Dualistic structure on Sasakian manifolds

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Abstract

Affine connections compatible with a contact structure are defined and conditions for two compatible connections on a Sasaki manifold to form a dualistic structure are given.

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1 Introduction

A fundamental theorem of Riemannian geometry states that given a metric, there is a unique connection (among the class of torsion-free connections) that "preserves" the metric, i.e., the following condition is satisfied

\[ X.g(Y, Z) = g(\widehat{\nabla} X Y, Z) + g(Y, \widehat{\nabla} X Z) \]  

for all vectors fields \( X, Y, Z \) on \( M \).

Such a connection, denoted as \( \widehat{\nabla} \), is called the Levi-Civita connection. It is compatible with the metric.

It turns out that one can define a kind of "compatibility" relation more general than expressed by (1), by introducing the notion of "conjugacy" (defined by *) between two connections.
Geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences [3].

Let \((M, g)\) be a Riemannian manifold and \(\nabla\) an affine connection on \(M\). A connection \(\nabla^*\) is called conjugate connection of \(\nabla\) with respect to the metric \(g\) if

\[
X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \quad (2)
\]

for arbitrary vectors fields \(X, Y, Z\) on \(M\). The triple of a Riemannian metric and a pair of conjugates connection \((g, \nabla, \nabla^*)\) satisfying (2) is called a dualistic structure on \(M\).

Clearly, \((\nabla^*)^* = \nabla\). Moreover, \(\hat{\nabla}\), which satifies (1), is special in the sense that it is self-conjugate \(\hat{\nabla}^* = \hat{\nabla}\).

In the present paper, we shall introduce the notion of affine connections compatible with a contact structure and provide conditions for two such compatible connections to form a dualistic structure on Sasaki manifold.

2 Preliminaries

For more background on almost contact metric manifolds, we recommend the reference [5, 7].

An odd-dimensional Riemannian manifold \((M, g)\) is said to be an almost contact metric manifold if there exist on \(M\) a \((1, 1)\) tensor field \(\varphi\), a vector field \(\xi\) (called the structure vector field) and a 1-form \(\eta\) such that \(\eta(\xi) = 1\), \(\phi^2(X) = -X + \eta(X)\xi\) and \(g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\), for any vector fields \(X, Y\) on \(M\). In particular, in an almost contact metric manifold we also have \(\varphi \xi = 0\) and \(\eta \circ \varphi = 0\).

Such a manifold is said to be a contact metric manifold if \(d\eta = \phi\), where \(\phi(X, Y) = g(X, \varphi Y)\) is called the fundamental 2-form of \(M\). If, in addition, \(\xi\) is a Killing vector field, then \(M\) is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if \(\nabla_X \xi = -\varphi X\), for any vector field \(X\) on \(M\). In a K-contact manifold, we have

\[
(\nabla_X \varphi)Y = R(\xi, X)Y
\]

for any \(X, Y\) ([5], pp. 92-94).

On the other hand, the almost contact metric structure of \(M\) is said to be normal if \([\varphi, \varphi](X, Y) = -2d\eta (X, Y)\xi\), for any \(X, Y\), where \([\varphi, \varphi]\) denotes the Nijenhuis torsion of \(\varphi\), given by

\[
[\varphi, \varphi](X, Y) = \phi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]
\]
A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if
\[(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.\]

\section{Main results}

Let \((M^{2n+1}, \varphi, \xi, \eta, \phi, g)\) be a contact metric manifold where
\[
\phi(X, Y) = d\eta(X, Y) = g(X, \varphi Y)
\]
the 2-form on \(M\) for all \(X, Y\) vectors field on \(M\).

We say that a pair of affine connections \(\nabla\) and \(\nabla'\) are compatible with the contact structure \(\phi\) if
\[
X.\phi(Y, Z) = \phi(\nabla_X Y, Z) + \phi(Y, \nabla'_X Z) + \eta(Y)g(X, Z) - \eta(Z)g(X, Y). \tag{3}
\]

In what follows, we shall prove some preliminary results needed to determine a dualistic structure on a Sasakian manifold (see Theorem 3.11).

