Int. J. Open Problems Compt. Math., Vol. 14, No. 2, June 2021
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Generalized Reverse Derivations on Prime and Semiprime Rings

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Received 20 March 2021; Accepted 3 May 2021

(Communicated by Ayhan Esi)

Abstract

The present paper investigates some properties of generalized reverse derivations on prime and semiprime rings. Firstly, the commutativity of a prime ring R is examined under the following differential identities provided by a generalized reverse derivation F associated with a reverse derivation d of R on a one-sided ideal of R and a mapping G: $(i)\, F([x,y]) \mp [G(z),y] \in Z(R)$, $(ii)\, F([x,y]) \mp G(z) \circ y \in Z(R)$, $(iii)\, F([x,y]) \mp G(z)x \in Z(R)$, $(iv)\, F([x,y]) \mp xG(z) \in Z(R)$, $(v)\, F([x,y]) \mp xz \in Z(R)$, $(vi)\, F(yx) \mp [G(z),y] \in Z(R)$, $(vii)\, F(yx) \mp G(z) \circ y \in Z(R)$, $(viii)\, F(xy) \mp G(z)x \in Z(R)$, $(ix)\, F(xy) \mp xG(z) \in Z(R)$, $(x)\, F(y \circ x) \mp [G(z),y] \in Z(R)$, $(xi)\, F(y \circ x) \mp G(z) \circ y \in Z(R)$, $(xii)\, F(x \circ y) \mp G(z)x \in Z(R)$, $(xiii)\, F(x \circ y) \mp xG(z) \in Z(R)$, $(xiv)\, F(x \circ y) \mp xz \in Z(R)$. ***Secondly, we study the relationships between $r-$generalized reverse derivations and $l-$generalized derivation and $l-$generalized reverse derivation and $r-$generalized derivations on a noncentral square-closed Lie ideal in a semiprime ring. Finally, we provide two open problems.***

Keywords: *Commutativity, Generalized Reverse Derivation, Prime Ring, Reverse Derivation, Semiprime Ring, Square-Closed Lie Ideal*

1 Introduction

R will denote an associative ring and $Z(R)$ denotes the center of R. Let U be a subset of R. The set $C_R(U) = \{x \in R \mid xa = ax, \text{ for all } a \in U\}$ is called centralizer of U. For each $r, s \in R$, commutator and anti-commutator are defined as $[r, s] = rs - sr$ and $r \circ s = rs + sr$, respectively. For any $a, b \in R$, if $aRb = (0)$ implies either $a = 0$ or $b = 0$, then R is said to be a prime ring and if $aRa = (0)$ implies $a = 0$, then R is called a semiprime ring. An additive subgroup U of R is called a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal U of R is said to be square-closed if $x^2 \in U$, for all $x \in U$. It is well known that if U is a square-closed Lie ideal, then $2xy \in U$, for all $x, y \in U$. In a ring, every ideal is a square-closed Lie ideal. However, the opposite is not always true.

Remember that an additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$ [1]. An additive mapping $d : R \to R$ is called a reverse derivation if $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$. The consept of reverse derivation of a prime ring R was introduced by Herstein in [2]. He has shown that if R is a prime ring and d is a reverse derivation, then R is a commutative integral domain and d is an ordinary derivation on R. In [3], Samman and Alyamani have provided some examples which examine all the cases between reverse derivations and derivations. Moreover, they have shown that if a prime ring R admits a nonzero reverse derivation, then R is commutative.

In [4], Bresăr has extended the concept of derivations to the concept of generalized derivations. An additive mapping $F : R \to R$ is called a generalized derivations on R associated with a derivation $d : R \to R$, if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. In [5], Gölbaşı and Kaya have presented the concepts of $l-$generalized derivations (generalized derivations [4]) and $r-$generalized derivations. An additive mapping $F : R \to R$ is called an $l-$generalized derivations on R associated with a derivation $d : R \to R$, if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $F : R \to R$ is called an $r-$generalized derivations on R associated with a derivation $d : R \to R$, if $F(xy) = d(x)y + xF(y)$, for all $x, y \in R$. In [6], Aboubakr and Gonzălez have extend the concept of reverse derivations to the concepts of $l-$generalized reverse derivations (generalized reverse derivations) and $r-$generalized reverse derivations. An additive mapping $F : R \to R$ is called an $l-$generalized reverse derivations (generalized reverse derivations) on R associated with a reverse derivation $d : R \to R$, if $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$. An additive mapping $F : R \to R$ is called an $r-$generalized reverse derivations on R associated with a reverse derivation $d : R \to R$, if $F(xy) = d(y)x + yF(x)$, for all $x, y \in R$. Furthermore, a mapping F is called a generalized reverse derivation if both an $l-$generalized reverse derivation and an $r-$generalized reverse derivation are. Additionally, the authors have provided some examples

