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Local well-posedness for a generalized two-component Degasperis-Procesi system

Nurhan Dündar

The Ministry of National Education, Turkey
e-mail:nurhandundar@hotmail.com

Necat Polat

Department of Mathematics, Dicle University, 21280 Diyarbakir, Turkey
e-mail:npolat@dicle.edu.tr

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Abstract

In this article, we study the local well-posedness of the Cauchy problem for the generalized two-component Degasperis-Procesi system by using Kato's theory in the Sobolev spaces.

Keywords: *The generalized two-component Degasperis-Procesi system, Local well-posedness, Kato's theory.*
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1 Introduction and overview

In the present work, we investigate the Cauchy problem of the following generalized two-component Degasperis-Procesi system:

$$\begin{cases} u_t - u_{xxt} + 4u^m u_x - 3u_x u_{xx} - uu_{xxx} + k\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + u\rho_x + 2u_x\rho = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), & x \in R. \end{cases} \tag{1}$$

where $m \geq 1$, $m \in N$ and k is an arbitrary real constant. For $m = 1$, the system (1) was firstly introduced by Popowicz in [14] as a generalization of the Degasperis-Procesi equation (DP) by use of the Dirac reduction of the generalized, but degenerated Hamiltonian operator of the Boussinesq equation. In system (1), if $m = 1$ and $\rho = 0$, we obtain the classical DP [3]

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{2}$$

Eq. (2) was proved formally integrable by constructing a Lax pair [4]. The authors also presented that Eq. (2) has bi- Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm peakons [4]. We obtain the Camassa-Holm equation (CH) [1] by changing the coefficients 4 and 3 in Eq. (2) with 3 and 2, respectively. CH and DP are two integrable equations with application in the theory of water waves ([2], [4], [5], [9]). Also, both equations have the same asymptotic accuracy. Although CH and DP are similar in many aspects, we want to emphasize that they are actually very different. For example, DP has not only peakon solutions [4] and periodic peakon solutions [19], but also shock peakons [13] and the periodic shock waves [6]. This is a different property of DP from CH.

After the DP was derived, it was studied by many researchers in several aspects ([8], [12], [17], [18], [19] and the citations therein). Furthermore, Tian and Li [15] studied the generalized DP (or called it the modified DP),

$$u_t - u_{xxt} + 4u^m u_x - 3u_x u_{xx} - uu_{xxx} = 0, \tag{3}$$

where $m > 0$, $m \in N$. They studied the local well-posedness of the Cauchy problem of Eq. (3).

In recent years, the system (1) has attracted the attention of many authors, and it has been studied by many authors, for $m = 1$. Yan and Yin [16] established local well-posedness in the nonhomogeneous Besov spaces. Then they derived precise blow-up scenario, proved the existence of strong solutions which blow up in finite time. In [20], Yu and Tian investigated the traveling wave solutions to the two component Degasperis-Procesi system. Jin and Guo [10] studied the blow-up mechanisms and persistence properties of strong solutions.

We note that the Cauchy problem of system (1) (for $m > 1$) has not been discussed yet. The main purpose of this article is to investigate the local well-posedness the system of (1), for $m \geq 1$.

2 Preliminaries

In the here, we will introduce some notations by summarizing them. Also, we will give the theorem and the lemma necessary for our proof.

$\|.\|_Z$ denotes the norm of Banach space Z. H^s is the classical Sobolev space with norm $\|.\|_{H^s} = \|.\|_s$, $s \in R$. For practical purposes, we will denote different positive constants with the same symbols c.

We will apply Kato's theory to establish the local well-posedness for the Cauchy problem of (1). In the following we will present a form suitable of Kato's general theory for our purpose. Consider the abstract quasi-linear initial value problem:

$$\frac{dv}{dt} + A(v)v = f(v), \qquad t \geq 0, \qquad v(0) = v_0. \tag{4}$$

Let (4) be in a Hilbert space X, and let Y be another Hilbert space which is continuously and densely embedded into X. Let $S : Y \to X$ be a topological isomorphism. $L(Y, X)$ denotes the space of all bounded linear operators from Y to X, particularly, it is denoted by $L(X)$, if $X = Y$. We write $G(X, 1, \beta)$ for the set of all linear operators A in X, where β is a real number, such that $-A$ generates a C_0-semigroup $T(t)$ on X and that $\|T(t)\|_{L(X)} \leq e^{t\beta}$ for all $t \geq 0$.