\textbf{Proposition 3.1} Let \(\nabla\) and \(\nabla'\) be an affine connections on the smooth manifold \(M\), compatible with the contact structure \(\phi\) and \(T, T'\) their torsion tensors respectively. Then:

(a)  
\[
\phi(T(X, Y) - T'(X, Y), Z) \quad = \quad (\nabla'\phi)(X, Y, Z) - (\nabla'\phi)(Y, X, Z) + \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]

(b)  
\[
(\nabla\phi)(Y, X, Z) - (\nabla\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z)
\]

if and only if
\[
(\nabla'\phi)(Y, X, Z) - (\nabla'\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]

for all \(X, Y, Z\) vectors fields on \(M\).

\textbf{Proof 3.2} Using the compatibility condition, we get
(a) \[
\phi(T(X,Y) - T'(X,Y), Z) := \phi(\nabla_X Y - \nabla_Y X - \nabla'_X Y + \nabla'_Y X, Z)
\]
\[
= X.\phi(Y, Z) - \phi(Y, \nabla_X Z) - \eta(Y)g(X, Z) \\
+ \eta(Z)g(X, Y) - Y.\phi(X, Z) + \phi(X, \nabla_Y Z) \\
+ \eta(X)g(Y, Z) - \eta(Z)g(X, Y) - \phi(\nabla'_X Y, Z) \\
+ \phi(\nabla'_Y X, Z) \\
= (\nabla'\phi)(X, Y, Z) - (\nabla'\phi)(Y, X, Z) \\
+ \eta(X)g(Y, Z) - \eta(Y)g(X, Z).
\]

(b) We have
\[
(\nabla'\phi)(X, Y, Z) := X.\phi(Y, Z) - \phi(\nabla'_X Y, Z) - \phi(Y, \nabla'_X Z),
\]
and we knowing that
\[
\phi(\nabla'_X Y, Z) = - \phi(Z, \nabla'_X Y) \\
= -X.\phi(Z, Y) + \phi(\nabla_X Z, Y) - \eta(Y)g(X, Z) + \eta(Z)g(X, Y) \\
= X.\phi(Y, Z) - \phi(Y, \nabla_X Z) - \eta(Y)g(X, Z) + \eta(Z)g(X, Y),
\]
then
\[
(\nabla'\phi)(X, Y, Z) := -X.\phi(Y, Z) + \phi(\nabla_X Y, Z) + \phi(Y, \nabla_X Z) \\
+ 2\eta(Y)g(X, Z) - 2\eta(Z)g(X, Y) \\
= -(\nabla\phi)(X, Y, Z) + 2\eta(Y)g(X, Z) - 2\eta(Z)g(X, Y).
\]

Now suppose that for all \(X, Y, Z\) vectors fields on \(M\),
\[
(\nabla'\phi)(X, Y, Z) - (\nabla'\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]
using the above equation, we obtain
\[
(\nabla\phi)(X, Y, Z) - (\nabla\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]
and conversely.

**Corollary 3.3** Let \(\nabla\) and \(\nabla'\) be an affine connections on the smooth manifold \(M\), compatible with the contact structure \(\phi\) and \(T, T'\) their torsion tensors respectively. Then:

(a) If \( (\nabla'\phi)(Y, X, Z) = (\nabla'\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z) \),
then \( T' = T \).
(b) If \((\nabla' \phi)(Y, X, Z) - (\nabla \phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z)\), and \(T' = 0\),
then \((\nabla \phi)(Y, X, Z) - (\nabla \phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z)\)
and \(T = 0\), too.

**Proof 3.4** Follows from Proposition 3.1

**Proposition 3.5** Let \(\nabla\) be an affine connection on the Sasakian manifold \(M\) such that
\[(\nabla X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad (\ast)\]
for all \(X, Y\) vectors fields on \(M\).
If \(\nabla\) and \(\nabla'\) are compatible with the contact structure \(\phi\), then
\[
\nabla' \varphi = \nabla \varphi.
\]