on matrices ring via generalized reverse derivations. Afterwards, Ibraheem [7] showed that R is a prime ring with a right ideal I of R such that $I \cap Z(R) \neq (0)$ and if f is a generalized reverse derivation on R with a nonzero reverse derivation d on R such that $[f(x), x] \in Z(R)$, for all $x, y \in I$, then R is commutative. Furthermore, Huang [8] has investigated the commutativity of a prime ring R with a generalized reverse derivation $F : R \to R$ under the following conditions: For all $x, y \in R$, $(i)\, F(xy) \mp xy \in Z(R)$, $(ii)\, F([x, y]) \mp [F(x), y] \in Z(R)$, $(iii)\, F([x, y]) \mp [F(x), F(y)] \in Z(R)$, $(iv)\, F(x \circ y) \mp F(x) \circ F(y) \in Z(R)$, $(v)\, [F(x), y] \mp [x, F(y)] \in Z(R)$, $(vi)\, F(x) \circ y \mp x \circ F(y) \in Z(R)$.
The first part of the present study is directly motivated by the work of Huang [8]. Thus, we proved the following theorem:

Theorem. Let R be a prime ring, I be a nonzero right (left) ideal of R, $G : R \to R$ be a mapping, and $F : R \to R$ be a generalized reverse derivation associated with a reverse derivation $d : R \to R$ such that $d(Z(R)) \neq (0)$. If one of the following conditions holds:

(i) $F([x, y]) \mp [G(z), y] \in Z(R)$ (ii) $F([x, y]) \mp G(z) \circ y \in Z(R)$
(iii) $F([x, y]) \mp G(z)x \in Z(R)$ (iv) $F([x, y]) \mp xG(z) \in Z(R)$
(v) $F([x, y]) \mp xz \in Z(R)$ (vi) $F(yx) \mp [G(z), y] \in Z(R)$
(vii) $F(yx) \mp G(z) \circ y \in Z(R)$ $(viii)$ $F(xy) \mp G(z)x \in Z(R)$
(ix) $F(xy) \mp xG(z) \in Z(R)$ (x) $F(y \circ x) \mp [G(z), y] \in Z(R)$
(xi) $F(y \circ x) \mp G(z) \circ y \in Z(R)$ (xii) $F(x \circ y) \mp G(z)x \in Z(R)$
$(xiii)$ $F(x \circ y) \mp xG(z) \in Z(R)$ (xiv) $F(x \circ y) \mp xz \in Z(R)$

for all $x, y, z \in I$, then R is commutative.
In [6], Aboubakr and Gonzălez have extended $r-$generalized derivation and $l-$generalized derivation definitions of Gölbaşı and Kaya [5] to $r-$generalized reverse derivation and $l-$generalized reverse derivation. Moreover, the same article includes many examples investigating the relationship between $l-$generalized reverse derivation and $r-$generalized reverse derivation. To be more specific, the theorems provided by Aboubakr and Gonzălez are as follow:
Theorem. [6, Theorem 3.1 and Theorem 3.2] Let R be a semiprime ring and I be an ideal of R. There exists $F : I \to R$, an $l-$generalized ($r-$generalized) reverse derivation associated with a nonzero reverse derivation $d : I \to R$, if and only if $d(I), F(I) \subseteq C_R(I)$, d is a derivation on I, and F is an $r-$generalized ($l-$generalized) derivation associated with d on I.
In the second part of the present article, we examined what happens for a noncentral square-closed Lie ideal of a semiprime ring. Therefore, we proved the following theorem:
Theorem. Let R be a $2-$torsion free semiprime ring and U be a noncentral square-closed Lie ideal of R. There exists $F : U \to R$, an $l-$generalized ($r-$generalized) reverse derivation associated with a nonzero reverse derivation $d : U \to R$, if and only if $d(U)$, $F(U) \subseteq Z(R)$, d is a derivation on U, and F

is an $r-$generalized ($l-$generalized) derivation associated with d on U. Moreover, we will use without explicit mention the following basic identities:

- $[xy, z] = x[y, z] + [x, z]y$
- $[x, yz] = y[x, z] + [x, y]z$
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$

The material in the study is a part of the first author's Master's Thesis supervised by Prof. Dr. Neşet Aydın.

