Value Distribution and Uniqueness of Some Difference-Differential Polynomials of Meromorphic Functions

Naveenkumar S H

Department of Mathematics, Assistant Professor, Bangalore campus, GITAM University, Bangalore 561203, INDIA
e-mail: naveenkumarsh.220@gmail.com

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Abstract

In this paper, we discuss the value distribution and uniqueness problems of some difference-differential polynomials of meromorphic functions sharing a small function. The results of the paper improve and generalize the results due to S.S. Bhoosnurmath and S.R. Kabbur[1].

Keywords: entire function, uniqueness, small function, difference-differential polynomials, Sharing value.

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1 Introduction

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna value distribution theory (see [2], [13]). Let \( f(z) \) and \( g(z) \) be two non constant meromorphic functions in the complex plane. By \( S(r,f) \), we mean any quantity satisfying \( S(r,f) = o\{T(r,f)\} \) as \( r \to \infty \), possibly outside a set of finite logarithmic measure.

A meromorphic function \( a(z) \) is called a small function with respect to \( f \), provided that \( T(r,a) = s(r,f) \). The order and hyper order of meromorphic
function $f$ are defined respectively by
\[
\sigma(f) = \lim_{r \to \infty} \sup \frac{\log T(r, f)}{\log r}
\]
and
\[
\sigma_2(f) = \lim_{r \to \infty} \sup \frac{\log \log T(r, f)}{\log r}
\]
Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Let and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity, we say that $f$ and $g$ share the value $a$ CM(counting multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide only in their locations, then we say that $f$ and $g$ share the value $a$ IM(ignoring multiplicities). For a positive integer $p$ we denote by $N_p(r, a; f)$ the counting function of $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq p$ and $p$ times if $m > p$.

We denote by $E_m(a, f)$ the set of all $a$-points of $f$ with multiplicities not exceeding $m$, where an $a$-point is counted according to its multiplicity. Also we denote by $\bar{E}_m(a, f)$ the set of distinct $a$-points of $f$ with multiplicities not greater than $m$. We denote by $N_k(r, \frac{1}{f-a})$ the counting function for zeros of $f - a$ with multiplicity $\leq k$, and by $\bar{N}_k(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $\bar{N}_k(r, \frac{1}{f-a})$ be the counting function for zeros of $f - a$ with multiplicity at least $k$ and $\bar{N}_k(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Set
\[
N_k(r, \frac{1}{f-a}) = \bar{N}(r, \frac{1}{f-a}) + \bar{N}_2(r, \frac{1}{f-a}) + \ldots + \bar{N}_k(r, \frac{1}{f-a}).
\]

The following two theorems proved by S.S. Bhoosnurmath and S.R. Kabbur [1].

**Theorem A.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $c$ is a nonzero complex constant and $n, m$ are positive integers such that $n \geq m + 6$. If $f^n(z)(f^m(z) - 1)f(z + c)$ and $g^n(z)(g^m(z) - 1)g(z + c)$ share $\alpha(z)$ CM, then $f(z) = tg(z)$ where $t^m = 1$.

**Theorem B.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $c$ is a nonzero complex constant and $n, m$ are positive integers satisfying $n \geq 4m + 12$. If $f^n(z)(f^m(z) - 1)f(z + c)$ and $g^n(z)(g^m(z) - 1)g(z + c)$ share $\alpha(z)$ IM, then $f(z) = tg(z)$ where $t^m = 1$. 

I. Lahiri introduced weighted sharing instead of sharing IM or CM, which measure how close a shared value is to being shared CM or to be shared IM. The definition are as follows.

**Definition 1 [7]**. Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with weight $k$.

The definition implies that if $f, g$ share a value $a$ with weight $k$, then $z_0$ is an $a$-point of $f$ with multiplicity $m(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $m(> k)$, where $m$ is not necessarily equal to $n$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$ then $f, g$ share $(a, p)$ for any integer $p, 0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

If $\alpha$ is a small function of $f$ and $g$, share $\alpha$ with weight $k$ means that $f - \alpha, g - \alpha$ share the value 0 with weight $k$.

## 2 Preliminary Lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by $H$ the following function:

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right)$$

**Lemma 1 (see[13])**. Let $f$ be a meromorphic function of finite order and $c$ is a non-zero complex constant. Then

$$n \left( r, \frac{f(z + c)}{f(z)} \right) = m \left( r, \frac{f(z)}{f(z + c)} \right) = S(r, f).$$

Arguing in a similar manner as in [3], we obtain the following lemma.

**Lemma 2**. Let $f$ be an meromorphic function of finite order. Then $T(r, f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + c_j)) = (n + m + \sigma)T(r, f) + S(r, f)$.

