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The q-Dunkl wavelet theory and localization operators

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Abstract

In this paper, we study some problems of time-frequency analysis associated with the q-Dunkl wavelet transform. We introduce the notion of two q-Dunkl wavelet. The resolution of the identity formula and Calderòn's type reproducing formula are proved. Next, we define the localization operators associated with the q-Dunkl wavelet transform. We prove for these operators the boundedness and compactness on the Schatten classes. Finally, the traces and the trace class norm inequalities are shown.

Keywords. *q-Dunkl wavelet transform, localization operators, Calderón's reproducing formula*

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1 Introduction

The q-theory, called also in some literature quantum calculus, began to arise. Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to the large number of its application domains.

Very recently, many authors have been investigating the behavior of the q-theory to several problems already studied for the Fourier Analysis; for instance, sampling theorem [1], Paley-Wiener theorems [2, 3], wavelet transform [14], uncertainty principles [15], wavelet packets [16], Ramanujan master theorem [17], Sobolev spaces [24], Gabor transform [23], wavelet multipliers [25], wave equation [27], Fock spaces [28] and so on.

One of the aims of the Fourier Analysis, is the study of the theory of localization operators. This theory has been initiated by Daubechies in [10, 11], developed in series of papers by Wong [8, 30, 31], and detailed in the book [32] also by Wong.

Nowadays, the localization operators have found many applications to time-frequency analysis, the theory of differential equations, quantum mechanics. Arguing from these point of view, many works were done on them, we refer in particular to the papers of Balazs et al. [4, 5], (see also [9, 12, 13, 19, 32]).

As the q-harmonic analysis has known remarkable development, the natural question to ask whether there exists the equivalent of the theory of localization operators in the framework of the q-theory.

Motivated by the recent works of Bettaibi and all [6, 7], where the harmonic analysis associated to the q-Dunkl theory, has known remarkable development, it is natural to solve some questions for the time-frequency analysis associated with the q-Dunkl wavelet transform.

The purpose of the present paper is twofold. On one hand, we want to study some results for the q-Dunkl wavelet transform. On the other hand we want to study the boundedness and compactness of localization operators associated with q-Dunkl wavelet transform on the Schatten classes.

The remainder of this paper is arranged as follows. In §2 we recall the main results about the harmonic analysis associated with the q-Dunkl operator. In §3, we prove the resolution identity and a Calderón's reproducing formulas in the cadre of the q-Dunkl two-wavelet theory. §4, is devoted to the study of boundedness and compactness of the localization operators for the q-Dunkl continuous wavelet transform on the Schatten classes.

2 Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer to the general references [6, 18, 20, 21, 26, 27, 32]. Throughout this paper, we assume that $q \in (0, 1)$.

2.1 Basic symbols

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1;\ (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k),\ n = 1,\ 2, ...;\ (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k), \tag{2.1}$$

and

$$\mathbb{R}_q = \Big\{ \pm q^n\ :\ n \in \mathbb{Z}\Big\},\ \widetilde{\mathbb{R}}_q = \Big\{ \pm q^n\ :\ n \in \mathbb{Z}\Big\} \cup \Big\{0\Big\}.$$

We also denote

$$[x]_q = \frac{1 - q^x}{1 - q},\ x \in \mathbb{C} \tag{2.2}$$

and

$$[n]_q! = \frac{(q;q)_n}{(1 - q)^n},\ n \in \mathbb{N}.$$

2.2 Operators and elementary special functions

The q^2-analogue differential operator is given by (see [26, 27]),

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}, & \text{if } z \neq 0 \\ \lim\limits_{x\to 0} \partial_q(f)(x) \quad (\text{ in } \quad \mathbb{R}_q) & \text{if } z = 0. \end{cases} \tag{2.3}$$

Note that if f is differentiable at z, then $\lim\limits_{q\to 1} \partial_q(f)(z) = f'(z)$.

The q-Gamma function is given by (see [20])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots.$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x),\ \mathscr{R}(x) > 0.$$

The q-trigonometric functions q-cosine and q-sine are defined by (see [26, 27])

$$\cos(x;q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}$$

and

$$\sin(x;q^2) == \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.$$

The q-analogue exponential function is given by

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{2.4}$$

These three functions are absolutely convergent for all z in the plane and when q tends to 1 they tend to the corresponding classical ones pointwise and uniformly on compacts.

Note that we have for all $x \in \mathbb{R}_q$

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \qquad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

and

$$|e(-ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{2.5}$$

Here, for a function f defined on $\mathbb{R}_q$. The q-Jakson integrals are defined by (see [20, 21])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \qquad \int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty} q^n\Big(f(bq^n) - f(aq^n)\Big),$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} \{f(q^n)q^n + f(-q^n)q^n\},$$

provided the sums converge absolutely.

2.3 Sets and spaces

By the use of the q^2-analogue differential operator ∂_q, we note:
• $\mathscr{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n \in \mathbb{N},\ a \geq 0,\ P_{n,a}(f) = \sup\left\{|\partial_q^k f(x)|;\ 0 \leq k \leq n, \quad x \in [-a, a] \cap \mathbb{R}_q\right\} < \infty$$

and

$$\lim_{x\to 0}(\partial_q^n f)(x) \quad (\text{ in } \quad \mathbb{R}_q) \ \text{ exists.}$$

We provide it with the topology defined by the semi norms $P_{n,a}$.
• $\mathscr{S}_q(\mathbb{R}_q)$ the space of function f defined on $\mathbb{R}_q$ satisfying

$$\forall n, m \in \mathbb{N}, \quad P_{n,m,q} = \sup_{x\in\mathbb{R}_q} |x^m \partial_q^n f(x)| < \infty$$

and

$$\lim_{x\to 0}(\partial_q^n f)(x) \ \ (\text{in } \ \mathbb{R}_q) \ \text{ exists.}$$

• $D_q(\mathbb{R}_q)$ the subspace of $\mathscr{S}_q(\mathbb{R}_q)$ constituted of functions with compact supports.
• $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$, the space of functions f on $\mathbb{R}_q$, satisfying

$$\begin{aligned} \|f\|_{L^p_{\alpha,q}(\mathbb{R}_q)} &= \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x\right)^{1/p} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_\alpha(\mathbb{R}_q)} &= ess \sup_{x\in\mathbb{R}_q} |f(x)| < \infty, \quad p = \infty. \end{aligned}$$

In particular, $L^2_{\alpha,q}(\mathbb{R}_q)$ denotes the Hilbert space with the inner product

$$\langle f, g\rangle_{\alpha,q} = \int_{\mathbb{R}_q} f(x)\overline{g(x)}|x|^{2\alpha+1} d_q x.$$

2.4 Elements of q-Dunkl Harmonic Analysis

In this section, we collect some notations and results on q-Dunkl operator and q-Dunkl transform studied in [6].
For $\alpha \geq \frac{1}{2}$, the q-Dunkl transform is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by:

$$\mathscr{F}_D^{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{-\infty}^{\infty} f(x)\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1} d_q x, \quad \text{for all } \lambda \in \widetilde{\mathbb{R}_q}, \tag{2.6}$$

where $c_{\alpha,q} = \dfrac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}$ and $\psi^{\alpha,q}_\lambda$ is the q-Dunkl kernel defined by

$$\psi^{\alpha,q}_\lambda(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2), \tag{2.7}$$

with $j_\alpha(x; q^2)$ is the normalized third Jackson's q-Bessel function given by:

$$j_\alpha(x; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(n+1)\Gamma_{q^2}(\alpha+n+1)} \Big(\frac{x}{1+q}\Big)^{2n}.$$

It was proved in [6] that for all $\lambda \in \mathbb{C}$, the function $x \mapsto \psi_\lambda^{\alpha,q}(x)$ is the unique solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q}(f) = i\lambda f \\ f(0) = 1, \end{cases} \tag{2.8}$$

where $\Lambda_{\alpha,q}$ is the q-Dunkl operator defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q[f_e + q^{2\alpha+1} f_o](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x}, \tag{2.9}$$

with f_e and f_o are respectively the even and the odd parts of f.
We recall that the q-Dunkl operator $\Lambda_{\alpha,q}$ lives the spaces $D_q(\mathbb{R}_q)$ and $\mathscr{S}_q(\mathbb{R}_q)$ invariant.

Remark 2.1. *(i) It is easy to see that in the even case $\mathscr{F}_D^{\alpha,q}$ reduces to the q-Bessel transform and in the case $\alpha = \frac{1}{2}$, it reduces to the q^2-analogue Fourier transform.*
(ii) It is worthy to claim that letting $q \uparrow 1$ subject to the condition

$$\frac{\ln(1-q)}{\ln(q)} \in 2\mathbb{Z}, \tag{2.10}$$

$\mathscr{F}_D^{\alpha,q}$ tends, at least formally, the classical Dunkl transform. (See [6]).
In the remainder of this paper, we assume that the condition (2.10) holds.

Some other properties of the q-Dunkl kernel and the q-Dunkl transform are given in the following results (see [6]).