Theorem 2.1 *[11] Assume that:*

(I) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\|(A(y) - A(z))\,\omega\|_X \leq \kappa_1 \|y - z\|_X \|\omega\|_Y, \qquad y, z, \omega \in Y,$$

and $A(y) \in G(X, 1, \beta)$, (i.e. $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(II) $SA(y)S^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y) - B(z))\,\omega\|_X \leq \kappa_2 \|y - z\|_Y \|\omega\|_X, \qquad y, z \in Y, \ \omega \in X.$$

(III) $f : Y \to Y$ and also extends to a map from X into X. f is bounded on bounded sets in Y, and satisfies

$$\begin{aligned} \|(f(y) - f(z))\|_Y &\leq \kappa_3 \|y - z\|_Y, \qquad y, z \in Y, \\ \|(f(y) - f(z))\|_X &\leq \kappa_4 \|y - z\|_X, \qquad y, z \in Y. \end{aligned}$$

Here $\kappa_i = (i = 1, 2, 3, 4)$ are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$. If the conditions (I), (II) and (III) hold, given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to (4) such that

$$v = v(., v_0) \in C([0, T); Y) \cap C^1([0, T); X).$$

Moreover, the map $v_0 \to v(., v_0)$ is continuous from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

Lemma 2.2 *[11] Let r, t be any real numbers such that $-r < t \leq r$. Then*

$$\begin{aligned} \|fg\|_{H^t} &\leq c\|f\|_{H^r}\|g\|_{H^t}, \qquad \text{if } r > \frac{1}{2}, \\ \|fg\|_{H^{r+t-\frac{1}{2}}} &\leq c\|f\|_{H^r}\|g\|_{H^t}, \qquad \text{if } r < \frac{1}{2} \end{aligned}$$

where c is a positive constant depending on r, t.

3 Local well-posedness

We note that if $p(x) = \frac{1}{2}e^{-|x|}$, $x \in R$, then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(R)$. Here we denote by $*$ the convolution. We can rewrite (1) as follows:

$$\begin{cases} u_t + uu_x = -\partial_x p * (\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2 + \frac{k}{2}\rho^2) & t > 0, x \in R, \\ \rho_t + u\rho_x = -2u_x\rho, & t > 0, x \in R, \\ u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), & x \in R, \end{cases} \tag{5}$$

or in the equivalent form:

$$\begin{cases} u_t + uu_x = -\partial_x(1-\partial_x^2)^{-1}(\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2 + \frac{k}{2}\rho^2) & t > 0, x \in R, \\ \rho_t + u\rho_x = -2u_x\rho, & t > 0, x \in R, \\ u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), & x \in R. \end{cases} \tag{6}$$

Theorem 3.1 *Given $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s \geq 2$, there exists a maximal $T = T\left(\|U_0\|_{H^s\times H^{s-1}}\right) > 0$, and a unique solution $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (6) (or (1)) such that*

$$U = U(., U_0) \in C\left([0,T); H^s \times H^{s-1}\right) \cap C^1\left([0,T); H^{s-1} \times H^{s-2}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$U_0 \to U(., U_0) : H^s \times H^{s-1} \to C\left([0,T); H^s \times H^{s-1}\right) \cap C^1\left([0,T); H^{s-1} \times H^{s-2}\right)$$

is continuous.

To prove this theorem, we will apply Theorem 2.1, with

$$\begin{aligned} U &= (u, \rho), \\ A(U) &= \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}, \\ f(U) &= \begin{pmatrix} -\partial_x (1-\partial_x^2)^{-1} \left(\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2 + \frac{k}{2}\rho^2\right) \\ -2u_x\rho \end{pmatrix}, \end{aligned}$$

$Y = H^s \times H^{s-1}$, $X = H^{s-1} \times H^{s-2}$, $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$ and $S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$. We know that S is an isomorphism of $H^s \times H^{s-1}$ onto $H^{s-1} \times H^{s-2}$. Thus, to get Theorem 3.1 by applying Theorem 2.1, we only need to check that $A(U)$ and $f(U)$ satisfy the conditions (I), (II) and (III).

Lemma 3.2 *The operator* $A(U) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$ *with* $U \in H^s \times H^{s-1}$, $s \geq 2$, *belongs to* $G(L^2 \times L^2, 1, \beta)$.

Lemma 3.3 *The operator* $A(U) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$ *with* $U \in H^s \times H^{s-1}$, $s \geq 2$, *belongs to* $G(H^{s-1} \times H^{s-2}, 1, \beta)$.

