 < p \leq q$, *and* $x_i > 0$, $i = 1, 2, .., n$, $n \in N$, *then*

$$\sum_{i=1}^{n} x_i^p \leq \; n^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}. \tag{15}$$

is valid. Equality in (15) holds if $x_i = p\, n^{\frac{-1}{q}}$ *for all* $i = 1, .., n$. *Thus the constant* $n^{1-\frac{p}{q}} \dfrac{p^p}{e^p}$ *is the best possible.*

Proof 3.4 *By applying the inequality (14) and Lemma 1.5 with* $x = (\sum_{i=1}^{n} x_i^q)^{\frac{1}{q}}$, $r = p$, *we obtain*

$$\sum_{i=1}^{n} x_i^p \leq n^{1-\frac{p}{q}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{p}{q}}$$

$$\leq n^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}.$$

Remark 3.5 *If* $q = 1$ *in (15) we obtain Theorem 1.6.*

We give the integral analogue of Theorem 1.7.

Theorem 3.6 *Let $p > 0$ be a real number and f be non negative continuous function on $[a, b]$ such that $0 < f(x) \leq p(b-a)^{-1}$, then*

1) If $p \geq 1$,

$$\left(\int_a^b f(x)\,dx\right)^p \leq \frac{p^p}{e^p}\exp\left(\int_a^b f(x)\,dx\right) \leq \frac{p^{2p}}{(b-a)^{1+p}}\int_a^b f^{-p}(x)\,dx. \tag{16}$$

2) If $0 < p \leq 1$,

$$\int_a^b f^p(x)\,dx \leq (b{-}a)^{1-p}\,\frac{p^p}{e^p}\exp\left(\int_a^b f(x)\,dx\right) \leq \frac{p^{2p}}{(b-a)^{2p}}\int_a^b f^{-p}(x)\,dx. \tag{17}$$

Equalities in (16) and (17) holds if $f = p(b-a)^{-1}$. Thus the constants $C_1 = \dfrac{p^{2p}}{(b-a)^{p+1}}$, $C_2 = \dfrac{p^{2p}}{(b-a)^{2p}}$, are the best posible.

Proof 3.7 *Let $p \geq 1$ if $0 < f(x) \leq p(b-a)^{-1}$, therefore $\int_a^b f(x)\,dx \leq p$, and*

$$p^{-p}(b-a)^p \leq f^{-p}(x),$$

it follows that

$$p^{-p}(b-a)^{p+1} \leq \int_a^b f^{-p}(x)\,dx.$$

By Lemma 1.5 with $x = \int_a^b f(x)\,dx$, $r = p$, we obtain

$$\begin{aligned}\frac{e^p}{p^p}\left(\int_a^b f(x)\,dx\right)^p &\leq \exp\left(\int_a^b f(x)\,dx\right)\\ &\leq e^p\,p^p(b-a)^{-1-p}\,p^{-p}\,(b-a)^{p+1}\\ &\leq \frac{p^p\,e^p}{(b-a)^{1+p}}\int_a^b f^{-p}(x)\,dx,\end{aligned}$$

thus

$$\left(\int_a^b f(x)\,dx\right)^p \leq \frac{p^p}{e^p}\exp\left(\int_a^b f(x)\,dx\right) \leq \frac{p^{2p}}{(b-a)^{1+p}}\int_a^b f^{-p}(x)\,dx.$$

For $0 < p \leq 1$, by using (7), similarly one can prove the following two sided inequality

$$\int_a^b f^p(x)\,dx \leq (b-a)^{1-p}\,\frac{p^p}{e^p}\exp\left(\int_a^b f(x)\,dx\right) \leq \frac{p^{2p}}{(b-a)^{2p}}\int_a^b f^{-p}(x)\,dx.$$

The following Theorem is the generalization of Theorem 1.8.