Proposition 2.1. *i) For all $\lambda, x \in \mathbb{R},\ a \in \mathbb{C}$, we have*

$$\psi_\lambda^{\alpha,q}(x) = \psi_x^{\alpha,q}(\lambda), \quad \psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax), \quad \overline{\psi_\lambda^{\alpha,q}(x)} = \psi_{-\lambda}^{\alpha,q}(x).$$

ii) If $\alpha = -\frac{1}{2}$, then $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$.
iii) For $\alpha > -\frac{1}{2}$, the q-Dunkl kernel $\psi_\lambda^{\alpha,q}$ has the following q-integral representation of Mehler type

$$\forall\, x \in \mathbb{R}_q, \quad \psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty}(1+t)e(i\lambda xt; q^2)d_qt. \tag{2.11}$$

iv) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q}$ is bounded on $\mathbb{R}_q$ and we have

$$\forall x \in \mathbb{R}_q, \quad |\psi_\lambda^{\alpha,q}(x)| \le \frac{4}{(q;q)_\infty}. \tag{2.12}$$

v) For all $\lambda \in \mathbb{R}_q$, the function $\psi_\lambda^{\alpha,q}$ belongs to $\mathscr{S}(\mathbb{R}_q)$.
vi) The function $\psi_\lambda^{\alpha,q}$ verifies the following orthogonality relation: For all $x,\ y \in \mathbb{R}_q$

$$\int_{-\infty}^{\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1} d_q\lambda = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)}{(1-q)|xy|^{\alpha+1}}\delta_{x,y}. \tag{2.13}$$

vii) If $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ then $\mathscr{F}_D^{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_q)$ and

$$\|\mathscr{F}_D^{\alpha,q}(f)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \le \frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q)}. \tag{2.14}$$

Moreover

$$\lim_{|\lambda|\to\infty}\mathscr{F}_D^{\alpha,q}(f)(\lambda)=0,\ \lambda\in\mathbb{R}_q;\quad \lim_{|\lambda|\to 0}\mathscr{F}_D^{\alpha,q}(f)(\lambda)=\mathscr{F}_D^{\alpha,q}(f)(0),\ \lambda\in\widetilde{\mathbb{R}_q}. \tag{2.15}$$

viii) For $f\in L^1_{\alpha,q}(\mathbb{R}_q)$, we have

$$\mathscr{F}_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda)=i\lambda\mathscr{F}_D^{\alpha,q}(f)(\lambda). \tag{2.16}$$

ix) For $f,\ g\in L^1_{\alpha,q}(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\lambda)g(\lambda)|\lambda|^{2\alpha+1}d_q\lambda=\int_{-\infty}^{\infty}f(x)\mathscr{F}_D^{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx. \tag{2.17}$$

Theorem 2.1. *For all $f\in L^1_{\alpha,q}(\mathbb{R}_q)$, we have*

$$\forall x\in\mathbb{R}_q,\ f(x)\ =\ c_{\alpha,q}\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)|\lambda|^{2\alpha+1}d_q\lambda=\overline{\mathscr{F}_D^{\alpha,q}(\overline{\mathscr{F}_D^{\alpha,q}(f)})}(x). \tag{2.18}$$

Theorem 2.2. *i) Plancherel's formula*
For $\alpha\geq-\frac{1}{2}$, the q-Dunkl transform $\mathscr{F}_D^{\alpha,q}$ is an isomorphism from $\mathscr{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f\in\mathscr{S}_q(\mathbb{R}_q)$, we have

$$\|\mathscr{F}_D^{\alpha,q}(f)\|_{L^2_{\alpha,q}(\mathbb{R}_q)}=\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{2.19}$$

ii) Plancherel's theorem
The q-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. Its inverse transform $(\mathscr{F}_D^{\alpha,q})^{-1}$ is given by :

$$(\mathscr{F}_D^{\alpha,q})^{-1}(f)(x)=c_{\alpha,q}\int_{-\infty}^{\infty}f(\lambda)\psi_\lambda^{\alpha,q}(x)|\lambda|^{2\alpha+1}d_q\lambda=\mathscr{F}_D^{\alpha,q}(f)(-x). \tag{2.20}$$

Proposition 2.2. *Parseval's formula for $\mathscr{F}_D^{\alpha,q}$.*
For all f,g in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty}f(\lambda)\overline{g(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda=\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(x)\overline{\mathscr{F}_D^{\alpha,q}(g)(x)}|x|^{2\alpha+1}d_qx. \tag{2.21}$$

Proposition 2.3. *For all f in $D(\mathbb{R}_q)$ (resp. $S(\mathbb{R}_q)$), we have the following relations*

$$\forall\ x\in\mathbb{R}_q,\quad \mathscr{F}_D^{\alpha,q}(\overline{f})(x)=\overline{\mathscr{F}_D^{\alpha,q}(\breve{f})(x)}, \tag{2.22}$$

$$\forall\ x\in\mathbb{R}_q,\quad \mathscr{F}_D^{\alpha,q}(f)(x)=\mathscr{F}_D^{\alpha,q}(\breve{f})(-x), \tag{2.23}$$

where $\breve{g}(x):=g(-x)$.

Definition 2.1. *The q-Dunkl translation operator is defined for $f\in L^2_{\alpha,q}(\mathbb{R}_q)$ and $x,\ y\in\mathbb{R}_q$ by*

$$\tau_y^{\alpha,q}(f)(x)=c_{\alpha,q}\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)|\lambda|^{2\alpha+1}d_q\lambda. \tag{2.24}$$

It verifies the following properties.

Proposition 2.4. *(i) For all* $x,\ y \in \mathbb{R}_q$, $\tau_y^{\alpha,q}(f)(x) = \tau_x^{\alpha,q}(f)(y)$.

(ii) If $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ *(resp.* $\mathscr{S}_q(\mathbb{R}_q)$*) then* $\tau_y^{\alpha,q}(f) \in L^2_{\alpha,q}(\mathbb{R}_q)$ *(resp.* $\mathscr{S}_q(\mathbb{R}_q)$*) and we have*

$$\|\tau_y^{\alpha,q}(f)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4}{(q;q)_\infty}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{2.25}$$

(iii) For all $x,\ y,\ \lambda \in \mathbb{R}_q$, $\tau_y^{\alpha,q}(\psi_\lambda^{\alpha,q})(x) = \psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)$.

(iv) For all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, $x,\ y \in \mathbb{R}_q$, *we have*

$$\mathscr{F}_D^{\alpha,q}(\tau_y^{\alpha,q}f)(\lambda) = \psi_\lambda^{\alpha,q}(y)\mathscr{F}_D^{\alpha,q}(f)(\lambda). \tag{2.26}$$

(v) For all $f \in \mathscr{S}_q(\mathbb{R}_q)$, *and for all* $y \in \mathbb{R}_q$, *we have*

$$\Lambda_{\alpha,q}\tau_y^{\alpha,q}f = \tau_y^{\alpha,q}\Lambda_{\alpha,q}f.$$

Definition 2.2. *The q-Dunkl convolution product is defined for* $f,\ g \in \mathscr{S}_q(\mathbb{R}_q)$ *by:*

$$f *_{\alpha,q} g(x) = c_{\alpha,q}\int_{-\infty}^{\infty}\tau_x^{\alpha,q}f(-y)g(y)|y|^{2\alpha+1}d_qy. \tag{2.27}$$

Proposition 2.5. *1) Let* $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ *and* $g \in L^2_{\alpha,q}(\mathbb{R}_q)$*, then* $f *_{\alpha,q} g \in L^2_{\alpha,q}(\mathbb{R}_q)$ *and we have*

$$\mathscr{F}_D^{\alpha,q}(f *_{\alpha,q} g) = \mathscr{F}_D^{\alpha,q}(f).\mathscr{F}_D^{\alpha,q}(g).$$

2) Let $f,\ g$ *be in* $L^2_{\alpha,q}(\mathbb{R}_q)$*, then* $f *_{\alpha,q} g \in L^2_{\alpha,q}(\mathbb{R}_q)$ *if and only if* $\mathscr{F}_D^{\alpha,q}(f)\mathscr{F}_D^{\alpha,q}(g)$ *is in* $L^2_{\alpha,q}(\mathbb{R}_q)$ *and we have* $\mathscr{F}_D^{\alpha,q}(f *_{\alpha,q} g) = \mathscr{F}_D^{\alpha,q}(f)\mathscr{F}_D^{\alpha,q}(g)$ *and*

$$\int_{-\infty}^{\infty}|f *_{\alpha,q} g(x)|^2|x|^{2\alpha+1}d_qx = \int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(f)|^2|\mathscr{F}_D^{\alpha,q}g(x)|^2|x|^{2\alpha+1}d_qx.$$

3 Two q-Dunkl wavelet theory

3.1 Resolution identity formula

Notation. We denote by

$L^p_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$ the space of all functions f defined on $\mathbb{R}_q \times \mathbb{R}_q$, $p \in [1,\infty]$, and satisfying

$$\|f\|_{L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)} := \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(a,x)|^p d\mu_{\alpha,q}(a,x)\right)^{\frac{1}{p}} < \infty,$$

$$\|f\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)} := ess \sup_{(a,x)\in\mathbb{R}_q\times\mathbb{R}_q}|f(a,x)| < \infty$$

where $d\mu_{\alpha,q}(a,x) = \dfrac{|x|^{2\alpha+1}d_qad_qx}{|a|^{2\alpha+3}}$.