Lemma 3.4 *Let* $A(U) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$ *with* $U \in H^s \times H^{s-1}$, $s \geq 2$. *Then*

$$A(U) \in L\left(H^s \times H^{s-1}, H^{s-1} \times H^{s-2}\right).$$

Moreover,

$$\|(A(U) - A(V))\, W\|_{H^{s-1}\times H^{s-2}} \leq \kappa_1 \|U - V\|_{H^{s-1}\times H^{s-2}} \|W\|_{H^s\times H^{s-1}},$$

for all $U, V, W \in H^s \times H^{s-1}$.

Lemma 3.5 *Let* $B(U) = SA(U)S^{-1} - A(U)$ *with* $U \in H^s \times H^{s-1}$, $s \geq 2$. *Then* $B(U) \in L(H^{s-1} \times H^{s-2})$ *and*

$$\|(B(U) - B(V))\, W\|_{H^{s-1}\times H^{s-2}} \leq \kappa_2 \|U - V\|_{H^s\times H^{s-1}} \|W\|_{H^{s-1}\times H^{s-2}},$$

for all $U, V \in H^s \times H^{s-1}$ *and* $W \in H^{s-1} \times H^{s-2}$.

The proof of these lemmas can be found in [7], therefore, we will skip the proof of these lemmas here. Thus, the conditions (I) and (II) are satisfied. We will now show that the condition (III) is satisfied. For this, we need to prove the following lemma.

Lemma 3.6 *Let* $U \in H^s \times H^{s-1}$, $s \geq 2$ *and let*

$$f(U) = \begin{pmatrix} -\partial_x (1-\partial_x^2)^{-1} \left(\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2 + \frac{k}{2}\rho^2\right) \\ -2u_x\rho \end{pmatrix}.$$

Then f *is bounded on bounded sets in* $H^s \times H^{s-1}$, *and satisfies*

(i) $\|f(U) - f(V)\|_{H^s\times H^{s-1}} \leq \kappa_3 \|U - V\|_{H^s\times H^{s-1}} \qquad U, V \in H^s \times H^{s-1}$,

(ii) $\|f(U) - f(V)\|_{H^{s-1}\times H^{s-2}} \leq \kappa_4 \|U - V\|_{H^{s-1}\times H^{s-2}} \qquad U, V \in H^s \times H^{s-1}$.

Proof: Let $U, V \in H^s \times H^{s-1}$, $s \geq 2$ and let $V = (v_1, v_2)$. Note that H^{s-1} is a Banach algebra. Then, we have

$$\begin{aligned}
& \|f(U) - f(V)\|_{H^s \times H^{s-1}} \\
\leq & \left\| -\partial_x \left(1 - \partial_x^2\right)^{-1} \left(\frac{4}{m+1} u^{m+1} - \frac{1}{2} u^2 - \frac{4}{m+1} v_1^{m+1} + \frac{1}{2} v_1^2 \right) \right\|_s \\
& + \left\| -\partial_x \left(1 - \partial_x^2\right)^{-1} \left(\frac{k}{2} \rho^2 - \frac{k}{2} v_2^2 \right) \right\|_s + \|-2u_x \rho + 2v_{1,x} v_2\|_{s-1} \\
\leq & c \left(\left\| u^{m+1} - v_1^{m+1} \right\|_{s-1} + \left\| u^2 - v_1^2 \right\|_{s-1} + \left\| \rho^2 - v_2^2 \right\|_{s-1} \right) \\
& + c \left(\|u_x \rho - v_{1,x} v_2\|_{s-1} \right). \qquad (7)
\end{aligned}$$

Using the imbedding property of Sobolev spaces H^s and Lemma 2.2, we have

$$\begin{aligned}
\left\| u^{m+1} - v_1^{m+1}) \right\|_{s-1} &= \left\| (u - v_1)(u^m + u^{m-1} v_1 + ... + v_1^m) \right\|_{s-1} \\
&\leq c \|u - v_1\|_s \left\| u^m + u^{m-1} v_1 + ... + v_1^m \right\|_s \\
&\leq c \|u - v_1\|_s \left(\|u\|_s^m + \|u\|_s^{m-1} \|v_1\|_s + ... + \|v_1\|_s^m \right) \\
&\leq c \|u - v_1\|_s . \qquad (8) \\
\left\| u^2 - v_1^2 \right\|_{s-1} &= \|(u - v_1)(u + v_1)\|_{s-1} \\
&\leq c \|u - v_1\|_s \|u + v_1\|_s \\
&\leq c (\|u\|_s + \|v_1\|_s) \|u - v_1\|_s \\
&\leq c \|u - v_1\|_s . \qquad (9)
\end{aligned}$$