**Lemma 3**. Let $f$ be an meromorphic function of finite order. Then $T(r, f^n(z)P(f) \prod_{j=1}^d f(z + c_j)) = (n + m + \sigma)T(r, f) + S(r, f)$.

**Proof.** Since $f$ is meromorphic function of finite order, we deduce from Lemma
1 and the standard Valiron-Mohen’ko theorem

\[(n + m + \sigma)T(r, f) = T(r, f^{n+m} f^\sigma) + S(r, f)
= m(r, f^{n+m} f^\sigma) + N(r, f^{n+m} f^\sigma) + S(r, f)
= m \left( r, f^{n+m} \prod_{j=1}^d f(z + c_j)^{v_j} \right) \frac{f^\sigma}{\prod_{j=1}^d f(z + c_j)^{c_j}}
+ N \left( r, f^{n+m} \prod_{j=1}^d f(z + c_j)^{c_j} \right) \frac{f^\sigma}{\prod_{j=1}^d f(z + c_j)^{c_j}} + S(r, f)
= m \left( r, f^{n+m} \prod_{j=1}^d f(z + c_j)^{v_j} \right) + m \left( r, \prod_{j=1}^d f(z + c_j)^{v_j} \right)
+ N \left( r, f^{n+m} \prod_{j=1}^d f(z + c_j)^{v_j} \right) + N \left( r, \prod_{j=1}^d f(z + c_j)^{v_j} \right) + S(r, f)
= T(r, f^{n+m} \prod_{j=1}^d f(z + c_j)^{v_j}) + S(r, f)
= T(r, F) + S(r, f).

Thus, we get the conclusion.

**Lemma 4 (see[13])**. Let \( f \) be a non-constant meromorphic function and \( p, k \) be two positive integers. Then

\[N_p(r, f) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f).
\]

**Lemma 5 (see[7])**. If \( F \) and \( G \) are two non-constant meromorphic functions and \( E_{3j}(1, F) = E_{3j}(1, G) \), then one of the following cases holds.

1) \( T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + 2N_2(r, f) + 2N_2(e, \frac{1}{G}) + 2N_2(r, G) + S(r, F) + S(r, G), \)
2) \( F \equiv G, \quad (3) \; FG \equiv 1. \)

**Lemma 6**. Let \( h \) be a transcendental meromorphic function of finite order. Then we have

\[T \left( r, h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j} \right) \geq (n + m - \sigma)T(r, h) + S(r, f),\]
Proof. From Lemma 1, we have

\[ (n + m + \sigma)T(r, h) = T(r, h^{n+m}(z)h^\sigma) + S(r, h) \]
\[ = m(r, h^{n+m}(z)h^\sigma) + N(r, h^{n+m}(z)h^\sigma) + S(r, h) \]
\[ = m \left( r, h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j} \frac{h^\sigma}{\prod_{j=1}^{d} h(z + c_j)^{v_j}} \right) \]
\[ + N \left( r, h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j} \frac{h^\sigma}{\prod_{j=1}^{d} h(z + c_j)^{v_j}} \right) + S(r, h) \]
\[ \leq T \left( r, h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j} \right) + 2\sigma T(r, h) + S(r, h). \]

Thus, we get the conclusion.

3 Main results

Regarding Theorem A-B, a natural question to ask is what can be said if we study the uniqueness of difference polynomials of the form \([f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f(z + c_j)^{v_j}]^{(k)}\) and \([g^n(z)(g^m(z) - 1)\prod_{j=1}^{d} g(z + c_j)^{v_j}]^{(k)}\) where \(c_j = 1, 2, ..., d\) are complex constants, \(v_j = 1, 2, ..., d\) are non-negative integers and \(\sigma = v_1 + v_2 + ... + v_d\) without the notion of weighted sharing? In this section, our main concern is to find the possible answer of the above question. We prove the following result.

Theorem 1. Let \(f\) and \(g\) be two transcendental meromorphic functions of finite order, and \(\alpha(z)(\neq 0)\) be a small function with respect to both \(f\) and \(g\). Suppose that \(c_j = 1, 2, ..., d\) are non-zero complex constants, \(v_j = 1, 2, ..., d\) are non-negative integers, \(n, m \geq 1\) and \(k(\geq 0)\) are integers satisfying \(n \geq 4k + m + \sigma + 5\). If \(E_{3j}(\alpha(z), [f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f(z + c_j)^{v_j}]^{(k)}) = E_{3j}(\alpha(z), [g^n(z)(g^m(z) - 1)\prod_{j=1}^{d} g(z + c_j)^{v_j}]^{(k)})\), then \(f = hg\), where \(h\) is a constant and \(h^m = 1\).