For $p \in [1,\infty]$, p' denotes as in all that follows, the conjugate exponent of p.

Definition 3.1. *([7]) A q-Dunkl wavelet is a square q-integrable function h on* $\mathbb{R}_q$ *satisfying the following admissibility condition*

$$0 < C_h^{\alpha,q} := \int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(h)(a\xi)|^2\frac{d_qa}{|a|} < \infty. \tag{3.1}$$

We generalize the notion of the q-Dunkl wavelet as follows.

Definition 3.2. *Let u and v be in $L^2_{\alpha,q}(\mathbb{R}_q)$. We say that the pair (u,v) is a two q-Dunkl wavelet on $\mathbb{R}_q$ if the following integral, noted by $C^{\alpha,q}_{u,v}$,*

$$\int_{-\infty}^{\infty} \mathscr{F}^{\alpha,q}_D(v)(a\xi)\overline{\mathscr{F}^{\alpha,q}_D(u)(a\xi)}\frac{d_qa}{|a|} \tag{3.2}$$

is constant for all $\xi \in \mathbb{R}_q$. We call the number $C^{\alpha,q}_{u,v}$ the two q-Dunkl wavelet constant associated to the functions u and v.

Remark 3.1. *It is obvious that if u is a q-Dunkl wavelet then the pair (u,u) is a two q-Dunkl wavelet, and $C^{\alpha}_{u,u}$ coincides with $C^{\alpha,q}_{u}$.*

Proposition 3.1. *For $a \in \mathbb{R}_q$ and $h \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$), the function h_a defined by*

$$\forall\, x \in \mathbb{R}_q, \quad h_a(x) := \frac{1}{|a|^{2\alpha+2}}h(\frac{x}{a}) \tag{3.3}$$

belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$) and we have

$$\|h_a\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \frac{1}{|a|^{\alpha+1}}\|h\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{3.4}$$

(ii) Let $a \in \mathbb{R}_q$ and h be in $L^1_{\alpha,q}(\mathbb{R}_q) \bigcup L^2_{\alpha,q}(\mathbb{R}_q)$. We have

$$\mathscr{F}^{\alpha,q}_D(h_a)(\xi) = \mathscr{F}^{\alpha,q}_D(h)(a\xi), \quad \xi \in \widetilde{\mathbb{R}}_q. \tag{3.5}$$

Let h be a q-Dunkl wavelet in $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$). We consider for all $a \in \mathbb{R}_q$ and b be in $\widetilde{\mathbb{R}}_q$, the family of q-Dunkl wavelets $h^{\alpha}_{a,x}$ defined in $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathscr{S}_q(\mathbb{R}_q)$) for $x \in \mathbb{R}_q$ by

$$h_{a,x}(y) := |a|^{\alpha+1}\tau^{\alpha,q}_x(h_a)(y), \tag{3.6}$$

where $\tau^{\alpha,q}_x$, $x \in \mathbb{R}_q$, is the q-Dunkl translation operator given by (2.24).

Remark 3.2. *Let h be in $L^2_{\alpha,q}(\mathbb{R}_q)$. We have*

$$\forall\, (a,x) \in \mathbb{R}_q \times \mathbb{R}_q, \quad \|h_{a,x}\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \frac{4}{(q;q)_\infty}\|h\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{3.7}$$

Definition 3.3. *Let h be a q-Dunkl wavelet on $\mathbb{R}_q$ in $L^2_{\alpha,q}(\mathbb{R}_q)$. The continuous q-Dunkl wavelet transform $\Psi^{\alpha}_{q,h}$ on $\mathbb{R}_q$ is defined for regular functions f on $\mathbb{R}_q$ by*

$$\forall\, (a,x) \in \mathbb{R}_q \times \mathbb{R}_q, \quad \Psi^{\alpha}_{q,h}(f)(a,x) = c_{\alpha,q}\int_{-\infty}^{\infty} f(y)\overline{h_{a,x}(y)}|y|^{2\alpha+1}d_qy = c_{\alpha,q}\langle f, h_{a,x}\rangle_{\alpha,q}. \tag{3.8}$$

This transform can also be written in the form

$$\Psi^{\alpha}_{q,h}(f)(a,x) = |a|^{\alpha+1}\check{f} *_{\alpha,q} \overline{h_a}(x), \tag{3.9}$$

where $*_{\alpha,q}$ is the q-Dunkl convolution product given by (2.27).

Remark 3.3. *Let h be a q-Dunkl wavelet, in $L^2_{\alpha,q}(\mathbb{R}_q)$. Then from the relations (3.7) and (3.8), for all f in $L^2_{\alpha,q}(\mathbb{R}_q)$ we have*

$$||\Psi^{\alpha}_{q,h}(f)||_{L^{\infty}_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}||f||_{L^2_{\alpha,q}(\mathbb{R}_q)}||h||_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{3.10}$$

Theorem 3.1. *Let (u, v) be a two q-Dunkl wavelet. Then for all f and g in $L^2_{\alpha,q}(\mathbb{R}_q)$, there holds*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)\overline{\Psi^{\alpha}_{q,v}(g)(a,x)}d\mu_{\alpha,q}(a,x) = C^{\alpha,q}_{u,v}\int_{-\infty}^{\infty} f(x)\overline{g(x)}|x|^{2\alpha+1}d_qx, \tag{3.11}$$

where

$$C^{\alpha,q}_{u,v} := \int_{-\infty}^{\infty}\overline{\mathscr{F}^{\alpha,q}_D(u)(a\xi)}\mathscr{F}^{\alpha,q}_D(v)(a\xi)\frac{d_qa}{|a|}. \tag{3.12}$$

Proof. The use of Fubini's Theorem, relation (3.9), Parseval's formula (2.21) and Proposition 2.3 give

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)\overline{\Psi^{\alpha}_{q,v}(g)(a,x)}d\mu_{\alpha,q}(a,x) = \int_{-\infty}^{\infty}|a|^{2\alpha+2}\int_{-\infty}^{\infty}\breve{f}*_{\alpha,q}\overline{u}_a(x)\overline{\breve{g}*_{\alpha,q}\overline{v}_a(x)}d\mu_{\alpha,q}(a,x)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathscr{F}^{\alpha,q}_D(\breve{f})(\xi)\overline{\mathscr{F}^{\alpha,q}_D(\breve{g})(\xi)}\mathscr{F}^{\alpha,q}_D(\overline{u}_a)(\xi)\overline{\mathscr{F}^{\alpha,q}_D(\overline{v_a})(\xi)}|\xi|^{2\alpha+1}d_q\xi\frac{d_qa}{|a|}$$

$$= \int_{-\infty}^{\infty}\mathscr{F}^{\alpha,q}_D(\breve{f})(\xi)\overline{\mathscr{F}^{\alpha,q}_D(\breve{g})(\xi)}\Big(\int_{-\infty}^{\infty}\mathscr{F}^{\alpha,q}_D(\overline{u})(a\xi)\mathscr{F}^{\alpha,q}_D(v)(-a\xi)\frac{d_qa}{|a|}\Big)|\xi|^{2\alpha+1}d_q\xi.$$

$$= \int_{-\infty}^{\infty}\mathscr{F}^{\alpha,q}_D(f)(\xi)\overline{\mathscr{F}^{\alpha,q}_D(g)(\xi)}\Big(\int_{-\infty}^{\infty}\overline{\mathscr{F}^{\alpha,q}_D(u)(a\xi)}\mathscr{F}^{\alpha,q}_D(v)(a\xi)\frac{d_qa}{|a|}\Big)|\xi|^{2\alpha+1}d_q\xi$$

$$= C^{\alpha,q}_{u,v}\int_{-\infty}^{\infty}\mathscr{F}^{\alpha,q}_D(f)(\xi)\overline{\mathscr{F}^{\alpha,q}_D(g)(\xi)}|\xi|^{2\alpha+1}d_q\xi.$$

Thus, the proof will be achieved by the use of Parseval's formula (2.21). □

Remark 3.4. *If $u = v$ is a q-Dunkl wavelet and $f = g$ we obtain the following Plancherel's formula*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi^{\alpha}_{q,u}(f)(a,x)|^2d\mu_{\alpha,q}(a,x) = C^{\alpha,q}_{u}\int_{-\infty}^{\infty}|f(x)|^2|x|^{2\alpha+1}d_qx, \tag{3.13}$$

where

$$C^{\alpha,q}_{u} = C^{\alpha,q}_{u,u} := \int_{-\infty}^{\infty}|\mathscr{F}^{\alpha,q}_D(u)(a\xi)|^2\frac{d_qa}{|a|}. \tag{3.14}$$

Corollary 3.1. *Let (u, v) be a two q-Dunkl wavelet. We have the following:*
If $C^{\alpha,q}_{u,v} = 0$, then $\Psi^{\alpha}_{q,u}(L^2_{\alpha,q}(\mathbb{R}_q))$ and $\Psi^{\alpha}_{q,v}(L^2_{\alpha,q}(\mathbb{R}_q))$ are orthogonal.