**Proof 3.6** By \((\ast)\), we have
\[
\phi(Z, (\nabla' X \varphi)Y - (\nabla X \varphi)Y) = \phi(Z, (\nabla' X \varphi)Y) - \phi(Z, (\nabla X \varphi)Y)
= \phi(Z, X \varphi Y) + \phi(\varphi Z, \nabla_X Y) + \eta(Y)\phi(Z, X),
\]
using the compatibility condition, we obtain
\[
\phi(Z, (\nabla' X \varphi)Y - (\nabla X \varphi)Y) = X.\phi(Z, \varphi Y) - \phi(\nabla X Z, \varphi Y) - \eta(Z)g(X, \varphi Y)
+ X.\phi(\varphi Z, Y) - \phi(\nabla X \varphi Z, Y)
= -\eta(Z)g(X, \varphi Y) - \phi((\nabla X \varphi)Z, Y),
\]
replacing \((\nabla X \varphi)Z = g(X, Z)\xi - \eta(Z)X\), we obtain
\[
\phi(Z, (\nabla' X \varphi)Y - (\nabla X \varphi)Y) = 0 \iff \nabla' \varphi = \nabla \varphi.
\]

**Lemma 3.7** Let \(\nabla\) be an affine connection on the Sasakian manifold \((M, \varphi, \xi, \eta, g)\).
If
\[(\nabla X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \]
for all \(X, Y\) vectors fields on \(M\) then,
1. \(\nabla X \xi = -\varphi X + \eta(\nabla X \xi)\xi,\)
2. \(\eta(\nabla X \varphi)X = \eta(\nabla X \varphi)\xi = 0,\)
3. \(\eta(\nabla X \xi) = g(X, \nabla \xi) = \eta(X)\eta(\nabla \xi)\).

**Proof 3.8** Suppose that for all \(X, Y\) vectors fields on \(M\),
\[(\nabla X \varphi)Y = g(X, Y)\xi - \eta(Y)X.\]
1. We can see that $\nabla_{\xi} \varphi = 0$ and for $Y = \xi$ we have

$$(\nabla_{X} \varphi)\xi = \eta(X)\xi - X \iff -\varphi \nabla_{X} \xi = \eta(X)\xi - X$$

$\iff \varphi^{2} \nabla_{X} \xi = \varphi X$

$\iff \nabla_{X} \xi = -\varphi X + \eta(\nabla_{X} \xi),$ 

2. We have

$$2d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]),$$

for $Y = \xi$ we obtain

$$d\eta(X,\xi) = -\xi\eta(X) - \eta([X,\xi]),$$

knowing that $d\eta(X,Y) = \phi(X,Y) = g(X,\varphi Y)$ then,

$$\xi\eta(X) = \eta([\xi,X]),$$

replacing $X$ by $\varphi X$ we obtain

$$\eta([\xi,\varphi X]) = 0 \iff \eta(\nabla_{\varphi X} \xi) \quad = \quad \eta(\nabla_{\xi} \varphi X)$$

$$\quad = \quad \eta((\nabla_{\xi} \varphi)X) + \eta(\varphi \nabla_{\xi} X)$$

$$\quad = \quad 0.$$

3. We have $X = -\varphi^{2}X + \eta(X)\xi$ so,

$$\eta(\nabla_{X} \xi) = -\eta(\nabla_{\varphi^{2}X} \xi) + \eta(X)\eta(\nabla_{\xi} \xi),$$

using lemma (3.7, 2) we obtain

$$\eta(\nabla_{X} \xi) = \eta(X)\eta(\nabla_{\xi} \xi).$$

On the other hand,

$$g(X,\nabla_{\xi} \xi) = g(-\varphi^{2}X + \eta(X)\xi,\nabla_{\xi} \xi)$$

$$\quad = \quad g(\varphi X, \varphi \nabla_{\xi} \xi) + \eta(X)\eta(\nabla_{\xi} \xi),$$

we can see that $\varphi \nabla_{\xi} \xi = 0$ so,

$$g(X,\nabla_{\xi} \xi) = \eta(X)\eta(\nabla_{\xi} \xi).$$

which prove our assertions.
Proposition 3.9 Let $\nabla$ be an affine connection on the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ such that for all $X, Y$ vectors fields on $M$,

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$ 

If

$$(\nabla \phi)(Y, X, Z) - (\nabla \phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),$$

for all $X, Y, Z$ vectors fields on $M$ then $\nabla g$ is symmetric.

