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Coefficient Estimates and Fekete-Szegö Inequality for Family of Bi-Univalent Functions Defined by the Second Kind Chebyshev Polynomial

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Abstract

The main object of the present paper is to use the second kind Chebyshev polynomial expansions to derive estimates on the initial coefficients for a certain family of analytic and bi-univalent functions with respect to symmetric conjugate points defined in the open unit disk. Also, we solve Fekete-Szegö problem for functions in this family. Furthermore, we give connections to some of the earlier known results.

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1 Introduction

The importance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Several researchers dealing with orthogonal polynomials of

Chebyshev family, contain mainly results of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications one can refer [8,10,15]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \qquad and \qquad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1),$$

where n indicates the polynomial degree and $t = \cos n\theta$.

The Fekete-Szeg$\ddot{o}$ functional $|a_3 - \mu a_2^2|$ for analytic univalent functions is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szeg$\ddot{o}$ [12] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity.

Let $\mathcal{A}$ stand for the family of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1}$$

Also, let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are univalent in U. According to the Koebe One-Quarter Theorem [9] "every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$. For the inverse function f^{-1}, we have

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ indicate the class of bi-univalent functions in U given by (1). In fact, Srivastava et al. [19] have apparently resuscitated the study of holomorphic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [13], Ali et al. [2], Deniz [7] and others (see, for example [1,3,4,5,6,14,17,18,20]). We notice that the family Σ is not empty. Some examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad -\log(1-z)$$

with the corresponding inverse functions

$$\frac{w}{1+w}, \quad \frac{e^{2w}-1}{e^{2w}+1} \quad \text{and} \quad \frac{e^w - 1}{e^w},$$

respectively. Other common examples of functions is not a member of Σ are

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}.$$

Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|, (n = 3,4,\ldots)$ for functions $f \in \Sigma$ is still an open problem.

El-Ashwah and Thomas [11] introduced the class S_{sc}^* of functions called starlike with respect to symmetric conjugate points, they are the functions $f \in S$ satisfy the condition

$$Re\left\{\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right\}>0\ ,\ \ z\in U.$$

A function $f \in S$ is called convex with respect to symmetric conjugate points, if

$$Re\left\{\frac{\left(zf'(z)\right)'}{\left(f(z)-\overline{f(-\bar{z})}\right)'}\right\}>0\ ,\ \ z\in U.$$

The class of all convex functions with respect to symmetric conjugate points is denote by C_{sc}.

"With a view to recalling the principal of subordination between analytic functions, let the functions f and g be analytic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g\big(w(z)\big)$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)(z \in U)$. It is well known that (see [16]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$."

We consider the function

$$H(z,t)=\frac{1}{1-2tz+z^2}\ ,\ \ t\in\left(\frac{1}{2},1\right],z\in U.$$

We note that if $t = \cos\beta$, where $\beta \in (-\frac{\pi}{3},\frac{\pi}{3})$, then

$$H(z,t)=\frac{1}{1-2\cos\beta\, z+z^2}=1+\sum_{n=1}^{\infty}\frac{\sin(n+1)\beta}{\sin\beta}z^n\,,z\in U.$$

Therefore

$$H(z,t)=1+2\cos\beta\, z+(3\cos^2\beta-\sin^2\beta)z^2+\cdots,z\in U.$$

In view of [22], we can write

$$H(z,t)=1+U_1(t)z+U_2(t)z^2+\cdots\ \ \big(z\in U,t\in(-1,1)\big),$$

where

$$U_{n-1}=\frac{\sin(n\ \arccos t)}{\sqrt{1-t^2}}\quad (n\in\mathbb{N}=\{1,2,\dots\})$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t)=2tU_{n-1}(t)-U_{n-2}(t)$$

and

$$U_1(t)=2t\ ,\ \ U_2(t)=4t^2-1\ ,\ \ U_3(t)=8t^3-4t\ ,\dots\ . \qquad (3)$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$ is given by

$$\sum_{n=0}^{\infty}T_n(t)z^n=\frac{1-tz}{1-2tz+z^2}\ \ ,z\in U.$$

The Chebyshev polynomials of first kind $T_n(t)$ and of the second kind $U_n(t)$ are connected by

$$\frac{d\,T_n(t)}{dt}=nU_{n-1}(t),\ \ T_n(t)=U_n(t)-tU_{n-1}(t),\ \ 2T_n(t)=U_n(t)-U_{n-2}(t).$$

2 Main Results

We begin this section by defining the family $S_\Sigma^c(\alpha,t)$ as follows:

Definition 2.1. *For $0 \le \alpha \le 1$ and $t \in \left(\frac{1}{2},1\right]$, a function $f \in \Sigma$ is said to be in the family $S_\Sigma^c(\alpha,t)$ if it fulfills the subordinations:*

$$\frac{2[\alpha z^3 f'''(z) + (\alpha+1)z^2 f''(z) + zf'(z)]}{\alpha\left(z^2\big(f(z)-\overline{f(-\bar{z})}\big)'' + \big(f(z)-\overline{f(-\bar{z})}\big)\right) + (1-\alpha)z\big(f(z)-\overline{f(-\bar{z})}\big)'} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$\frac{2[\alpha w^3 g'''(w) + (\alpha+1)w^2 g''(w) + wg'(w)]}{\alpha\left(w^2\big(g(w)-\overline{g(-\bar{w})}\big)'' + \big(g(w)-\overline{g(-\bar{w})}\big)\right) + (1-\alpha)w\big(g(w)-\overline{g(-\bar{w})}\big)'} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (2).