Theorem 3.8 *Let* $0 < p < q \leq 1$, $r > 0$ *and* f, w *be measurable non negative functions on* $[a, b]$ *such that* $\int_a^b f^q\, w\, dx < \infty$, *then*

$$\left[\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}}\right]^r \leq \left(\int_a^b w\, dx\right)^{\frac{r}{p}-\frac{r}{q}} \frac{r^r}{e^r} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{18}$$

Equality in (18) holds if $f = r\left(\int_a^b w\right)^{\frac{-1}{q}}$. *Thus the constant* $\frac{r^r}{e^r}\left(\int_a^b w\right)^{\frac{r}{p}-\frac{r}{q}}$ *in (18) is the best possible.*

Proof 3.9 *Let* $0 < p < q \leq 1$. *By Lemma 1.5 with* $x = \left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}$, $r > 0$ *we get*

$$\left[\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}\right]^r \leq \frac{r^r}{e^r} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{19}$$

Now by applying inequality (11) and (19), we obtain

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{r}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{r}{p}-\frac{r}{q}} \left(\int_a^b f^q\, w\, dx\right)^{\frac{r}{q}}$$

$$\leq \left(\int_a^b w\, dx\right)^{\frac{r}{p}-\frac{r}{q}} \frac{r^2}{e^r} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}},$$

thus we get inequality (18).

If in inequality (18) we repalce f *by* $r\left(\int_a^b w\right)^{\frac{-1}{q}}$, *we get equality.*

Under the same assumptions of theorem 3.8, we have the following Corollaries.

Corollary 3.10 *We consider inequality (18).*

1) If in (18) $r = p$, *then*

$$\int_a^b f^p\, w\, dx \leq \frac{p^p}{e^p}\left(\int_a^b w\, dx\right)^{1-\frac{p}{q}} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{20}$$

2) If in (18) $r = q$, *then*

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{q}{p}} \leq \frac{q^q}{e^q}\left(\int_a^b w\, dx\right)^{\frac{q}{p}-1} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{21}$$

3) If in (18) $r = 1$, then

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \frac{1}{e}\left(\int_a^b w\, dx\right)^{\frac{1}{p}-\frac{1}{q}} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{22}$$

Corollary 3.11 *By putting $q = 1$ in the inequality (18), we have*

$$\left[\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}}\right]^r \leq \left(\int_a^b w\, dx\right)^{\frac{r}{p}-r} \frac{r^r}{e^r} \exp\left(\int_a^b f\, w\, dx\right). \tag{23}$$

1) If in (23) $r = p$, then

$$\int_a^b f^p\, w\, dx \leq \frac{p^p}{e^p}\left(\int_a^b w\, dx\right)^{1-p} \exp\left(\int_a^b f\, w\, dx\right). \tag{24}$$

2) If in (23) $r = 1$, then

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \frac{1}{e}\left(\int_a^b w\, dx\right)^{\frac{1}{p}-1} \exp\left(\int_a^b f\, w\, dx\right). \tag{25}$$

Remark 3.12 *Note that Corollaries 3.10 and 3.11 present generalizations of Theorem 1.8. In particalar if in (24) $w = 1$, we obtain Theorem 1.8 with the same best constant $\frac{p^p}{e^p}(b-a)^{1-p}$.*

Remark 3.13

1) If in (18) $p = 1$, then $1 < q \leq \infty$ and

$$\left(\int_a^b f\, w\, dx\right)^r \leq \left(\int_a^b w\, dx\right)^{r-\frac{r}{q}} \frac{r^r}{e^r} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{26}$$

2) If in (26) $r = 1$, then

$$\left(\int_a^b f\, w\, dx\right) \leq \left(\int_a^b w\, dx\right)^{1-\frac{1}{q}} \frac{1}{e} \exp\left(\int_a^b f^q\, w\, dx\right)^{\frac{1}{q}}. \tag{27}$$

If we put $q = \infty$ in (18) we get the following Corollary.

Corollary 3.14 *Let $0 < p \leq 1$, $r > 0$ and f, w be measurable non negative functions on $[a, b]$ such that $\|f\|_\infty < \infty$, the next inequality is valid*

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{r}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{r}{p}} \frac{r^r}{e^r} \exp\left(\|f\|_\infty\right). \tag{28}$$

1) If in (28) $r = p$, then

$$\int_a^b f^p\, w\, dx \le \left(\int_a^b w\, dx \right) \frac{p^p}{e^p} \exp\left(\|f\|_\infty \right). \tag{29}$$

2) If in (28) $r = 1$, then

$$\left(\int_a^b f^p\, w\, dx \right)^{\frac{1}{p}} \le \left(\int_a^b w\, dx \right)^{\frac{1}{p}} \frac{1}{e} \exp\left(\|f\|_\infty \right). \tag{30}$$

Remark 3.15 *1) We note that if in (29) $w = 1$, we obtain the following inequalities.*

$$\int_a^b f^p\, dx \le (b-a)^{1-p} \frac{p^p}{e^p} \exp\left(\int_a^b f\, dx \right) \le (b-a) \frac{p^p}{e^p} \exp\left(\|f\|_\infty \right). \tag{31}$$