2 Generalized Reverse Derivations on One-Sided Ideals in Prime Rings

Lemma 2.1. [8, Lemma 1] *Let R be a prime ring with center $Z(R)$. If d is a reverse derivation of R, then $d(Z(R)) \subseteq Z(R)$.*

Lemma 2.2. [9, Remark 1] *Let R be a prime ring with center $Z(R)$. If $a, ab \in Z(R)$, for some $a, b \in R$, then either $a = 0$ or $b \in Z(R)$.*

Lemma 2.3. [10, Lemma 3] *If a prime ring R contains a nonzero commutative right (left) ideal, then R is commutative.*

Lemma 2.4. *Let R be a prime ring and I be a nonzero right (left) ideal of the ring R. If $[I, I] \subseteq Z(R)$, then R is commutative.*

Proof. Let I be a nonzero right ideal of R and $[I, I] \subseteq Z(R)$. Let $x, y, \in I, r \in R$. Then, $[[x, xy], r] = 0$. If the equation is rearranged, then $[x, r][x, y] = 0$. Replacing r by yr, we get $y[x, r][x, y]+[x, y]r[x, y] = 0$. Because of $[x, r][x, y] = 0$, we have

$$[x, y]r[x, y] = 0, \text{ for all } x, y \in I, r \in R$$

Because the ring R is a prime ring,

$$[x, y] = 0, \text{ for all } x, y \in I$$

Therefore, I is commutative. Thus, R is commutative according to Lemma 2.3. Besides, the same proof is done for a nonzero left ideal by means of the above-mentioned proof. □

Lemma 2.5. *Let R be a prime ring and I be a nonzero right (left) ideal of the ring R. If $x \circ y = 0$, for all $x, y \in I$, then R is commutative.*

Proof. Let I be a right ideal of R. Assume that

$$x \circ y = 0, \text{ for all } x, y \in I$$

Replacing x by xz such that $z \in I$, we get $(x \circ y)z + x[z, y] = 0$. Our hypothesis reduces it to

$$x[z, y] = 0, \text{ for all } x, y, z \in I$$

For $r \in R$, replacing x by xr, we obtain

$$xr[z, y] = 0, \text{ for all } x, y, z \in I, r \in R$$

By using primeness of R,

$$x = 0 \text{ or } [z, y] = 0, \text{ for all } x, y, z \in I$$

Since I is a nonzero right ideal,

$$[z, y] = 0, \text{ for all } y, z \in I$$

Therefore, I is commutative. Thus, the ring R is commutative from Lemma 2.3. Moreover, the same proof is done for a nonzero left ideal via the above-mentioned proof. □

Lemma 2.6. *Let R be a prime ring and I be a nonzero right (left) ideal of the ring R. If $x \circ y \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. Let I be a nonzero right ideal. From the hypothesis,

$$[x \circ y, r] = 0, \text{ for all } x, y \in I, r \in R$$

In this equation, if xy is written instead of y, then $0 = [x, r](x \circ y) + x[x \circ y, r]$. Using $[x \circ y, r] = 0$, we can write the last equation as $[x, r](x \circ y) = 0$. For $s \in R$, replacing r by rs, then

$$[x, r]s(x \circ y) = 0, \text{ for all } x, y \in I, r, s \in R$$

Since R is a prime ring,

$$[x, r] = 0 \text{ or } x \circ y = 0, \text{ for all } x, y \in I, r \in R$$

The sets $I_1 = \{x \in I : [x, r] = 0, r \in R\}$ and $I_2 = \{x \in I : x \circ y = 0, y \in I\}$ are subgroups of I. According to Brauer, either $I_1 = I$ or $I_2 = I$ becasue a group cannot be written as a union of proper subgroups.
If $I_1 = I$, then

$$[x, r] = 0, \text{ for all } x \in I, r \in R$$

That is, $I \subseteq Z(R)$. Hence, R is commutative by Lemma 2.3.
If $I_2 = I$, then

$$x \circ y = 0, \text{ for all } x, y \in I$$

Hence, the ring R is commutative according to Lemma 2.5. Furthermore, it is clear that if I is a nonzero left ideal, then the ring R is commutative. □