Theorem 3.2. *(Inversion formula). Let (u, v) be a two q-Dunkl wavelet such that $C^{\alpha,q}_{u,v} \neq 0$. For all f belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$, we have*

$$C^{\alpha,q}_{u,v}f(y) = c_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)v^{\alpha}_{a,x}(y)|x|^{2\alpha+1}\frac{d_qxd_qa}{|a|^{2\alpha+3}},\ y\in\mathbb{R}_q. \tag{3.15}$$

Proof. Let $y \in \mathbb{R}_q$ and put $k = \delta_y$. Then, the use of relation (3.11) and the definition of $\Psi^{\alpha}_{q,u}$, give that

$$\begin{aligned}
(1-q)|y|^{2\alpha+2}f(y) &= \int_{-\infty}^{\infty} f(t)\bar{k}(t)|t|^{2\alpha+1}d_qt \\
&= \frac{1}{C^{\alpha,q}_{u,v}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)\overline{\Psi^{\alpha}_{q,v}(k)(a,x)}|x|^{2\alpha+1}\frac{d_qxd_qa}{|a|^{2\alpha+3}} \\
&= \frac{c_{\alpha,q}}{C^{\alpha,q}_{u,v}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)\Big(\int_{-\infty}^{\infty}\bar{k}(t)v^{\alpha}_{a,x}(t)|t|^{2\alpha+1}d_qt\Big)|x|^{2\alpha+1}\frac{d_qxd_qa}{|a|^{2\alpha+3}} \\
&= (1-q)|y|^{2\alpha+2}\frac{c_{\alpha,q}}{C^{\alpha,q}_{u,v}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(f)(a,x)v^{\alpha}_{a,x}(y)|x|^{2\alpha+1}\frac{d_qxd_qa}{|a|^{2\alpha+3}}
\end{aligned}$$

the last line of this proof is obtained by the use of the q-Jackson integral. □

3.2 Calderón's reproducing formula

Theorem 3.3. *(Calderón's formula). Let u and v be the q-Dunkl wavelets in $L^2_{\alpha,q}(\mathbb{R}_q)$ such that (u, v) is a two q-Dunkl wavelet, $\mathscr{F}^{\alpha,q}_D(u)$, $\mathscr{F}_k(v)$ belong to $L^{\infty}_{\alpha,q}(\mathbb{R}_q)$ and $C^{\alpha,q}_{u,v} \neq 0$. Then, for f in $L^2_{\alpha,q}(\mathbb{R}_q)$ and $\varepsilon, \delta \in \mathbb{R}_q$ such that $\epsilon < \delta$, the function*

$$f^{\epsilon,\delta}(x) = \frac{c_{\alpha,q}}{C^{\alpha,q}_{u,v}}\int_{\epsilon\le|a|\le\delta}\int_{-\infty}^{\infty}\Psi^{\alpha}_{q,u}(a,b)v_{a,b}(x)d_qb\frac{d_qa}{|a|^{2\alpha+3}}, \quad x \in \mathbb{R}_q, \tag{3.16}$$

belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$, and satisfies

$$\lim_{\epsilon\to 0,\delta\to\infty}\|f^{\epsilon,\delta} - f\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 0. \tag{3.17}$$

To prove this theorem we need the following Lemmas.

Lemma 3.1. *Let u and v be two q-Dunkl wavelets satisfying the conditions of Theorem 3.3 and f in $L^2_{\alpha,q}(\mathbb{R}_q)$. Then,*

i) The functions $(\check{f} *_{\alpha,q} \overline{u_a})\check{}$ and $(\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a$ are in $L^2_{\alpha,q}(\mathbb{R}_q)$, and we have

$$\mathscr{F}^{\alpha,q}_D((\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a)(\xi) = \mathscr{F}^{\alpha,q}_D(f)(\xi)\overline{\mathscr{F}^{\alpha,q}_D(u_a)(\xi)}\mathscr{F}^{\alpha,q}_D(v_a)(\xi), \quad \xi \in \mathbb{R}_q. \tag{3.18}$$

ii) We have

$$\|(\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \le \|\mathscr{F}^{\alpha,q}_D(u)\|_{L^{\infty}_{\alpha,q}(\mathbb{R}_q)}\|\mathscr{F}^{\alpha,q}_D(v)\|_{L^{\infty}_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{3.19}$$

Proof. i) From Proposition 2.3 and Proposition 2.5 we have

$$\begin{aligned}
\mathscr{F}^{\alpha,q}_D((\check{f} *_{\alpha,q} \overline{u_a})\check{}\,)(\xi) &= \mathscr{F}^{\alpha,q}_D(\check{f} *_{\alpha,q} \overline{u_a})(-\xi) \\
&= \mathscr{F}^{\alpha,q}_D(\check{f})(-\xi)\mathscr{F}^{\alpha,q}_D(\overline{u_a})(-\xi) \\
&= \mathscr{F}^{\alpha,q}_D(f)(\xi)\overline{\mathscr{F}^{\alpha,q}_D(u_a)(\xi)}.
\end{aligned} \tag{3.20}$$

For $x \in \mathbb{R}_q$, we put $Z(x) = (\check{f} *_{\alpha,q} \overline{u_a})\check{}(x)$. It's easy to see that Z belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$. Using Proposition 2.5 and the fact that

$$\mathscr{F}^{\alpha,q}_D(Z *_{\alpha,q} v_a)(\xi) = \mathscr{F}^{\alpha,q}_D((\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a)(\xi) = \mathscr{F}^{\alpha,q}_D(Z)(\xi)\mathscr{F}^{\alpha,q}_D(v_a)(\xi), \quad \xi \in \mathbb{R}_q \tag{3.21}$$

the relation (3.18) is obtained by using relations (3.20) and (3.21).

ii) From i) we have

$$\int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}((\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a)(\xi)|^2|\xi|^{2\alpha+1}d_q\xi = \int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2|\mathscr{F}_D^{\alpha,q}(u_a)(\xi)|^2|\mathscr{F}_D^{\alpha,q}(v_a)(\xi)|^2|\xi|^{2\alpha+1}d_q\xi.$$

Then, from the Plancherel formula (2.19) and the fact that $\mathscr{F}_D^{\alpha,q}(u_a)$ and $\mathscr{F}_D^{\alpha,q}(v_a)$ belong to $L^\infty_{\alpha,q}(\mathbb{R}_q)$, we obtain

$$\|(\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \leq \|\mathscr{F}_D^{\alpha,q}(u_a)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathscr{F}_D^{\alpha,q}(v_a)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}.$$

This end the proof by the use of relation (3.5). □

Lemma 3.2. *Let u, v and f be as above. For $\epsilon,\ \delta \in \mathbb{R}_q$ such that $\epsilon < \delta$, define the two function*

$$K_{\epsilon,\delta}(\xi) = \frac{1}{C_{u,v}^{\alpha,q}} \int_{\epsilon\leq|a|\leq\delta} \overline{\mathscr{F}_D^{\alpha,q}(u_a)(\xi)}\mathscr{F}_D^{\alpha,q}(v_a)(\xi)\frac{d_qa}{|a|},\ \ \xi \in \mathbb{R}_q, \tag{3.22}$$

and

$$f^{\epsilon,\delta}(y) = \frac{c_{\alpha,q}}{C_{u,v}^{\alpha,q}} \int_{\epsilon\leq|a|\leq\delta}\int_{-\infty}^{\infty} \Psi^\alpha_{q,u}(a,x)v_{a,x}(y)|x|^{2\alpha+1}d_qx\frac{d_qa}{|a|^{2\alpha+3}},\ \ y \in \mathbb{R}_q. \tag{3.23}$$

Then,
i)

$$0 < |K_{\epsilon,\delta}(\xi)| \leq \frac{\sqrt{C_u^{\alpha,q}C_v^{\alpha,q}}}{|C_{u,v}^{\alpha,q}|}, \tag{3.24}$$

and

$$\lim_{\epsilon\to0,\delta\to\infty} K_{\epsilon,\delta}(\xi) = 1. \tag{3.25}$$

ii) The function $f^{\epsilon,\delta} \in\ L^2_{\alpha,q}(\mathbb{R}_q)$ and

$$\mathscr{F}_D^{\alpha,q}(f^{\epsilon,\delta}) = \mathscr{F}_D^{\alpha,q}(f)K_{\epsilon,\delta}. \tag{3.26}$$

Proof. i) It follows from the Cauchy-Schwarz inequality and the relation (3.14), that

$$\begin{aligned} |K_{\epsilon,\delta}(\xi)| &\leq \frac{1}{|C_{u,v}^{\alpha,q}|}\Big(\int_{\epsilon\leq|a|\leq\delta}|\mathscr{F}_D^{\alpha,q}(u_a)(\xi)|^2\frac{d_qa}{|a|}\Big)^{\frac12}\Big(\int_{\epsilon\leq|a|\leq\delta}|\mathscr{F}_D^{\alpha,q}(v_a)(\xi)|^2\frac{d_qa}{|a|}\Big)^{\frac12} \\ &\leq \frac{1}{|C_{u,v}^{\alpha,q}|}\Big(\int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(u_a)(\xi)|^2\frac{d_qa}{|a|}\Big)^{\frac12}\Big(\int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(v_a)(\xi)|^2\frac{d_qa}{|a|}\Big)^{\frac12} = \frac{\sqrt{C_u^{\alpha,q}C_v^{\alpha,q}}}{|C_{u,v}^{\alpha,q}|}. \end{aligned}$$