Proof.
Let \(F_1 = f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f(z + c_j)^{v_j}, \quad G_1 = g^n(z)(g^m(z) - 1)\prod_{j=1}^{d} g(z + c_j)^{v_j}\),
\[ F = \frac{F_1^{(k)}}{\alpha(z)} G = \frac{G_1^{(k)}}{\alpha(z)}. \]
Then \(F\) and \(G\) are transcendental meromorphic functions and \(E_{3j}(1, F) = E_{3j}(1, G)\) except the zeros and poles of \(\alpha(z)\). By Lemma 2 and Lemma 4 we have
\[ N_2(r, \frac{1}{F}) \leq N_2(r, \frac{1}{F^{(k)}}) + S(r, f) \]
\[ \leq T(r, F^{(k)}) - T(r, F_1) + N_{2+k}(r, \frac{1}{F_1}) + S(r, f) \]  \hspace{1cm} (1)
\[ \leq T(r, F) - (n + m + \sigma)T(r, f) + N_{2+k}(r, \frac{1}{F_1}) + S(r, f). \]

So we get
\[ (n + m + \sigma)T(r, f) \leq T(r, F) + N_{2+k}(r, \frac{1}{F_1}) - N_2(r, \frac{1}{F}) + S(r, f). \]  \hspace{1cm} (2)

According to Lemma 4, we can deduce
\[ N_2(r, \frac{1}{F}) \leq N_2(r, \frac{1}{F^{(k)}}) + S(r, f) \]
\[ \leq N_{2+k}(r, \frac{1}{F_1}) + kN(r, f) + S(r, f). \]  \hspace{1cm} (3)

Similarly we have
\[ (n + m + \sigma)T(r, g) \leq T(r, G) + N_{2+k}(r, \frac{1}{G_1}) - N_2(r, \frac{1}{G}) + S(r, g). \]  \hspace{1cm} (4)

And
\[ N_2(r, \frac{1}{G}) \leq N_{2+k}(r, \frac{1}{G_1}) + kN(r, g) + S(r, g). \]  \hspace{1cm} (5)

Suppose, if possible the (1) of Lemma 5 holds, that is
\[ T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + 2N_2(r, F) + 2N_2(r, \frac{1}{G}) \]
\[ + 2N_2(r, G) + S(r, f) + S(r, g) \]  \hspace{1cm} (6)

By (2),(3),(4),(5) and (6), we have
\[ (n + m + \sigma)(T(r, f) + T(r, g)) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_{2+k}(r, \frac{1}{F_1}) \]
\[ + N_{2+k}(r, \frac{1}{G_1}) + S(r, f) + S(r, g) \]
\[ \leq 2N_{2+k}(r, \frac{1}{F_1}) + 2N_{2+k}(r, \frac{1}{G_1}) + 2k(N(r, f) + N(r, g)) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq (4k + 4 + 2m + 2\sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]  \hspace{1cm} (7)
\[ (n - 4k - m - \sigma - 4)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]  
which contradicts the fact that \( n \geq 4k + m + \sigma + 5 \). Therefore, by Lemma 5 we have either \( FG = 1 \) or \( F = G \).

If \( FG = 1 \), that is

\[
[f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z + c_j)]^{(k)}, [g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g(z + c_j)]^{(k)} = \alpha^2, \tag{9}
\]

We can deduce from above that

\[
N(r, \frac{1}{f^n}) = N(r, \frac{1}{f^{n-1}}) = S(r, f), \tag{10}
\]

which is impossible. So we have \( F = G \), that is

\[
[f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z + c_j)]^{(k)} = [g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g(z + c_j)]^{(k)}. \tag{11}
\]

Integrating above, we deduce

\[
[f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z + c_j)]^{(k-1)} = [g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g(z + c_j)]^{(k-1)} + c, \tag{12}
\]

where \( c \) is a constant. If \( c \neq 0 \), by the second fundamental theorem of Nevanlinna, we have

\[
T(r, F_1^{(k-1)}) \leq \overline{N}(r, \frac{1}{F_1^{(k-1)}}) + S(r, F) \\
\leq \overline{N}(r, \frac{1}{F_1^{(k-1)}}) + \overline{N}(r, \frac{1}{G_1^{(k-1)}}) + S(r, F). \tag{13}
\]