Theorem 2.7. *Let R be a prime ring, I be a nonzero right (left) ideal of R, $G : R \to R$ be a mapping, and $F : R \to R$ be a generalized reverse derivation associated with reverse derivation $d : R \to R$ such that $d(Z(R)) \neq (0)$. If one of the following conditions holds:*

(i) $F([x,y]) \mp [G(z),y] \in Z(R)$	(ii) $F([x,y]) \mp G(z) \circ y \in Z(R)$
(iii) $F([x,y]) \mp G(z)x \in Z(R)$	(iv) $F([x,y]) \mp xG(z) \in Z(R)$
(v) $F([x,y]) \mp xz \in Z(R)$	(vi) $F(yx) \mp [G(z),y] \in Z(R)$
(vii) $F(yx) \mp G(z) \circ y \in Z(R)$	$(viii)$ $F(xy) \mp G(z)x \in Z(R)$
(ix) $F(xy) \mp xG(z) \in Z(R)$	

then R is commutative.

Proof. Suppose that I is a right ideal.
(i) From the hypothesis,

$$F([x,y]) - [G(z),y] \in Z(R), \text{ for all } x,y,z \in I \quad (1)$$

Replacing y by $cy, c \in Z(R)$ in (1), we get

$$(F([x,y]) - [G(z),y])c + [x,y]d(c) \in Z(R), \text{ for all } x,y,z \in I, c \in Z(R) \quad (2)$$

Using (1) and (2), $[x,y]d(c) \in Z(R)$. Since $[x,y]d(c) \in Z(R)$ and $d(c) \in Z(R)$, from the Lemma 2.2, we have

$$d(c) = 0 \text{ or } [x,y] \in Z(R), \text{ for all } x,y \in I, c \in Z(R)$$

Because of $d(Z(R)) \neq (0)$,

$$[x,y] \in Z(R), \text{ for all } x,y \in I$$

Thus, the ring R is commutative from Lemma 2.4.
Moreover, suppose that

$$F([x,y]) + [G(z),y] \in Z(R), \text{ for all } x,y,z \in I$$

Replacing the generalized reverse derivation F associated with reverse derivation d by the generalized reverse derivation $-F$ associated with reverse derivation $-d$, then the following expression is obtained:

$$F([x,y]) - [G(z),y] \in Z(R), \text{ for all } x,y,z \in I$$

As a result, the ring R is commutative from the above proof. Similarly, proofs for (ii), (iii), (iv), (v), (vi), (vii), $(viii)$, and (ix) are made using Lemma 2.2 and Lemma 2.4. Moreover, similar proofs for a nonzero left ideal are also done via the above-mentioned proof of (i). □

Theorem 2.8. *Let R be a prime ring, I be a nonzero right (left) ideal of R, $G : R \to R$ be a mapping, and $F : R \to R$ be a generalized reverse derivation associated with reverse derivation $d : R \to R$ such that $d(Z(R)) \neq (0)$. If one of the following conditions holds:*

(i) $F(y \circ x) \mp [G(z), y] \in Z(R)$ (ii) $F(y \circ x) \mp G(z) \circ y \in Z(R)$
(iii) $F(x \circ y) \mp G(z)x \in Z(R)$ (iv) $F(x \circ y) \mp xG(z) \in Z(R)$
(v) $F(x \circ y) \mp xz \in Z(R)$

for all $x, y, z \in I$, then R is commutative.