And we get easily that

$$\lim_{\epsilon\to0,\delta\to\infty} K_{\epsilon,\delta}(\xi) = 1.$$

ii) It's not difficult, by the use of the definition of the q-Dunkl convolution product, to see that

$$f^{\epsilon,\delta}(y) = \frac{1}{C_{u,v}^{\alpha,q}} \int_{\epsilon\leq|a|\leq\delta} (\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a(y)\frac{d_qa}{|a|}. \tag{3.27}$$

Indeed,

$$\begin{aligned} f^{\epsilon,\delta}(y) &= \frac{c_{\alpha,q}}{C_{u,v}^{\alpha,q}} \int_{\epsilon\le|a|\le\delta} \int_{-\infty}^{\infty} (\check{f} *_{\alpha,q} \overline{u_a})(x)\tau_y^{\alpha,q}(v_a)(x)|x|^{2\alpha+1} d_qx \frac{d_qa}{|a|} \\ &= \frac{c_{\alpha,q}}{C_{u,v}^{\alpha,q}} \int_{\epsilon\le|a|\le\delta} \int_{-\infty}^{\infty} (\check{f} *_{\alpha,q} \overline{u_a})\check{}(x)\tau_y^{\alpha,q}(v_a)(-x)|x|^{2\alpha+1} d_qx \frac{d_qa}{|a|} \\ &= \frac{1}{C_{u,v}^{\alpha,q}} \int_{\epsilon\le|a|\le\delta} (\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a(y) \frac{d_qa}{|a|}. \end{aligned}$$

By using Hölder's inequality for the measure $\frac{d_qa}{|a|}$, we get

$$|f^{\epsilon,\delta}(y)|^2 \le \frac{1}{|C_{u,v}^{\alpha,q}|^2} \Big(\int_{\epsilon\le|a|\le\delta} \frac{d_qa}{|a|} \Big) \int_{\epsilon\le|a|\le\delta} \Big| \Big(\check{f} *_{\alpha,q} \overline{u_a}\Big)\check{} *_{\alpha,q} v_a(y) \Big|^2 \frac{d_qa}{|a|}.$$

So, by applying Fubuni-Tonelli's theorem, we obtain

$$\int_{-\infty}^{\infty} |f^{\epsilon,\delta}(y)|^2 |y|^{2\alpha+1} d_qy \le \frac{1}{|C_{u,v}^{\alpha,q}|^2} \Big(\int_{\epsilon\le|a|\le\delta} \frac{d_qa}{|a|} \Big) \int_{\epsilon\le|a|\le\delta} \Big(\int_{-\infty}^{\infty} |(\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a(y)|^2 |y|^{2\alpha+1} d_qy \Big) \frac{d_qa}{|a|}.$$

From Parseval's formula (2.21) and relation (3.18), we deduce that

$$\int_{-\infty}^{\infty} |f^{\epsilon,\delta}(y)|^2 |y|^{2\alpha+1} d_qy \le$$

$$\frac{1}{|C_{u,v}^{\alpha,q}|^2} \Big(\int_{\epsilon\le|a|\le\delta} \frac{d_qa}{|a|} \Big) \int_{-\infty}^{\infty} |\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2 \Big(\int_{\epsilon\le|a|\le\delta} |\mathscr{F}_D^{\alpha,q}(u_a)(\xi)|^2 |\mathscr{F}_D^{\alpha,q}(v_a)(\xi)|^2 \frac{d_qa}{|a|} \Big) |\xi|^{2\alpha+1} d_q\xi.$$

On the other hand, from the relations (3.14) and (3.5), we have

$$\int_{\epsilon\le|a|\le\delta} |\mathscr{F}_D^{\alpha,q}(u_a)(\xi)|^2 |\mathscr{F}_D^{\alpha,q}(v_a)(\xi)|^2 \frac{d_qa}{|a|} \le C_v^{\alpha,q} \|\mathscr{F}_D^{\alpha,q}(u)\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)}.$$

Thus,

$$\int_{-\infty}^{\infty} |f^{\epsilon,\delta}(y)|^2 |y|^{2\alpha+1} d_qy \le \frac{C_v^{\alpha,q}}{|C_{u,v}^{\alpha,q}|^2} \Big(\int_{\epsilon\le|a|\le\delta} \frac{d_qa}{|a|} \Big) \|\mathscr{F}_D^{\alpha,q}(u)\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|\mathscr{F}_D^{\alpha,q}(f)\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)},$$

and the Plancherel formula (2.19) implies

$$\int_{-\infty}^{\infty} |f^{\epsilon,\delta}(y)|^2 |y|^{2\alpha+1} d_qy \le \frac{C_v^{\alpha,q}}{|C_{u,v}^{\alpha,q}|^2} \Big(\int_{\epsilon\le|a|\le\delta} \frac{d_qa}{a} \Big) \|\mathscr{F}_D^{\alpha,q}(u)\|^2_{L^\infty_{\alpha,q}(\mathbb{R}_q)} \|f\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)} < \infty.$$

Then, $f^{\epsilon,\delta}$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$.
Now we prove the relation (3.26). Let χ in $\mathscr{S}(\mathbb{R}_q)$. We have the function $(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)$ is in $\mathscr{S}(\mathbb{R}_q)$. From the relation (3.27), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} f^{\epsilon,\delta}(y) \overline{(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)} |y|^{2\alpha+1} d_qy \\ &= \int_{-\infty}^{\infty} \Big(\frac{1}{C_{u,v}^{\alpha,q}} \int_{\epsilon\le|a|\le\delta} (\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a(y) \frac{d_qa}{|a|} \Big) \overline{(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)} |y|^{2\alpha+1} d_qy \\ &= \frac{1}{C_{u,v}^{\alpha,q}} \int_{\epsilon\le|a|\le\delta} \Big(\int_{-\infty}^{\infty} (\check{f} *_{\alpha,q} \overline{u_a})\check{} *_{\alpha,q} v_a(y) \overline{(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)} |y|^{2\alpha+1} d_qy \Big) \frac{d_qa}{|a|}. \qquad (3.28) \end{aligned}$$

To justify the use of the Fubini's theorem in the last sequence of the equalities, observe that

$$\Big|\frac{1}{C_{u,v}^{\alpha,q}}\Big|\int_{-\infty}^{\infty}\int_{\epsilon\le|a|\le\delta}|(\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a(y)\overline{(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)}|\frac{d_qa}{|a|}|y|^{2\alpha+1}d_qy =$$
$$\Big|\frac{1}{C_{u,v}^{\alpha,q}}\Big|\int_{\epsilon\le|a|\le\delta}\Big[\int_{-\infty}^{\infty}|(\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a(y)||(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)||y|^{2\alpha+1}d_qy\Big]\frac{d_qa}{|a|}.$$

By applying Hölder's inequality to the second member, we get

$$\Big|\frac{1}{C_{u,v}^{\alpha,q}}\Big|\int_{\epsilon\le|a|\le\delta}\Big[\int_{-\infty}^{\infty}|(\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a(y)||(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)||y|^{2\alpha+1}d_qy\Big]\frac{d_qa}{|a|}\le$$
$$\Big|\frac{1}{C_{u,v}^{\alpha,q}}\Big|\int_{\epsilon\le|a|\le\delta}\|(\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\frac{d_qa}{|a|}.$$

From the relation (3.19) and the Plancherel formula (2.19), we obtain

$$\Big|\frac{1}{C_{u,v}}\Big|\int_{\epsilon\le|a|\le\delta}\Big[\int_{-\infty}^{\infty}|(\check{f}*_{\alpha,q}\overline{u_a})\check{}*_{\alpha,q}v_a(y)||(\mathscr{F}_D^{\alpha,q})^{-1}(\chi)(y)||y|^{2\alpha+1}d_qy\Big]\frac{d_qa}{|a|}\le$$
$$\frac{1}{\Big|C_{u,v}^{\alpha,q}\Big|}\Big(\int_{\epsilon\le|a|\le\delta}\frac{d_qa}{|a|}\Big)\|\mathscr{F}_D^{\alpha,q}(u)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\mathscr{F}_D^{\alpha,q}(v)\|_{L^\infty_{\alpha,q}(\mathbb{R}_q)}\|\psi\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}<\infty.$$