Similarly, we get

$$\left\| \rho^2 - v_2^2 \right\|_{s-1} \leq c \|\rho - v_2\|_{s-1} \qquad (10)$$

and

$$\begin{aligned}
\|u_x \rho - v_{1,x} v_2\|_{s-1} &\leq \|u_x \rho - u_x v_2\|_{s-1} + \|u_x v_2 - v_{1,x} v_2\|_{s-1} \\
&\leq c \|u\|_s \|\rho - v_2\|_{s-1} + c \|v_2\|_{s-1} \|u - v_1\|_s . \qquad (11)
\end{aligned}$$

So, from (7)-(11), we have

$$\begin{aligned}
\|f(U) - f(V)\|_{H^s \times H^{s-1}} &\leq c \|u - v_1\|_s + c \|\rho - v_2\|_{s-1} \\
&\leq \kappa_3 \|U - V\|_{H^s \times H^{s-1}} .
\end{aligned}$$

This completes (i). Choosing $V = 0$ in the above inequality, we get that f is bounded on bounded set in $H^s \times H^{s-1}$.

Now, we prove (ii).

$$
\begin{aligned}
&\|f(U)-f(V)\|_{H^{s-1}\times H^{s-2}} \\
\leq\ & \left\|-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{4}{m+1}u^{m+1}-\frac{1}{2}u^2-\frac{4}{m+1}v_1^{m+1}+\frac{1}{2}v_1^2\right)\right\|_{s-1} \\
&+\left\|-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{k}{2}\rho^2-\frac{k}{2}v_2^2\right)\right\|_{s-1}+\|-2u_x\rho+2v_{1,x}v_2\|_{s-2} \\
\leq\ & c\left(\left\|u^{m+1}-v_1^{m+1}\right\|_{s-2}+\left\|u^2-v_1^2\right\|_{s-2}+\left\|\rho^2-v_2^2\right\|_{s-2}\right) \\
&+c\left(\|u_x\rho-v_{1,x}v_2\|_{s-2}\right). \qquad (12)
\end{aligned}
$$

Again, by using the imbedding property of Sobolev spaces H^s and Lemma 2.2, we have

$$
\begin{aligned}
\left\|u^{m+1}-v_1^{m+1})\right\|_{s-2} &= \left\|(u-v_1)(u^m+u^{m-1}v_1+...+v_1^m)\right\|_{s-2} \\
&\leq c\|u-v_1\|_{s-1}\left\|u^m+u^{m-1}v_1+...+v_1^m\right\|_{s-2} \\
&\leq c\|u-v_1\|_{s-1}\left(\|u\|_s^m+\|u\|_s^{m-1}\|v_1\|_s+...+\|v_1\|_s^m\right) \\
&\leq c\|u-v_1\|_{s-1}. \qquad (13) \\
\left\|u^2-v_1^2\right\|_{s-2} &= \|(u-v_1)(u+v_1)\|_{s-2} \\
&\leq c\|u-v_1\|_{s-1}\|u+v_1\|_{s-2} \\
&\leq c\|u-v_1\|_{s-1}. \qquad (14)
\end{aligned}
$$

In an analogous way, we have

$$
\left\|\rho^2-v_2^2\right\|_{s-2}\leq c\|\rho-v_2\|_{s-2} \qquad (15)
$$

and

$$
\begin{aligned}
\|u_x\rho-v_{1,x}v_2\|_{s-2} &\leq \|u_x\rho-u_xv_2\|_{s-2}+\|u_xv_2-v_{1,x}v_2\|_{s-2} \\
&\leq c\|\rho-v_2\|_{s-2}+c\|u-v_1\|_{s-1}. \qquad (16)
\end{aligned}
$$

So, from (12)-(16), we get

$$
\begin{aligned}
\|f(U)-f(V)\|_{H^{s-1}\times H^{s-2}} &\leq c\|u-v_1\|_{s-1}+c\|\rho-v_2\|_{s-2} \\
&\leq \kappa_4\|U-V\|_{H^{s-1}\times H^{s-2}}.
\end{aligned}
$$

This completes the proof of Lemma 3.6.

Proof of Theorem 3.1. The proof of Theorem 3.1 is obtained by combining Theorem 2.1 and Lemma 3.2-Lemma 3.6.

4 Open Problem

In this article, we obtained the local well-posedness for the generalized Degasperis-Procesi system by using Kato's theory in Sobolev spaces. The open problems here are listed below:
1) Can we obtain the local well-posedness for the system (1) in the Besov spaces (which generalize the Sobolev spaces)?
2) Are there global solutions for the system (1)?

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