2) If in (28) $p = 1$, then we get

$$\left(\int_a^b f\, w\, dx \right)^r \le \left(\int_a^b w\, dx \right)^r \frac{r^r}{e^r} \exp\left(\|f\|_\infty \right). \tag{32}$$

3) If in (32) $r = p$, then

$$\left(\int_a^b f\, w\, dx \right)^p \le \left(\int_a^b w\, dx \right)^p \frac{p^p}{e^p} \exp\left(\|f\|_\infty \right). \tag{33}$$

4) If in (32) $r = 1$, then

$$\int_a^b f\, w\, dx \le \left(\int_a^b w\, dx \right) \frac{1}{e} \exp\left(\|f\|_\infty \right). \tag{34}$$

3.1 Some functions classes

In this section we give a partial answer to the proposed open problem in [2] by considering some functions classes.

Lemma 3.16 *Let $1 \le p, q$ conjugate real numbers and f, g be two non negative measurable functions satisfying the condition $0 < m \le \frac{f^p}{g^q} \le M$ where m, M are positive real numbers, then*

$$\left(\int_a^b f^p\, dx \right)^{\frac{1}{p}} \left(\int_a^b g^q\, dx \right)^{\frac{1}{q}} \le \left(\frac{M}{m} \right)^{\frac{1}{pq}} \left(\int_a^b f\, g\, dx \right). \tag{35}$$

Proof 3.17 *By the condition* $0 < m \leq \dfrac{f^p}{g^q} \leq M$, *we have*

$$f\,g \geq M^{\frac{-1}{q}}\,f^p$$

which yields

$$\left(\int_a^b f^p\,dx\right)^{\frac{1}{p}} \leq M^{\frac{1}{pq}}\left(\int_a^b f\,g\,dx\right)^{\frac{1}{p}}. \tag{36}$$

On the other hand $0 < m \leq \dfrac{f^p}{g^q} \leq M$, *thus*

$$f\,g \geq m^{\frac{1}{p}}\,g^q,$$

then

$$\left(\int_a^b g^q\,dx\right)^{\frac{1}{q}} \leq m^{\frac{-1}{pq}}\left(\int_a^b f\,g\,dx\right)^{\frac{1}{q}}. \tag{37}$$

By (36) and (37) we get

$$\left(\int_a^b f^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b g^q\,dx\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}\left(\int_a^b f\,g\,dx\right). \tag{38}$$

Theorem 3.18 *Let* $1 \leq p < \infty$, q *its conjugate and* f *be a non negative continuous function on* $[a,b]$ *satisfying* $0 < m \leq f^p \leq M$, *where* m, M *are positive real numbers, then*

$$\int_a^b f^p\,dx \leq \alpha\,\frac{p^p}{e^p}\,\exp\left(\int_a^b f\,dx\right), \tag{39}$$

where $\alpha = (b-a)^{1-p}\left(\dfrac{M}{m}\right)^{1-\frac{1}{p}}$ *is the best constant.*

Proof 3.19 *Since that function* f *is continuous on* $[a,b]$, *then* f *is measurable in the same interval, consequently we can apply the previous Lemma. By putting in (35)* $g = 1$, *one obtains*

$$\left(\int_a^b f^p\,dx\right)^{\frac{1}{p}}(b-a)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}\left(\int_a^b f\,dx\right),$$

thus

$$\int_a^b f^p\,dx \leq (b-a)^{1-p}\left(\frac{M}{m}\right)^{1-\frac{1}{p}}\left(\int_a^b f\,dx\right)^p. \tag{40}$$

By applying Lemma 1.5 with $x = \int_a^b f\,dx, r = p$, *we get*

$$\int_a^b f^p\,dx \le (b-a)^{1-p}\left(\frac{M}{m}\right)^{1-\frac{1}{p}}\frac{p^p}{e^p}\exp\left(\int_a^b f\,dx\right)$$
$$= \alpha\,\frac{p^p}{e^p}\exp\left(\int_a^b f\,dx\right).$$

Equality in (39) holds if $f = p(b-a)^{-1}$ *with* $m = M$.

In the following Theorem we consider another function classe.

