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Trapezoid type inequalities whose derivatives are $\log -$MT-convex

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Abstract

In this note, we first introduce the class of logarithmically MT-convex functions, and then we establish some trapezoid type inequalities for function whose derivatives are in this novel class.

Keywords: *Trapezoid inequality;* $\log -MT$*-convex function; Hölder inequality; power mean inequality.*
MSC (2010): 26D15, 26D20, 26A51.

1 Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if the following inequality

$$f(tx + (1-t)y) \ \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (see [3]).

It is well known that the convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

Varošanec [5] introduced the concept of the h-convexity which is a generalization of the classical convexity defined as follows:

A nonnegative function $f : I \to \mathbb{R}$ is said to be h-convex if

$$f\left(tx + (1-t)\,y\right) \leq h\left(t\right) f(x) + h\left(1-t\right) f(y)$$

holds for all $x, y \in I$, and $t \in (0,1)$, where $h : (0,1) \subset I \to \mathbb{R}$ is positive function.

Noor et al. [2] gave the concept of the logarithmically h-convex functions as follows

A positive function $f : I \to \mathbb{R}$ is said to be log-h-convex if

$$f\left(tx + (1-t)\,y\right) \leq \left[f(x)\right]^{h(t)} \left[f(y)\right]^{h(1-t)}$$

holds for all $x, y \in I$, and $t \in (0,1)$, where $h : (0,1) \subset I \to \mathbb{R}$ is positive function.

Tunç et al. [4] introduced a novel kinds of convexity called MT-convexity which represent a special case of the h-convexity defined as follows:

A nonnegative function $f : I \to \mathbb{R}$ is said to be MT-convex if

$$f\left(tx + (1-t)\,y\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)$$

holds for all $x, y \in I$, and $t \in (0,1)$.

In 1998, Dragomir and Agarwal [1] establishes the following identity

Lemma 1.1. *[1] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L\left([a,b]\right)$, then the following equality holds*

$$\frac{f\left(a\right) + f\left(b\right)}{2} - \frac{1}{b-a}\int_a^b f\left(x\right) dx = \frac{b-a}{2}\int_0^1 \left(1-2t\right) f'\left(ta + (1-t)\,b\right) dt.$$

We also recall that the ν-weighted arithmetic–geometric mean inequality can be says that for $a, b \geq 0$ and $0 \leq \nu \leq 1$

$$a^\nu b^{1-\nu} \leq \nu a + \left(1-\nu\right) b.$$

The incomplete beta function is defined as follows:

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dx$$

for $x\in[0,1]$ and $\alpha,\beta>0$, where $B_1(\alpha,\beta)=B(\alpha,\beta)$ be a beta function.

The hypergeometric function is defined as follows:

$$ {}_2F_1\left(a,b,c;z\right)=\frac{1}{B\left(b,c-b\right)}\int\limits_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}dt. $$

In this note, we first introduce the class of logarithmically MT-convex functions, and then we establish some trapezoid type inequalities for function whose derivatives are in this novel class.

2 Main results

We first, introduce the notion of logarithmically MT-convex function

Definition 2.1. *A positive function $f:I\to\mathbb{R}$ is said to be* log- *MT-convex if*

$$ f\left(tx+\left(1-t\right)y\right)\leq\ \left[f(x)\right]^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left[f(y)\right]^{\frac{\sqrt{1-t}}{2\sqrt{t}}} $$

holds for all $x,y\in I$, and $t\in(0,1)$.

Theorem 2.2. *Let $f:[a,b]\subset R\to\mathbb{R}$ be a differentiable mapping on $[a,b]$ such that $f'\left(x\right)\neq0$ for all $x\in[a,b]$. If $|f'|$ is* log-*MT- convex, then we have*

$$ \left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int\limits_a^b f\left(x\right)dx\right| \tag{1} $$
$$ \leq\tfrac{b-a}{8\sqrt{2}}\left(\sqrt{2}\left(\left|f'\left(b\right)\right|^2+\left|f'\left(a\right)\right|^2\right)-\left(\left|f'\left(a\right)\right|-\left|f'\left(b\right)\right|\right)^2\left({}_2F_1\left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{7}{2};\tfrac{1}{2}\right)\right)\right), $$

where ${}_2F_1\left(.,.,.;.\right)$ is the hypergeometric function.