Now, by using the Parseval formula (2.21) and the relation (3.18) and Fubini's theorem, the integral given by relation(3.28) becomes

$$\begin{aligned}
&\frac{1}{C_{u,v}^{\alpha,q}}\int_{\epsilon\le|a|\le\delta}\Big(\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\xi)\overline{\mathscr{F}_D^{\alpha,q}(u_a)(\xi)}\mathscr{F}_D^{\alpha,q}(v_a)(\xi)\overline{\chi(\xi)}|\xi|^{2\alpha+1}d_q\xi\Big)\frac{d_qa}{|a|}\\
=&\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\xi)\Big(\frac{1}{C_{u,v}^{\alpha,q}}\int_{\epsilon\le|a|\le\delta}\overline{\mathscr{F}_D^{\alpha,q}(u_a)(\xi)}\mathscr{F}_D^{\alpha,q}(v_a)(\xi)\frac{d_qa}{|a|}\Big)\overline{\chi(\xi)}|\xi|^{2\alpha+1}d_q\xi\\
=&\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f)(\xi)K_{\epsilon,\delta}(\xi)\overline{\chi(\xi)}|\xi|^{2\alpha+1}d_q\xi. \qquad (3.29)
\end{aligned}$$

On the other hand, by applying the Parseval formula (2.21) to the first member of the relation (3.28), we get

$$\int_{-\infty}^{\infty}\mathscr{F}_D^{\alpha,q}(f^{\epsilon,\delta})(\xi)\overline{\chi(\xi)}|\xi|^{2\alpha+1}d_q\xi. \qquad (3.30)$$

From the relations (3.29) and (3.30), we obtain for all χ in $\mathscr{S}(\mathbb{R}_q)$

$$\int_{-\infty}^{\infty}\Big(\mathscr{F}_D^{\alpha,q}(f^{\epsilon,\delta})(\xi)-\mathscr{F}_D^{\alpha,q}(f)(\xi)K_{\epsilon,\delta}(\xi)\Big)\overline{\chi(\xi)}|\xi|^{2\alpha+1}d_q\xi=0.$$

Thus

$$\mathscr{F}_D^{\alpha,q}(f^{\epsilon,\delta})(\xi)=\mathscr{F}_D^{\alpha,q}(f)(\xi)K_{\epsilon,\delta}(\xi),\quad \xi\in\mathbb{R}_q.$$

□

We are now ready to prove the main result of this subsection.

Proof. of Theorem 3.3. From Lemma 3.2 ii), the function $f^{\epsilon,\delta}$ belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$. By using the Plancherel formula (2.19) and Lemma 3.2 ii), we obtain

$$\begin{aligned}||f^{\epsilon,\delta}-f||_{L^2_{\alpha,q}(\mathbb{R}_q)} &= \int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(f^{\epsilon,\delta}-f)(\xi)|^2|\xi|^{2\alpha+1}d_q\xi\\ &= \int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(f)(\xi)(K_{\epsilon,\delta}(\xi)-1)|^2|\xi|^{2\alpha+1}d_q\xi\\ &= \int_{-\infty}^{\infty}|\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2|1-K_{\epsilon,\delta}(\xi)|^2|\xi|^{2\alpha+1}d_q\xi.\end{aligned}$$

But from Lemma 3.2 ii) again, for all $\xi\in\mathbb{R}_q$, we have

$$\lim_{\epsilon\to0,\delta\to\infty}|\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2|1-K_{\epsilon,\delta}(\xi)|^2=0,$$

and

$$|\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2|1-K_{\epsilon,\delta}(\xi)|^2\le C|\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2,$$

with $|\mathscr{F}_D^{\alpha,q}(f)(\xi)|^2$ in $L^1_{\alpha,q}(\mathbb{R}_q)$. So, the relation (3.17) follows from the dominated convergence theorem. □

4 Localization operators for the q-Dunkl wavelet transform

4.1 Preliminaries

Notation. We denote by

- $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $x:=(x_j)_{j\in\mathbb{N}}$, such that

$$\begin{aligned}||x||_p &:= \Big(\sum_{j=1}^{\infty}|x_j|^p\Big)^{\frac{1}{p}}<\infty, \quad \text{if} \quad 1\le p<\infty,\\ ||x||_\infty &:= \sup_{j\in\mathbb{N}}|x_j|<\infty.\end{aligned}$$

  For $p=2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x,y\rangle_2:=\sum_{j=1}^{\infty}x_j\overline{y_j}.$$

- $B(L^2_{\alpha,q}(\mathbb{R}_q))$ the space of bounded operators from $L^2_{\alpha,q}(\mathbb{R}_q)$ into itself.

Definition 4.1. *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{\alpha,q}(\mathbb{R}_q))$ are the eigenvalues of the positive self-adjoint operator $|A|=\sqrt{A^*A}$.*

(ii) For $1\le p<\infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p}:=\Big(\sum_{n=1}^{\infty}(s_n(A))^p\Big)^{\frac{1}{p}}. \tag{4.1}$$

Remark 4.1. *We note that S_2 is the space of Hilbert-Schmidt operators, whereas S_1 is the space of trace class operators.*

Definition 4.2. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n\rangle_{\alpha,q} \tag{4.2}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_{\alpha,q}(\mathbb{R}_q)$.

Remark 4.2. *If A is positive, then*

$$tr(A) = \|A\|_{S_1}. \tag{4.3}$$

Moreover, a compact operator A on the Hilbert space $L^2_{\alpha,q}(\mathbb{R}_q)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} \|Av_n\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)} \tag{4.4}$$

for any orthonormal basis $(v_n)_n$ of $L^2_{\alpha,q}(\mathbb{R}_q)$.

Definition 4.3. *We define $S_\infty := B(L^2_{\alpha,q}(\mathbb{R}_q))$, equipped with the norm,*

$$\|A\|_{S_\infty} := \sup_{v\in L^2_{\alpha,q}(\mathbb{R}_q):\|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)}=1} \|Av\|_{L^2_{\alpha,q}(\mathbb{R}_q)}. \tag{4.5}$$

Definition 4.4. *Let u, v be measurable functions on $\mathbb{R}_q$, σ be measurable function on $\mathbb{R}_q \times \mathbb{R}_q$, we define the two-wavelet localization operator noted by $\mathscr{L}_{u,v}(\sigma)$, on $L^p_{\alpha,q}(\mathbb{R}_q)$, $1 \le p \le \infty$, by*

$$\mathscr{L}_{u,v}(\sigma)(f)(y) = c_{\alpha,q} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(a,x)\Psi^\alpha_{q,u}(f)(a,x)\ v_{a,x}(y)d\mu_{\alpha,q}(a,x),\ y \in \mathbb{R}_q. \tag{4.6}$$

In accordance with the different choices of the symbols σ and the different continuities required, we need to impose different conditions on u and v, and then we obtain an operator on $L^p_{\alpha,q}(\mathbb{R}_q)$.

It is often more convenient to interpret the definition of $\mathscr{L}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^p_{\alpha,q}(\mathbb{R}_q)$, $p \in [1,\infty]$, and g in $L^{p'}_{\alpha,q}(\mathbb{R}_q)$,

$$\langle \mathscr{L}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(a,x)\Psi^\alpha_{q,u}(f)(a,x)\overline{\Psi^\alpha_{q,v}(g)(a,x)}d\mu_{\alpha,q}(a,x). \tag{4.7}$$

Proposition 4.1. *Let $p \in [1,\infty)$. The adjoint of linear operator $\mathscr{L}_{u,v}(\sigma) : L^p_{\alpha,q}(\mathbb{R}_q) \to L^p_{\alpha,q}(\mathbb{R}_q)$ is $\mathscr{L}_{v,u}(\overline{\sigma}) : L^{p'}_{\alpha,q}(\mathbb{R}_q) \to L^{p'}_{\alpha,q}(\mathbb{R}_q)$.*

Proof. For all f in $L^p_{\alpha,q}(\mathbb{R}_q)$ and g in $L^{p'}_{\alpha,q}(\mathbb{R}_q)$ it immediately follows from (4.7)

$$\begin{aligned}
\langle \mathscr{L}_{u,v}(\sigma)(f), g\rangle_{\alpha,q} &= \int_{\mathbb{R}_q\times\mathbb{R}_q} \sigma(a,x)\Psi^\alpha_{q,u}(f)(a,x)\overline{\Psi^\alpha_{q,v}(g)(a,x)}d\mu_{\alpha,q}(a,x) \\
&= \overline{\int_{\mathbb{R}_q\times\mathbb{R}_q} \overline{\sigma(a,x)}\Psi^\alpha_{q,v}(g)(a,x)\overline{\Psi^\alpha_{q,u}(f)(a,x)}d\mu_{\alpha,q}(a,x)} \\
&= \overline{\langle \mathscr{L}_{v,u}(\overline{\sigma})(g), f\rangle_{\alpha,q}} = \langle f, \mathscr{L}_{v,u}(\overline{\sigma})(g)\rangle_{\alpha,q}.
\end{aligned}$$

Thus we get

$$\mathscr{L}^*_{u,v}(\sigma) = \mathscr{L}_{v,u}(\overline{\sigma}). \tag{4.8}$$

□

In what follows, such operator $\mathscr{L}_{u,v}(\sigma)$ will be named localization operator for the sake of simplicity. In this section, u and v will be two q-Dunkl wavelets on $\mathbb{R}_q$ such that

$$\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|v\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1.$$

4.2 Boundedness for $\mathscr{L}_{u,v}(\sigma)$ on S_∞

The main result of this subsection is to prove that the linear operators $\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ are bounded for all symbols $\sigma \in L^p_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, $1 \leq p \leq \infty$. We first consider this problem for σ in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$ and next in $L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$ and we then conclude by using interpolation theory.