Proof. Let I be a nonzero right ideal.
(i) For $x, y, z \in I$,

$$F(y \circ x) - [G(z), y] \in Z(R) \tag{3}$$

In (3), if cy is written instead of y such that $c \in Z(R)$, then

$$(F(y \circ x) - [G(z), y])c + (y \circ x)d(c) \in Z(R) \tag{4}$$

Using (3) and (4), $(y \circ x)d(c) \in Z(R)$. Since $(y \circ x)d(c) \in Z(R)$ and $d(c) \in Z(R)$, from the Lemma (2.2), we have

$$d(c) = 0 \text{ or } y \circ x \in Z(R), \text{ for all } x, y \in I, c \in Z(R)$$

Since $d(Z(R)) \neq (0)$, R is commutative according to Lemma 2.6. Additionally, it is clear that if

$$F(y \circ x) + [G(z), y] \in Z(R), \text{ for all } x, y, z \in I$$

then the ring R is commutative similar to the proof (i). Similarly, proofs of (ii), (iii), (iv), and (v) are made using Lemma 2.2 and Lemma 2.6. Moreover, similar proofs for a nonzero left ideal are also done via the above-mentioned proofs of (i). □

Example 2.9. *Let S be a ring, $R = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{bmatrix} : x, y, z \in S \right\}$ and $I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{bmatrix} : x, y \in S \right\}$. Then, it is clear that R is a ring and I is an ideal of the ring R. Let $G : R \to R$ be a mapping. Define maps $F : R \to R$ and $d : R \to R$ as follows:*

$$F\left(\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix}$$

and

$$d\left(\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}$$

Then, it is straightforward to check that F is a generalized reverse derivation associated with the nonzero reverse derivation d on the ring R and so $d(Z(R)) \neq (0)$. Moreover,

(i) $F([x,y]) \mp [G(z),y] \in Z(R)$	(ii) $F([x,y]) \mp G(z) \circ y \in Z(R)$
(iii) $F([x,y]) \mp G(z)x \in Z(R)$	(iv) $F([x,y]) \mp xG(z) \in Z(R)$
(v) $F([x,y]) \mp xz \in Z(R)$	(vi) $F(yx) \mp [G(z),y] \in Z(R)$
(vii) $F(yx) \mp G(z) \circ y \in Z(R)$	$(viii)$ $F(xy) \mp G(z)x \in Z(R)$
(ix) $F(xy) \mp xG(z) \in Z(R)$	(x) $F(y \circ x) \mp [G(z),y] \in Z(R)$
(xi) $F(y \circ x) \mp G(z) \circ y \in Z(R)$	(xii) $F(x \circ y) \mp G(z)x \in Z(R)$
$(xiii)$ $F(x \circ y) \mp xG(z) \in Z(R)$	(xiv) $F(x \circ y) \mp xz \in Z(R)$

for all $x, y, z \in I$. However, since R is a not prime ring, R is non-commutative. In other words, the condition primeness in theorems is not superfluous.

3 Generalized Reverse Derivations on Noncentral Square-Closed Lie Ideals in Semiprime Rings

Lemma 3.1. [11, Lemma 2.3] *Let R be a 2-torsion free semiprime ring and U be a noncentral square-closed Lie ideal of R. Then, there exists a nonzero two-sided ideal $M = R[U,U]R$ of R such that $2M \subseteq U$.*

Lemma 3.2. [11, Lemma 2.6] *Let R be a 2-torsion free semiprime ring and U be a nonzero Lie ideal of R. Then, $C_R(U) = Z(R)$.*

Lemma 3.3. [12, Lemma 2.1] *Let R be a semiprime ring, I be a nonzero two-sided ideal of R, and $a \in R$. If $aIa = (0)$, then $a = 0$.*

Theorem 3.4. *Let R be a $2-$torsion free semiprime ring and U be a noncentral square-closed Lie ideal of R. There exists $F : U \to R$, an $l-$generalized reverse derivation associated with a nonzero reverse derivation $d : U \to R$, if and only if $d(U), F(U) \subseteq Z(R)$, d is a derivation on U, and F is an r-generalized derivation associated with d on U.*

Proof. For each $x, y, z \in U$,

$$F(4x(yz)) = 4(F(yz)x + yzd(x)) = 4((F(z)y + zd(y))x + yzd(x))$$

and so

$$F(4x(yz)) = 4F(z)yx + 4zd(y)x + 4yzd(x), \text{ for all } x, y, z \in U \tag{5}$$

Moreover, for all $x, y, z \in U$

$$F(4(xy)z) = 4(F(z)xy + zd(xy)) = 4(F(z)xy + z(d(y)x + yd(x)))$$

Hence,

$$F(4(xy)z) = 4F(z)xy + 4zd(y)x + 4zyd(x), \text{ for all } x, y, z \in U \tag{6}$$