Proof. From Lemma 1.1, properties of modulus, and log-MT-convexity of $|f'|$, we have

$$ \left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int\limits_a^b f\left(x\right)dx\right|\leq\tfrac{b-a}{2}\left(\int\limits_0^{\frac{1}{2}}\left(1-2t\right)\left|f'\left(a\right)\right|^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left|f'\left(b\right)\right|^{\frac{\sqrt{1-t}}{2\sqrt{t}}}dt\right. $$
$$ \left.+\int\limits_{\frac{1}{2}}^1\left(2t-1\right)\left|f'\left(a\right)\right|^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left|f'\left(b\right)\right|^{\frac{\sqrt{1-t}}{2\sqrt{t}}}dt\right). \tag{2} $$

Clearly, we have

$$ \begin{cases}\frac{\sqrt{t}}{2\sqrt{1-t}}\leq\frac{1}{2}\leq\frac{\sqrt{1-t}}{2\sqrt{t}}<1\text{ if }0<t\leq\frac{1}{2}\\ \frac{\sqrt{1-t}}{2\sqrt{t}}\leq\frac{1}{2}\leq\frac{\sqrt{t}}{2\sqrt{1-t}}<1\text{ if }\frac{1}{2}\leq t<1.\end{cases} \tag{3} $$

From (3), we deduce

$$\begin{cases} \frac{\sqrt{1-t}}{2\sqrt{t}} < 2 - \frac{\sqrt{t}}{2\sqrt{1-t}} \text{ if } 0 < t \leq \frac{1}{2} \\ \frac{\sqrt{t}}{2\sqrt{1-t}} < 2 - \frac{\sqrt{1-t}}{2\sqrt{t}} \text{ if } \frac{1}{2} \leq t < 1. \end{cases} \tag{4}$$

Using (4) in (2) we get

$$\begin{aligned} &\left| \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a} \int_a^b f(x)\, dx \right| \\ \leq &\tfrac{b-a}{2} \left(|f'(b)| \int_0^{\frac{1}{2}} (1-2t) |f'(a)|^{\frac{\sqrt{t}}{2\sqrt{1-t}}} |f'(b)|^{1-\frac{\sqrt{t}}{2\sqrt{1-t}}} dt \right. \\ &\left. + |f'(a)| \int_{\frac{1}{2}}^1 (2t-1) |f'(a)|^{1-\frac{\sqrt{1-t}}{2\sqrt{t}}} |f'(b)|^{\frac{\sqrt{1-t}}{2\sqrt{t}}} dt \right). \end{aligned} \tag{5}$$

Now, applying the well known Young's inequality for (5) yields

$$\begin{aligned} &\left| \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a} \int_a^b f(x)\, dx \right| \\ \leq &\tfrac{b-a}{2} \left(|f'(b)| \int_0^{\frac{1}{2}} (1-2t) \left(\tfrac{\sqrt{t}}{2\sqrt{1-t}} |f'(a)| + \left(1 - \tfrac{\sqrt{t}}{2\sqrt{1-t}}\right) |f'(b)| \right) dt \right. \\ &\left. + |f'(a)| \int_{\frac{1}{2}}^1 (2t-1) \left(\left(1 - \tfrac{\sqrt{1-t}}{2\sqrt{t}}\right) |f'(a)| + \tfrac{\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right) dt \right) \\ = &\tfrac{b-a}{2} \left(\tfrac{1}{4\sqrt{2}} \left(|f'(a)| |f'(b)| - |f'(b)|^2 \right) \int_0^1 t^{\frac{1}{2}} (1-t) \left(1 - \frac{1}{2} t\right)^{-\frac{1}{2}} dt \right. \\ &+ \tfrac{1}{4} |f'(b)|^2 + \tfrac{1}{4} |f'(a)|^2 \\ &\left. + \tfrac{1}{4\sqrt{2}} \left(|f'(a)| |f'(b)| - |f'(a)|^2 \right) \int_0^1 t^{\frac{1}{2}} (1-t) \left(1 - \frac{1}{2} t\right)^{-\frac{1}{2}} dt \right) \\ = &\tfrac{b-a}{8\sqrt{2}} \left(\sqrt{2} \left(|f'(b)|^2 + |f'(a)|^2 \right) - \left(|f'(a)| - |f'(b)| \right)^2 \left({}_2F_1 \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}; \tfrac{1}{2} \right) \right) \right), \end{aligned}$$