Proposition 4.2. *Let σ be in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, then the localization operator $\mathscr{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}. \tag{4.9}$$

Proof. For every functions f and g in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have from the relations (4.7) and (3.10),

$$\begin{aligned}
|\langle \mathscr{L}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| &\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\sigma(a,x)||\Psi^\alpha_{q,u}(f)(a,x)||\overline{\Psi^\alpha_{q,v}(g)(a,x)}|d\mu_{\alpha,q}(a,x) \\
&\leqslant \|\Psi^\alpha_{q,u}(f)\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}\|\Psi^\alpha_{q,v}(g)\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)} \\
&\leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.
\end{aligned}$$

Thus,

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

□

Proposition 4.3. *Let σ be in $L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, then the localization operator $\mathscr{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leqslant \sqrt{C^{\alpha,q}_u C^{\alpha,q}_v}\|\sigma\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Proof. For all functions f and g in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have from Hölder's inequality

$$\begin{aligned}
|\langle \mathscr{L}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| &\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\sigma(a,x)||\Psi^\alpha_{q,u}(f)(a,x)||\overline{\Psi^\alpha_{q,v}(g)(a,x)}|d\mu_{\alpha,q}(a,x) \\
&\leqslant \|\sigma\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}\|\Psi^\alpha_{q,u}(f)\|_{L^2_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}\|\Psi^\alpha_{q,v}(g)\|_{L^2_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.
\end{aligned}$$

Using Plancherel's formula for $\Psi^\alpha_{q,u}$ and $\Psi^\alpha_{q,v}$, given by the relation (3.13), we get

$$|\langle \mathscr{L}_{u,v}(\sigma)(f), g\rangle_{\alpha,q}| \leqslant \sqrt{C^{\alpha,q}_u C^{\alpha,q}_v}\|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|g\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\sigma\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Thus,

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leqslant \sqrt{C^{\alpha,q}_u C^{\alpha,q}_v}\|\sigma\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

□

We can now associate a localization operator $\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ to every symbol σ in $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, $1 \le p \le \infty$ and prove that $\mathscr{L}_{u,v}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 4.1. *Let σ be in $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, $1 \le p \le \infty$. Then there exists a unique bounded linear operator $\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$, such that*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leqslant \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}} (C^{\alpha,q}_u C^{\alpha,q}_v)^{\frac{p-1}{2p}} \|\sigma\|_{L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Proof. Let f be in $L^2_{\alpha,q}(\mathbb{R}_q)$. We consider the following operator

$$\mathscr{T} : L^1_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q) \bigcap L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q) \quad \to \quad L^2_{\alpha,q}(\mathbb{R}_q),$$

given by

$$\mathscr{T}(\sigma) := \mathscr{L}_{u,v}(\sigma)(f).$$

Then by Proposition 4.2 and Proposition 4.3

$$\|\mathscr{T}(\sigma)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty} \|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)} \tag{4.10}$$

and

$$\|\mathscr{T}(\sigma)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \le \sqrt{C^{\alpha,q}_u C^{\alpha,q}_v} \|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)}(\mathbb{R}_q) \|\sigma\|_{L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}. \tag{4.11}$$

Therefore, by (4.10), (4.11) and the Riesz-Thorin interpolation theorem (see [[29], Theorem 2] and [[32], Theorem 2.11]), $\mathscr{T}$ may be uniquely extended to a linear operator on $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, $1 \le p \le \infty$ and we have

$$\|\mathscr{L}_{u,v}(\sigma)(f)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = \|\mathscr{T}(\sigma)\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \le \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}} (C^{\alpha,q}_u C^{\alpha,q}_v)^{\frac{p-1}{2p}} \|f\|_{L^2_{\alpha,q}(\mathbb{R}_q)} \|\sigma\|_{L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}. \tag{4.12}$$

Since (4.12) is true for arbitrary functions f in $L^2_{\alpha,q}(\mathbb{R}_q)$, then we obtain the desired result. □

4.3 Schatten-von Neumann properties for $\mathscr{L}_{u,v}(\sigma)$

The main result of this subsection is to prove that, the localization operator $\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_{\alpha,q}(\mathbb{R}_q)$ is in the Schatten class S_p.

Proposition 4.4. *Let σ be in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q \times \mathbb{R}_q)$, then the localization operator $\mathscr{L}_{u,v}(\sigma)$ is in S_2 and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_2} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty} \|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Proof. Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\alpha,q}(\mathbb{R}_q)$. Then by (4.7), Fubini's theorem, Parseval's identity and the relations (3.8) and (4.8), we have

$$\begin{aligned}
\sum_{j=1}^\infty \|\mathscr{L}_{u,v}(\sigma)(\phi_j)\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)} &= \sum_{j=1}^\infty \langle \mathscr{L}_{u,v}(\sigma)(\phi_j), \mathscr{L}_{u,v}(\sigma)(\phi_j)\rangle_{\alpha,q} \\
&= c^2_{\alpha,q} \sum_{j=1}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \sigma(a,x)\langle \phi_j,\, u_{a,x}\rangle_{\alpha,q} \overline{\langle \mathscr{L}_{u,v}(\sigma)(\phi_j),\, v_{a,x}\rangle}_q d\mu_{\alpha,q}(a,x) \\
&= c^2_{\alpha,q} \int_{-\infty}^\infty \int_{-\infty}^\infty \sigma(a,x) \sum_{j=1}^\infty \langle \phi_j,\, u_{a,x}\rangle_{\alpha,q} \langle \mathscr{L}^*_{u,v}(\sigma)(v_{a,x}), \phi_j\rangle_{\alpha,q} d\mu_{\alpha,q}(a,x) \\
&= c^2_{\alpha,q} \int_{-\infty}^\infty \int_{-\infty}^\infty \sigma(a,x)\langle \mathscr{L}^*_{u,v}(\sigma)\, v_{a,x}, u_{a,x}\rangle_{\alpha,q} d\mu_{\alpha,q}(a,x).
\end{aligned}$$

Thus, from (4.8),(4.9) and (3.7) we get

$$\begin{aligned}\sum_{j=1}^{\infty}\|\mathscr{L}_{u,v}(\sigma)(\phi_j)\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)} &\leq \frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma(a,x)|\,\|\mathscr{L}^*_{u,v}(\sigma)\|_{S_\infty}d\mu_{\alpha,q}(a,x)\\ &\leq \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^2\|\sigma\|^2_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}<\infty.\end{aligned} \tag{4.13}$$

So, by (4.13) and Proposition 2.8 in the book [32], by Wong,

$$\mathscr{L}_{u,v}(\sigma):L^2_{\alpha,q}(\mathbb{R}_q)\to L^2_{\alpha,q}(\mathbb{R}_q)$$

is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 4.5. *Let σ be a symbol in $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$, $1\leqslant p<\infty$. Then the localization operator $\mathscr{L}_{u,v}(\sigma)$ is compact.*

Proof. Let $\sigma\in L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$ and let $(\sigma_n)_{n\in\mathbb{N}}\in L^1_{\mu_k}(\mathbb{R}_q\times\mathbb{R}_q)\bigcap L^\infty_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$ be a sequence of functions such that $\sigma_n\to\sigma$ in $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$ as $n\to\infty$. Then by Theorem 4.1

$$\|\mathscr{L}_{u,v}(\sigma_n)-\mathscr{L}_{u,v}(\sigma)\|_{S_\infty}\leq\Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}}(C^{\alpha,q}_uC^{\alpha,q}_v)^{\frac{p-1}{2p}}\|\sigma_n-\sigma\|_{L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Hence $\mathscr{L}_{u,v}(\sigma_n)\to\mathscr{L}_{u,v}(\sigma)$ in S_∞ as $n\to\infty$. On the other hand, as by Proposition 4.4, $\mathscr{L}_{u,v}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathscr{L}_{u,v}(\sigma)$ is compact. □

Theorem 4.2. *Let σ be in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$. Then $\mathscr{L}_{u,v}(\sigma):L^2_{\alpha,q}(\mathbb{R}_q)\to L^2_{\alpha,q}(\mathbb{R}_q)$ is in S_1 and we have*

$$\frac{2c^2_{\alpha,q}}{C^{\alpha,q}_u+C^{\alpha,q}_v}\|\widetilde{\sigma}\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}\leqslant\|\mathscr{L}_{u,v}(\sigma)\|_{S_1}\leqslant\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}, \tag{4.14}$$

where $\widetilde{\sigma}$ is given by

$$\forall\,(a,x)\in\mathbb{R}_q\times\mathbb{R}_q,\quad \widetilde{\sigma}(a,x)=\langle\mathscr{L}_{u,v}(\sigma)\,u_{a,x},\ v_{a,x}\rangle_{\alpha,q}.$$