In 1939 L. Berwald [6] proved, via generalization of a mean value inequality of J. Favard, that if f is a non negative concave continuous function on $[0,1]$ and $0 < p \le q < \infty$, then

$$\left(\int_0^1 |f|^q\,dx\right)^{\frac{1}{q}} \le (p+1)^{\frac{1}{p}}(q+1)^{\frac{-1}{q}}\left(\int_0^1 |f|^p\,dx\right)^{\frac{1}{p}}, \tag{41}$$

where the constant $(p+1)^{\frac{1}{p}}(q+1)^{\frac{-1}{q}}$ is sharp. If $p = 1$ this is called Favard's inequality.

A coordinate transformation in (41) $y = (b-a)x + a$ brings us to the case of the interval $[a,b]$.

Lemma 3.20 *Let* $1 \le p \le q < \infty$ *and* f *be a non negative continuous concave function on* $[a,b]$, *then*

$$\left(\int_a^b f^q\,dx\right)^{\frac{1}{q}} \le (b-a)^{\frac{1}{q}-\frac{1}{p}}\frac{(1+p)^{\frac{1}{p}}}{(1+q)^{\frac{1}{q}}}\left(\int_a^b f^p\,dx\right)^{\frac{1}{p}}, \tag{42}$$

where the constant $(b-a)^{\frac{1}{q}-\frac{1}{p}}\dfrac{(1+p)^{\frac{1}{p}}}{(1+q)^{\frac{1}{q}}}$ *is sharp.*

Remark 3.21 *If* $p = 1$ *in (42), we get for all* $1 \le q$

$$\left(\int_a^b f^q\,dx\right)^{\frac{1}{q}} \le (b-a)^{\frac{1}{q}-1}\frac{2}{(1+q)^{\frac{1}{q}}}\int_a^b f\,dx, \tag{43}$$

in particular $(b-a)^{\frac{1}{q}-1}\dfrac{2}{(1+q)^{\frac{1}{q}}}$ *is also sharp constant.*

Theorem 3.22 *Let $1 \leq p < \infty$ and f be a non negative continuous concave function on $[a, b]$, then*

$$\int_a^b f^p\,dx \leq \alpha \frac{p^p}{e^p} \exp\left(\int_a^b f\,dx\right). \tag{44}$$

Where $\alpha = (b-a)^{1-p}\dfrac{2^p}{(1+p)}$ is the best possible.

Proof 3.23 *Let $1 \leq p < \infty$. By (43) we have*

$$\left(\int_a^b f^p\,dx\right)^{\frac{1}{p}} \leq (b-a)^{\frac{1}{p}-1}\frac{2}{(1+p)^{\frac{1}{p}}}\int_a^b f\,dx. \tag{45}$$

By applying lemma 1.5 with $x = \int_a^b f\,dx, r = p$, we get

$$\begin{aligned}\int_a^b f^p\,dx &\leq (b-a)^{1-p}\,\frac{2^p}{1+p}\left(\int_a^b f\,dx\right)^p\\ &\leq (b-a)^{1-p}\,\frac{2^p}{(1+p)}\frac{p^p}{e^p}\exp\left(\int_a^b f\,dx\right)\\ &= \alpha\,\frac{p^p}{e^p}\exp\left(\int_a^b f\,dx\right).\end{aligned}$$

From (43) it follows that $\alpha, \dfrac{p^p}{e^p}$ is the sharp constant in (44), consequently α is the best constant in (44).

Remark 3.24 *The following inequality is the well known Hardy inequality (for more details see [4], [1]), $\int_a^b (Hf)^p(x)\,dx \leq \left(\dfrac{p}{p-1}\right)^p \int_a^b f^p(x)\,dx$, where $p > 1$, $Hf(x) = \frac{1}{x}\int_a^x f(x)\,dx$, the constant $\left(\dfrac{p}{p-1}\right)^p$ is the best possible.*
By inequality (44) we get

$$\int_a^b (Hf)^p\,dx \leq \left(\frac{p}{p-1}\right)^p \alpha\,\frac{p^p}{e^p}\exp\left(\int_a^b f\,dx\right). \tag{46}$$

4 Open problems

Problem 4.1 *In Theorem 3.6, replace the condition $0 < f(x) \leq p(b-a)^{-1}$ by weaker one.*

Problem 4.2 *For $p < 0$ a real number, determine the best possible constant $\alpha \in \mathbb{R}$ such that*

$$\alpha \cdot \frac{e^p}{p^p} \int_a^b |f(x)|^p dx \leq \exp\left(\int_a^b |f(x)| dx\right).$$

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