which is the desired result. □

Theorem 2.3. *Let $f:[a,b]\subset R\rightarrow\mathbb{R}$ be a differentiable mapping on $[a,b]$ such that $f'(x)\neq 0$ for all $x\in[a,b]$. If $|f'|^q$ is* log*-MT- convex where $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then we have*

$$\left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{6}$$
$$\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(+\left(|f'(b)|^q|f'(a)|^q-\left(|f'(a)|^q\right)^2\right)B\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right.$$
$$\left.+\left(\left(|f'(a)|^q\right)^2-\left(|f'(b)|^q\right)^2\right)B_{\frac{1}{2}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)+\left(|f'(b)|^q\right)^2+\left(|f'(a)|^q\right)^2\right),$$

where $B(.,.)$ and $B_{\frac{1}{2}}(.,.)$ are the beta and the incomplete beta functions respectively.

Proof. From Lemma 1.1, properties of modulus, Hölder inequality, log-MT-convexity of $|f'|^q$, (4), and Young's inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}\,dt\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^{\frac{1}{2}}\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}\,dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}\,dt\right)$$
$$\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q\int_0^{\frac{1}{2}}\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{1-\frac{\sqrt{t}}{2\sqrt{1-t}}}\,dt\right.$$
$$\left.+|f'(a)|^q\int_{\frac{1}{2}}^1\left(|f'(a)|^q\right)^{1-\frac{\sqrt{1-t}}{2\sqrt{t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}\,dt\right)$$

$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left(\left(\left|f'(a)\right|^q\right)\left|f'(b)\right|^q-\left(\left|f'(b)\right|^q\right)^2\right)\int_0^{\frac{1}{2}} t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt\right.$$

$$+2\left(\left|f'(b)\right|^q\right)^2\int_0^{\frac{1}{2}} dt+2\left(\left|f'(a)\right|^q\right)^2\int_{\frac{1}{2}}^1 dt$$

$$\left.+\left(\left|f'(b)\right|^q\left|f'(a)\right|^q-\left(\left|f'(a)\right|^q\right)^2\right)\int_{\frac{1}{2}}^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt\right)$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left(\left(\left|f'(a)\right|^q\right)\left|f'(b)\right|^q-\left(\left|f'(b)\right|^q\right)^2\right)B_{\frac{1}{2}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right.$$

$$+\left(\left|f'(b)\right|^q\right)^2+\left(\left|f'(a)\right|^q\right)^2-\left(\left|f'(b)\right|^q\left|f'(a)\right|^q-\left(\left|f'(a)\right|^q\right)^2\right)$$

$$\left.\times B_{\frac{1}{2}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)+\left(\left|f'(b)\right|^q\left|f'(a)\right|^q-\left(\left|f'(a)\right|^q\right)^2\right)B\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right)$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(+\left(\left|f'(b)\right|^q\left|f'(a)\right|^q-\left(\left|f'(a)\right|^q\right)^2\right)B\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right.$$

$$\left.+\left(\left(\left|f'(a)\right|^q\right)^2-\left(\left|f'(b)\right|^q\right)^2\right)B_{\frac{1}{2}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)+\left(\left|f'(b)\right|^q\right)^2+\left(\left|f'(a)\right|^q\right)^2\right),$$

which is the desired result. □

Theorem 2.4. *Under the hypotheses of Theorem 2.3, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{2^{2+\frac{1}{q}+\frac{1}{2q}}}\left(\sqrt{2}\left(\left(\left|f'(a)\right|^q\right)^2+\left(\left|f'(b)\right|^q\right)^2\right)\right.\tag{7}$$

$$\left.-\left(\left(\left|f'(a)\right|^q\right)-\left(\left|f'(b)\right|^q\right)\right)^2\left({}_2F_1\left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{7}{2};\tfrac{1}{2}\right)\right)\right)^{\frac{1}{q}},$$

where ${}_2F_1(.,.,.;.)$ *is the Hypergeometric function.*

Proof. From Lemma 1.1, properties of modulus, power mean inequality, log-MT-convexity of $|f'|^q$, (4), and Young's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq\frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\int_0^{\frac{1}{2}}(1-2t)\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(2t-1)\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\left(|f'(b)|^q\right)\int_0^{\frac{1}{2}}(1-2t)\left(|f'(a)|^q\right)^{\frac{\sqrt{t}}{2\sqrt{1-t}}}\left(|f'(b)|^q\right)^{1-\frac{\sqrt{t}}{2\sqrt{1-t}}}dt\right.$$