In particular, if we choose $\alpha = 0$ in Definition 2.1, the family $S_\Sigma^c(\alpha,t)$ reduce to the family $\mathcal{F}_\Sigma^{sc}(t)$ which was given by Wanas and Majeed (see [21]) and defined as follows:

Definition 2.2 [21]. *For $t \in \left(\frac{1}{2},1\right]$, a function $f \in \Sigma$ is said to be in the family $\mathcal{F}_\Sigma^{sc}(t)$ if it fulfills the subordinations:*

$$\frac{2\big(zf'(z)\big)'}{\big(f(z)-\overline{f(-\bar{z})}\big)'} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$\frac{2\big(wg'(w)\big)'}{\big(g(w)-\overline{g(-\bar{w})}\big)'} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where $g = f^{-1}$ is given by (2).

Theorem 2.1. *For $0 \le \alpha \le 1$ and $t \in \left(\frac{1}{2},1\right]$, let $f \in \mathcal{A}$ be in the family $S_\Sigma^c(\alpha,t)$. Then*

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{|(\alpha+2)^2 - 2(2\alpha^2+4\alpha+5)t^2|}}$$

and

$$|a_3| \le \frac{t^2}{(\alpha+2)^2} + \frac{t}{4\alpha+3}.$$

Proof. Let $f \in S_\Sigma^c(\alpha,t)$. Then there exists two analytic functions $u,v\colon U \longrightarrow U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in U) \tag{4}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in U), \tag{5}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

$$\frac{2[\alpha z^3 f'''(z) + (\alpha+1) z^2 f''(z) + z f'(z)]}{\alpha\left(z^2\big(f(z) - \overline{f(-\bar{z})}\big)'' + \big(f(z) - \overline{f(-\bar{z})}\big)\right) + (1-\alpha) z\big(f(z) - \overline{f(-\bar{z})}\big)'}$$

$$= 1 + U_1(t) u(z) + U_2(t) u^2(z) + \cdots \tag{6}$$

and

$$\frac{2[\alpha w^3 g'''(w) + (\alpha+1) w^2 g''(w) + w g'(w)]}{\alpha\left(w^2\big(g(w) - \overline{g(-\bar{w})}\big)'' + \big(g(w) - \overline{g(-\bar{w})}\big)\right) + (1-\alpha) w\big(g(w) - \overline{g(-\bar{w})}\big)'}$$

$$= 1 + U_1(t) v(w) + U_2(t) v^2(w) + \cdots \ . \tag{7}$$

Combining (4), (5), (6) and (7), we obtain

$$\frac{2[\alpha z^3 f'''(z) + (\alpha+1) z^2 f''(z) + z f'(z)]}{\alpha\left(z^2\big(f(z) - \overline{f(-\bar{z})}\big)'' + \big(f(z) - \overline{f(-\bar{z})}\big)\right) + (1-\alpha) z\big(f(z) - \overline{f(-\bar{z})}\big)'}$$

$$= 1 + U_1(t) u_1 z + [U_1(t) u_2 + U_2(t) u_1^2] z^2 + \cdots \tag{8}$$

and

$$\frac{2[\alpha w^3 g'''(w) + (\alpha+1) w^2 g''(w) + w g'(w)]}{\alpha\left(w^2\big(g(w) - \overline{g(-\bar{w})}\big)'' + \big(g(w) - \overline{g(-\bar{w})}\big)\right) + (1-\alpha) w\big(g(w) - \overline{g(-\bar{w})}\big)'}$$

$$= 1 + U_1(t) v_1 w + [U_1(t) v_2 + U_2(t) v_1^2] w^2 + \cdots \ . \tag{9}$$

It is quite well-known that if $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in U$, then

$$|u_i| \le 1 \quad \text{and} \quad |v_i| \le 1 \quad \text{for all } i \in \mathbb{N}. \tag{10}$$