On substracting (5) from (6) and using the condition 2−torsion free, we have

$$F(z)[x, y] = [y, z]d(x), \text{ for all } x, y, z \in U \tag{7}$$

Replacing x by y in (7), we have

$$[x, z]d(x) = 0, \text{ for all } x, z \in U \tag{8}$$

For $m \in M$, $2m \in 2M \subseteq U$, from Lemma 3.1. Thus replacing z by $2m$ in (8) and using the condition 2−torsion free, we get

$$[x, m]d(x) = 0, \text{ for all } x \in U, m \in M \tag{9}$$

Substituting m by rm such that $r \in R$ in (9),

$$[x, r]md(x) = 0, \text{ for all } x \in U, m \in M, r \in R \tag{10}$$

Linearizing (10) and using (10),

$$[y, r]md(x) + [x, r]md(y) = 0, \text{ for all } x, y \in U, m \in M, r \in R$$

Thus,

$$[y, r]md(x) = -[x, r]md(y), \text{ for all } x, y \in U, m \in M, r \in R \tag{11}$$

For $s \in R$, replacing m by $md(y)s[y, r]m$ in (11),

$$[x, r]md(y)s[y, r]md(x) = 0, \text{ for all } x, y \in U, m \in M, r \in R \tag{12}$$

Using (11) in (12),

$$[x, r]md(y)R[x, r]md(y) = (0), \text{ for all } x, y \in U, m \in M, r, s \in R$$

Since R is a semiprime ring,

$$[x, r]md(y) = 0, \text{ for all } x, y \in U, m \in M, r \in R \tag{13}$$

Writing $d(y)$ by r in (13), we obtain

$$[x, d(y)]md(y) = 0, \text{ for all } x, y \in U, m \in M \tag{14}$$

Replacing m by mx in (14), we have

$$[x, d(y)]mxd(y) = 0, \text{ for all } x, y \in U, m \in M \tag{15}$$

Multiplying (15) by x on the right, we get

$$[x, d(y)]md(y)x = 0, \text{ for all } x, y \in U, m \in M \tag{16}$$

Subtracting (16) from (15),

$$[x, d(y)]M[x, d(y)] = (0), \text{ for all } x, y \in U$$

According to Lemma 3.3,

$$[x, d(y)] = 0, \text{ for all } x, y \in U$$

Thus, $d(U) \subseteq C_R(U) = Z(R)$ according to Lemma 3.2. Hence,

$$d(xy) = d(y)x + yd(x) = d(x)y + xd(y), \text{ for all } x, y \in U$$

which proves that d is a derivation on U.
On the other hand, substituting y by z in (7), we have

$$F(z)[x, z] = 0, \text{ for all } x, y \in U \tag{17}$$

Using same techniques as above, we get $F(U) \subseteq C_R(U) = Z(R)$. Hence,

$$F(xy) = F(y)x + yd(x) = d(x)y + xF(y), \text{ for all } x, y \in U$$

and F is an $r-$generalized derivation with respect to d. The converse is trivial. □

Theorem 3.5. *Let R be a $2-$torsion free semiprime ring and U be a non-central square-closed Lie ideal of R. There exists $F : U \to R$, an $r-$generalized reverse derivation associated with a nonzero reverse derivation $d : U \to R$, if and only if $d(U), F(U) \subseteq Z(R)$, d is a derivation on U, and F is an $l-$generalized derivation associated with d on U.*

Proof. This proof follows steps of the proof of Theorem 3.4. □

As a consequence of Theorem 3.4 and Theorem 3.5, the following corollary is valid.

Corollary 3.6. *Let R be a $2-$torsion free semiprime ring and U be a non-central square-closed Lie ideal of R. If $F : U \to R$, a generalized reverse derivation associated with a nonzero reverse derivation $d : U \to R$, then d is a central derivation on U and F is a central generalized derivation associated with d on U.*

4 Open Problem

How to generalize lemmas and theorems provided in Section 2 for a semiprime ring or a square-closed Lie ideal of a semiprime ring? How to generalize theorems provided in Section 3 for a Lie ideal or Jordan İdeal of semiprime ring?

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