Proof 3.10 For all $X, Y, Z$ vectors fields on $M$, we have

$$(\nabla g)(X, Y, \varphi Z) := X.g(Y, \varphi Z) - g(\nabla_X Y, \varphi Z) - g(Y, \nabla_X (\varphi Z))$$

$$= X.\phi(Y, Z) - \phi(\nabla_X Y, Z) - g(Y, (\nabla_X \varphi)Z + \varphi \nabla_X Z)$$

$$= X.\phi(Y, Z) - \phi(\nabla_X Y, Z) - \phi(Y, \nabla_X Z) - g(Y, (\nabla_X \varphi)Z),$$

using again $(\nabla_X \varphi)Z = g(X, Z)\xi - \eta(Z)X$, we get

$$(\nabla g)(X, Y, \varphi Z) = (\nabla \phi)(X, Y, Z) - \eta(Y)g(X, Z) + \eta(Z)g(Y, X), \quad (4)$$

and we have also

$$(\nabla g)(Y, X, \varphi Z) = (\nabla \phi)(Y, X, Z) - \eta(X)g(Y, Z) + \eta(Z)g(X, Y), \quad (5)$$

differentiating 4 from 5 we obtain

$$(\nabla g)(Y, X, \varphi Z) - (\nabla g)(X, Y, \varphi Z) = (\nabla \phi)(Y, X, Z) - (\nabla \phi)(X, Y, Z)$$

$$- \eta(X)g(Y, Z) + \eta(Y)g(X, Z).$$

so, if

$$(\nabla \phi)(Y, X, Z) - (\nabla \phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),$$

then

$$(\nabla g)(X, Y, \varphi Z) = (\nabla g)(Y, X, \varphi Z).$$

On the other hand, we have

$$(\nabla g)(X, Y, \xi) = X.\eta(Y) - \eta(\nabla_X Y) - g(Y, \nabla_X \xi),$$

using lemma (3.7,1) then lemma (3.7,3) we obtain,

$$(\nabla g)(X, Y, \xi) = X.\eta(Y) - \eta(\nabla_X Y) + d\eta(Y, X) - \eta(Y)\eta(X)\eta(\nabla_\xi \xi).$$

In the same way, we have

$$(\nabla g)(Y, X, \xi) = Y.\eta(X) - \eta(\nabla_Y X) + d\eta(X, Y) - \eta(X)\eta(Y)\eta(\nabla_\xi \xi).$$

differentiating the two above equations we obtain

$$(\nabla g)(X, Y, \xi) - (\nabla g)(X, Y, \xi) = 0.$$

This completes the proof of the proposition.
Recall that \((M, \nabla, g)\) is statistical manifold if \(\nabla\) is a torsion free affine connection and \(\nabla g\) is symmetric, for \(g\) a pseudo-Riemannian metric on \(M\). If \(\nabla'\) is another affine connection on \(M\) such that \(\nabla\) and \(\nabla'\) are compatible with \(g\), then \(\nabla'\) is also torsion free and \(\nabla' g\) is symmetric. In this case, \((M, \nabla', g)\) is also statistical manifold called the dual statistical manifold of \((M, \nabla, g)\) and we say that \((\nabla, \nabla', g)\) is a dualistic structure on \(M\) [2].

**Theorem 3.11** Let \((M, \varphi, \xi, \eta, \phi, g)\) be a Sasakian manifold and let \(\nabla'\) be a torsion free affine connection on \(M\) such that
\[
(\nabla'_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,
\]
\[
(\nabla'\phi)(Y, X, Z) - (\nabla'\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]
for all \(X, Y, Z\) vectors field on \(M\) and \(\nabla\) and \(\nabla'\) are compatible with \(\phi\). Then \((M, \nabla', g)\) and \((M, \nabla, g)\) are statistical manifolds and \((\nabla, \nabla', g)\) is a dualistic structure on \(M\).

**Proof 3.12** Follows from propositions (3.1), (3.5) and (3.9).

4 Open Problem

By considering propositions 4 and 7 in [4], it is proved that there are Kähler and Sasakian manifolds of any dimensions. We start with \(\mathbb{R}^2\) and its natural Kähler structure. There is a 3-dimension Sasakian manifold \(M = \mathbb{R} \times \mathbb{R}^2\). Then, by the method of Oubină [6] and consequently we have a Kählerian manifold of dimension 4. Continuing the current method, we produce Kähler and Sasaki manifolds of any dimensions \(n \geq 2\).

regarding this close relation, one ask if it is possible to construct a dualistic structure on a Kählerian manifold starting from a dualistic structure on a Sasakian manifold and vice versa, based on this paper and work [2] presented by M. Blaga.

References