$$\left.+\left(|f'(a)|^q\right)\int_0^{\frac{1}{2}}(2t-1)\left(|f'(a)|^q\right)^{1-\frac{\sqrt{1-t}}{2\sqrt{t}}}\left(|f'(b)|^q\right)^{\frac{\sqrt{1-t}}{2\sqrt{t}}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\left(|f'(b)|^q\right)\int_0^{\frac{1}{2}}(1-2t)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left(|f'(a)|^q\right)+\left(1-\frac{\sqrt{t}}{2\sqrt{1-t}}\right)\left(|f'(b)|^q\right)\right)dt\right.$$

$$\left.+\left(|f'(a)|^q\right)\int_{\frac{1}{2}}^{1}(2t-1)\left(\left(1-\frac{\sqrt{1-t}}{2\sqrt{t}}\right)\left(|f'(a)|^q\right)+\frac{\sqrt{1-t}}{2\sqrt{t}}\left(|f'(b)|^q\right)\right)dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\left(\left(|f'(a)|^q\right)\left(|f'(b)|^q\right)-\left(|f'(b)|^q\right)^2\right)\int_0^{\frac{1}{2}}(1-2t)\frac{\sqrt{t}}{2\sqrt{1-t}}dt\right.$$

$$+\left(|f'(b)|^q\right)^2\int_0^{\frac{1}{2}}(1-2t)+\left(|f'(a)|^q\right)^2\int_{\frac{1}{2}}^{1}(2t-1)\,dt$$

$$\left.+\left(\left(|f'(a)|^q\right)\left(|f'(b)|^q\right)-\left(|f'(a)|^q\right)^2\right)\int_{\frac{1}{2}}^{1}(2t-1)\frac{\sqrt{1-t}}{2\sqrt{t}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{4\sqrt{2}}\left(\left(|f'(a)|^q\right)\left(|f'(b)|^q\right)-\left(|f'(b)|^q\right)^2\right)\int_0^{1}t^{\frac{1}{2}}(1-t)\left(1-\tfrac{1}{2}t\right)^{-\frac{1}{2}}dt\right.$$

$$+\frac{1}{4\sqrt{2}}\sqrt{2}\left(\left(|f'(a)|^q\right)^2+\left(|f'(b)|^q\right)^2\right)$$

$$+\frac{1}{4\sqrt{2}}\left(\left(\left|f'(a)\right|^{q}\right)\left(\left|f'(b)\right|^{q}\right)-\left(\left|f'(a)\right|^{q}\right)^{2}\right)\int_{0}^{1}t^{\frac{1}{2}}(1-t)\left(1-\frac{1}{2}t\right)^{-\frac{1}{2}}dt\Bigg)^{\frac{1}{q}}$$

$$=\frac{b-a}{2^{2+\frac{1}{q}+\frac{1}{2q}}}\left(-\left(\left(\left|f'(a)\right|^{q}\right)-\left(\left|f'(b)\right|^{q}\right)\right)^{2}\left({}_{2}F_{1}\left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{7}{2};\tfrac{1}{2}\right)\right)\right.$$

$$\left.+\sqrt{2}\left(\left(\left|f'(a)\right|^{q}\right)^{2}+\left(\left|f'(b)\right|^{q}\right)^{2}\right)\right)^{\frac{1}{q}}.$$

Thus, the proof is achieved. □

Open Problems

Can we find finer estimates for inequalities (1), (6), and (7)?

Can we prove the previous theorems without having resort to use the Young's inequality?

References

[1] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*. Appl. Math. Lett. 11 (1998), no. 5, 91–95.

[2] M. A. Noor, F. Qi and M. U. Awan, *Some Hermite-Hadamard type inequalities for log-h-convex functions*. Analysis (Berlin) 33 (2013), no. 4, 367–375.

[3] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[4] M. Tunç, H. Yildirim, *On MT-convexity*, arXiv:1205.5453v1(2012) [math.CA].

[5] S. Varošanec, *On h-convexity*. J. Math. Anal. Appl. 326 (2007), no. 1, 303–311.