Proof. Since σ is in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$, by Proposition 4.4, $\mathscr{L}_{u,v}(\sigma)$ is in S_2, then from the canonical form for compact operators given in [32, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j=1,2...\}$ for the orthogonal complement of the kernel of the operator $\mathscr{L}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathscr{L}_{u,v}(\sigma)|$ and $\{\varphi_j,\ j=1,2...\}$ an orthonormal set in $L^2_{\alpha,q}(\mathbb{R}_q)$, such that

$$\mathscr{L}_{u,v}(\sigma)(f)=\sum_{j=1}^{\infty}s_j\langle f,\phi_j\rangle_{\alpha,q}\varphi_j, \tag{4.15}$$

where $s_j,\ j=1,2...$ are the positive singular values of $\mathscr{L}_{u,v}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_1}=\sum_{j=1}^{\infty}s_j=\sum_{j=1}^{\infty}\langle\mathscr{L}_{u,v}(\sigma)(\phi_j),\varphi_j\rangle_{\alpha,q}.$$

Thus, by Fubini's theorem, Cauchy-Schwarz's inequality, Bessel inequality, relations (3.8) and (3.7), we get

$$\begin{aligned}
\|\mathscr{L}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}\langle \mathscr{L}_{u,v}(\sigma)(\phi_j), \varphi_j\rangle_{\alpha,q} \\
&= \sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\Psi^{\alpha}_{q,u}(\phi_j)(a,x)\overline{\Psi^{\alpha}_{q,v}(\varphi_j)(a,x)}d\mu_{\alpha,q}(a,x) \\
&\leqslant \int_{\mathbb{R}_q\times\mathbb{R}_q}|\sigma(a,x)|\left(\sum_{j=1}^{\infty}|\Psi^{\alpha}_{q,u}(\phi_j)(a,x)|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}|\Psi^{\alpha}_{q,v}(\varphi_j)(a,x)|^2\right)^{\frac{1}{2}}d\mu_{\alpha,q}(a,x) \\
&\leqslant c^2_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma(a,x)|\|\, u_{a,x}\|_{L^2_{\alpha,q}(\mathbb{R}_q)}\|\, v_{a,x}\|_{L^2_{\alpha,q}(\mathbb{R}_q)}d\mu_{\alpha,q}(a,x) \\
&\leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_{\infty}}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.
\end{aligned}$$

Thus

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_1} \leqslant \frac{16c^2_{\alpha,q}}{(q;q)^2_{\infty}}\|\sigma\|_{L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

We now prove that $\mathscr{L}_{u,v}(\sigma)$ satisfies the first member of (4.14). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{\alpha,q}(\mathbb{R}_q)$, and using formula (4.15), we get

$$\begin{aligned}
|\widetilde{\sigma}(a,x)| &= \left|\langle \mathscr{L}_{u,v}(\sigma)(\, u_{a,x}),\ v_{a,x}\rangle_{\alpha,q}\right| \\
&= \Big|\sum_{j=1}^{\infty}s_j\langle\, u_{a,x},\phi_j\rangle_{\alpha,q}\langle\varphi_j,\ v_{a,x}\rangle_{\alpha,q}\Big| \\
&\leqslant \frac{1}{2}\sum_{j=1}^{\infty}s_j\Big(\left|\langle\, u_{a,x},\phi_j\rangle_{\alpha,q}\right|^2+\left|\langle\, v_{a,x},\varphi_j\rangle_{\alpha,q}\right|^2\Big).
\end{aligned}$$

Then from Fubini's theorem, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\widetilde{\sigma}(a,x)|d\mu_{\alpha,q}(a,x) \leq \frac{1}{2}\sum_{j=1}^{\infty}s_j\Big(&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\langle\, u_{a,x},\phi_j\rangle_{\alpha,q}|^2d\mu_{\alpha,q}(a,x) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\langle\, v_{a,x},\varphi_j\rangle_{\alpha,q}|^2d\mu_{\alpha,q}(a,x)\Big).
\end{aligned}$$

Thus using Plancherel's formula for $\Psi^{\alpha}_{q,u}$, $\Psi^{\alpha}_{q,v}$, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\widetilde{\sigma}(a,x)|d\mu_{\alpha,q}(a,x) \leq \frac{C^{\alpha,q}_u+C^{\alpha,q}_v}{2c^2_{\alpha,q}}\sum_{j=1}^{\infty}s_j = \frac{C^{\alpha,q}_u+C^{\alpha,q}_v}{2c^2_{\alpha,q}}\|\mathscr{L}_{u,v}(\sigma)\|_{S_1}.$$

The proof is completed. □

Corollary 4.1. *For σ in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$, we have the following trace formula*

$$tr(\mathscr{L}_{u,v}(\sigma)) = c^2_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\langle\, v_{a,x},\ u_{a,x}\rangle_{\alpha,q}d\mu_{\alpha,q}(a,x). \tag{4.16}$$

Proof. Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\alpha,q}(\mathbb{R}_q)$. From Theorem 4.2, the localization operator $\mathscr{L}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (4.2), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned} tr(\mathscr{L}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty}\langle \mathscr{L}_{u,v}(\sigma)(\phi_j), \phi_j\rangle_{\alpha,q} \\ &= c^2_{\alpha,q}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\langle \phi_j, u_{a,x}\rangle_{\alpha,q}\overline{\langle \phi_j, v_{a,x}\rangle_q}d\mu_{\alpha,q}(a,x) \\ &= c^2_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\sum_{j=1}^{\infty}\langle \phi_j, u_{a,x}\rangle_{\alpha,q}\overline{\langle \phi_j, v_{a,x}\rangle_q}d\mu_{\alpha,q}(a,x) \\ &= c^2_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\langle v_{a,x}, u_{a,x}\rangle_{\alpha,q}\, d\mu_{\alpha,q}(a,x), \end{aligned}$$

and the proof is completed. □

In the following we give the main result of this subsection.

Corollary 4.2. *Let σ be in $L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$, $1 \leqslant p \leqslant \infty$. Then, the localization operator $\mathscr{L}_{u,v}(\sigma)$: $L^2_{\alpha,q}(\mathbb{R}_q) \longrightarrow L^2_{\alpha,q}(\mathbb{R}_q)$ is in S_p and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_p} \leqslant \Big(\frac{16c^2_{\alpha,q}}{(q;q)^2_\infty}\Big)^{\frac{1}{p}}(C^{\alpha,q}_u C^{\alpha,q}_v)^{\frac{p-1}{2p}}\|\sigma\|_{L^p_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)}.$$

Proof. The result follows from Proposition 4.3, Theorem 4.2 and by interpolation (See [32, Theorem 2.10 and Theorem 2.11]). □

Remark 4.3. *If $u = v$ and if σ is a real valued and nonnegative function in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$ then $\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,q}(\mathbb{R}_q) \to L^2_k(\mathbb{R}^d)$ is a positive operator. So, by (4.3) and Corollary 4.1*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_1} = c^2_{\alpha,q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(a,x)\|u_{a,x}\|^2_{L^2_{\alpha,q}(\mathbb{R}_q)}d\mu_{\alpha,q}(a,x). \tag{4.17}$$

Now we state a result concerning the trace of products of localization operators.

Corollary 4.3. *Let σ_1 and σ_2 be any real-valued and non-negative functions in $L^1_{\mu_{\alpha,q}}(\mathbb{R}_q\times\mathbb{R}_q)$. We assume that $u = v$ and u is a function in $L^2_{\alpha,q}(\mathbb{R}_q)$ such that $\|u\|_{L^2_{\alpha,q}(\mathbb{R}_q)} = 1$. Then, the localization operators $\mathscr{L}_{u,v}(\sigma_1)$, $\mathscr{L}_{u,v}(\sigma_2)$ are positive trace class operators and*

$$\begin{aligned} \Big\|\Big(\mathscr{L}_{u,v}(\sigma_1)\,\mathscr{L}_{u,v}(\sigma_2)\Big)^n\Big\|_{S_1} &= tr\Big(\mathscr{L}_{u,v}(\sigma_1)\,\mathscr{L}_{u,v}(\sigma_2)\Big)^n \\ &\le \Big(tr(\mathscr{L}_{u,v}(\sigma_1))\Big)^n\Big(tr(\mathscr{L}_{u,v}(\sigma_2))\Big)^n \\ &= \Big\|\mathscr{L}_{u,v}(\sigma_1)\Big\|^n_{S_1}\Big\|\mathscr{L}_{u,v}(\sigma_2)\Big\|^n_{S_1}, \end{aligned}$$

for any natural number n.

Proof. By Theorem 1 in the paper [22] by Liu we know that if A and B are in the trace class S_1 and are positive operators, then

$$\forall\, n \in \mathbb{N}, \quad tr(AB)^n \le \Big(tr(A)\Big)^n\Big(tr(B)\Big)^n.$$

So, if we take $A = \mathscr{L}_{u,v}(\sigma_1)$, $B = \mathscr{L}_{u,v}(\sigma_2)$ and we invoke the previous remark, the desired result is obtained and the proof is completed. □

5 Open Problem

In the present paper, we have successfully studied the localization theory associated with the q-Dunkl wavelet transforms on the Schatten classes. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to study the boundedness and compactness of these localization operators on $L^p_{\nu,q}(\mathbb{R}_q)$, $1 \leq p \leq \infty$.

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