By Lemma 4, we obtain

\[
(n + m + \sigma)T(r, f) \leq T(r, F_1^{(k-1)}) - \overline{N}(r, \frac{1}{F_1^{(k-1)}}) + N_k(r, \frac{1}{F_1}) + S(r, f) \\
\leq \overline{N}(r, \frac{1}{G_1^{(k-1)}}) = N_k(r, \frac{1}{F_1}) + S(r, f) \\
\leq N_k(r, \frac{1}{F_1}) + N_k(r, \frac{1}{G_1}) + S(r, f) + S(r, g) \\
\leq (k + m + \sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \tag{14}
\]
Similarly
\[(n + m + \sigma)T(r, g) \leq (k + m + \sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (15)\]

Combining (14) and (15), we obtain
\[(n - 2k - m - \sigma)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \quad (16)\]

which contradicts with \(n \geq 2k + m + \sigma + 5\). Hence \(c = 0\). Integrating the (12) \(k - 1\) times, we can deduce
\[f^n(z)(f^m(z) - 1) \prod_{j=1}^d f(z + c_j)^{v_j} = g^n(z)(g^m(z) - 1) \prod_{j=1}^d g(z + c_j)^{v_j}. \quad (17)\]

Set \(h = f/g\). If \(h\) is not a constant, from (17) we have
\[g^m(z) = \frac{h^{(n)} \prod_{j=1}^d h(z + c_j)^{v_j} - 1}{h^{(n+m)} \prod_{j=1}^d h(z + c_j)^{v_j} - 1}. \quad (18)\]

If 1 is a picard value of \(h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j}\), applying the Nevanlinna second fundamental theorem, we get
\[T(r, h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j}) \leq N_r(h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j} + N \left(r, \frac{1}{h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j}}\right) + S(r, h)\]
\[\leq (2d + 2)T(r, h) + S(r, h). \quad (19)\]

On the other hand, combining the standard Valiron-Mohon’ko theorem, we get
\[(n + m + \sigma)T(r, h) = T(r, h^{n+m} h^\sigma) + S(r, h)\]
\[\leq T(r, h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j}) + T(r, \prod_{j=1}^d h(z + c_j)^{v_j})\]
\[\leq (2d + 3)T(r, h) + S(r, h)\]

Therefore, 1 is not a picard exceptional value of \(h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j}\). Thus \(\exists z_0\) such that \(h^{n+m}(z_0) \prod_{j=1}^d h(z_0 + c_j)^{v_j} = 1\) by (18), we have \(h^{n+m}(z_0) \prod_{j=1}^d h(z + c_j)^{v_j} = 1\). Hence \(h_0^m = 1\), and
\[\frac{1}{N_r(h^{n+m} \prod_{j=1}^d h(z + c_j)^{v_j} - 1)} \leq \frac{1}{N_r(h^m - 1)}\]
\[\leq mT(r, h) + S(r, h). \quad (20)\]
From the above inequality and by the second fundamental theorem by Nevanlinna, we have

\[
T(r, h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j}) \leq N(r, h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j})
+ \overline{N}(r, \frac{1}{h^{n+m}(z) \prod_{j=1}^{d} h(z + c_j)^{v_j} - 1}) + S(r, h)
\leq (m + 2d + 2)T(r, h) + S(r, h)
\]

(21)

which is a contradiction with \( n \geq 2k + m + \sigma + 5 \). Therefore \( h \) is a constant. Substituting \( f = gh \) into (17), we can get

\[
\prod_{j=1}^{d} g(z + c_j)^{v_j} (g^{n+m}(z)(h^{n+m+\sigma} - 1) + g^{n}(z)(h^{n+\sigma} - 1)) = 0. \quad (22)
\]

Since \( g \) is an entire function, we have \( \prod_{j=1}^{d} g(z + c_j)^{v_j} \neq 0 \). Thus

\[
g^{n+m}(z)(h^{n+m+\sigma} - 1) + g^{n}(z)(h^{n+\sigma} - 1) = 0. \quad (23)
\]

If \( h^{n+\sigma} \neq 1 \), by (23) we can deduce \( T(r, g) = S(r, g) \), which contradicts with a transcendental function \( g \). So \( h^{n+\sigma} = 1 \). We can also deduce that \( h^{n+m+\sigma} = 1 \). Then \( h^m = 1 \). This completes the proof of Theorem 1.

4 Open Problem

1. What can be said if we consider the difference-differential polynomials of the form \( f^{(k)} P(f) \prod_{j=1}^{d} f(z + c_j)^{v_j} \), where \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0, a_0(\neq 0), a_1, \ldots a_{m-1}, a_m(\neq 0) \) and \( c_j(j = 1, 2, \ldots, d) \)
2. Whether it is possible to replace the sharing value small function by polynomial.
3. Is it possible to reduce the condition of the theorem.

References