Comparing the corresponding coefficients in (8) and (9), after simplifying, we have

$$2(\alpha+2) a_2 = U_1(t) u_1, \tag{11}$$

$$2(4\alpha+3) a_3 = U_1(t) u_2 + U_2(t) u_1^2, \tag{12}$$

$$-2(\alpha+2) a_2 = U_1(t) v_1 \tag{13}$$

and

$$2(4\alpha+3)(2a_2^2 - a_3) = U_1(t) v_2 + U_2(t) v_1^2. \tag{14}$$

It follows from (11) and (13) that

$$u_1 = -v_1 \tag{15}$$

and

$$8(\alpha+2)^2 a_2^2 = U_1^2(t)(u_1^2 + v_1^2). \tag{16}$$

If we add (12) to (14), we find that

$$4(4\alpha+3) a_2^2 = U_1(t)(u_2 + v_2) + U_2(t)(u_1^2 + v_1^2). \tag{17}$$

Substituting the value of $u_1^2 + v_1^2$ from (16) in the right hand side of (17), we get

$$\left[4(4\alpha+3) - \frac{8U_2(t)}{U_1^2(t)}(\alpha+2)^2\right] a_2^2 = U_1(t)(u_2 + v_2),$$

or equivalently

$$a_2^2 = \frac{U_1^3(t)(u_2+v_2)}{4(4\alpha+3)U_1^2(t)-8(\alpha+2)^2U_2(t)}, \tag{18}$$

Further computations using (3), (10) and (18), we obtain

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{|(\alpha+2)^2-2(2\alpha^2+4\alpha+5)t^2|}}.$$

Next, if we subtract (14) from (12), we deduce that

$$4(4\alpha+3)(a_3-a_2^2) = U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{19}$$

In view of (15) and (16), we get from (19)

$$a_3 = \frac{U_1^2(t)}{8(\alpha+2)^2}(u_1^2+v_1^2)+\frac{U_1(t)}{4(4\alpha+3)}\ (u_2-v_2).$$

Thus applying (3), we obtain

$$|a_3| \le \frac{t^2}{(\alpha+2)^2}+\frac{t}{4\alpha+3}.$$

If we choose $\alpha=0$ in Theorem 2.1, we conclude the result for well-known family $\mathcal{F}_\Sigma^{sc}(t)$ which was considered recently by Wanas and Majeed [21].

Corollary 2.1 [21]. *For* $t \in \left(\frac{1}{2},1\right]$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{F}_\Sigma^{sc}(t)$. *Then*

$$|a_2| \le \frac{t\sqrt{t}}{\sqrt{|2-5t^2|}}$$

and

$$|a_3| \le \frac{t(3t+4)}{12}.$$

In the next theorem, we discuss the "Fekete-Szegö problem" for the family $S_\Sigma^c(\alpha,t)$.

Theorem 2.2. *For* $0 \le \alpha \le 1$, $t \in \left(\frac{1}{2},1\right]$ *and* $\mu \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $S_\Sigma^c(\alpha,t)$. *Then*

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{t}{4\alpha+3}; \\ \qquad for\ \ |\mu-1| \le \dfrac{1}{2(4\alpha+3)}\left|\dfrac{(\alpha+2)^2}{t^2}-2(2\alpha^2+4\alpha+5)\right| \\ \dfrac{2t^3|\mu-1|}{|2(4\alpha+3)t^2-(\alpha+2)^2(4t^2-1)|}; \\ \qquad for\ \ |\mu-1| \ge \dfrac{1}{2(4\alpha+3)}\left|\dfrac{(\alpha+2)^2}{t^2}-2(2\alpha^2+4\alpha+5)\right| \end{cases}.$$

Proof. In the light of (18) and (19), we find that

$$a_3-\mu a_2^2 = (1-\mu)\frac{U_1^3(t)(u_2+v_2)}{4(4\alpha+3)U_1^2(t)-8(\alpha+2)^2U_2(t)}+\frac{U_1(t)(u_2-v_2)}{4(4\alpha+3)}$$
$$= U_1(t)\left[\left(\psi(\mu)+\frac{1}{4(4\alpha+3)}\right)u_2+\left(\psi(\mu)-\frac{1}{4(4\alpha+3)}\right)v_2\right],$$

where

$$\psi(\mu)=\frac{U_1^2(t)(1-\mu)}{4[(4\alpha+3)U_1^2(t)-2(\alpha+2)^2U_2(t)]}.$$

According to (3), we deduce that

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{t}{4\alpha+3}, & 0\le|\psi(\mu)|\le\dfrac{1}{4(4\alpha+3)}\\ \\ 4t|\psi(\mu)|, & |\psi(\mu)|\ge\dfrac{1}{4(4\alpha+3)}\end{cases}.$$

After some computations, we obtain the desired result.

Putting $\mu=1$ in Theorem 2.2, we conclude the following result:

Corollary 2.2. *For* $0\le\alpha\le1$ *and* $t\in\left(\frac{1}{2},1\right]$, *let* $f\in\mathcal{A}$ *be in the family* $S_\Sigma^c(\alpha,t)$. *Then*

$$|a_3-a_2^2|\le\frac{t}{4\alpha+3}.$$

3 Open Problem

The open problem is to find an upper bound for the second and Third Hankel determinants for functions belongs to the family $S_\Sigma^c(\alpha,t)$ and hence new results can be obtained.

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