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Existence of weak solutions with different initial energy levels to an equation modeling shallow-water waves

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Abstract

The present paper considers the Cauchy problem of a generalized shallow water wave equation with two different initial energy levels. We investigate the existence of global weak solutions in critical and supercritical energy levels by sign preserving property of some functionals introduced for the potential well method.

Keywords: *Critical initial energy, High energy initial data, Shallow water wave equation, Potential well.*

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1 Introduction

The models about the propagation of waves on the water surface play significant role in applied mathematics and physics. The governing equations of water waves, namely the Euler equations produce some difficulties in the theoretical and numerical studies. The difficulties encountered when working with the full system of equations that describes surface flow lead to derive simplified models. The Benney-Luke equation is an approximation of shallow water to the Euler equation that arises as an approximation to the water waves naturally [5, 14].

Isotropic Benney-Luke equation is

$$v_{tt} - \Delta v + \alpha \left(a\Delta^2 v - b\Delta v_{tt}\right) + \epsilon \left(2\nabla v \cdot \nabla v_t + v_t \Delta v\right) = 0, \tag{1}$$

where a, b, α, and ϵ are real valued positive constants, $v(x, y, t)$ is a real valued function which defines the velocity potential on the region. Here ϵ is the nonlinearity coefficient (amplitude parameter) and α is the dispersion coefficient (long-wave parameter). The parameters a, $b > 0$ are such that $a - b = \rho - \frac{1}{3}$, where ρ is the Bond number. Eq. (1) was derived for describing two-way water wave propagation with small amplitude in the existence of surface tension in [14]. Taking $a = 1/6$, $b = 1/2$ with no surface tension ($\rho = 0$) one can obtain the original Benney-Luke equation [5]. With a suitable renormalization the Benney-Luke equation reduces to the Kadomtsev-Petviashvilli (KP) and Korteweg-de Vries equations [14, 10] . Changing the variables of (1) with $\tau = \epsilon t/2$, $\xi = x - t$, $\eta = \epsilon^{1/2} y$ and $v(x, y, t) = h(\xi, \eta, \tau)$, and omitting the $O(\epsilon)$ terms one deduce that $\theta = h_\xi$ satisfies the KP equation [14]

$$\left(\theta_t - \left(\rho - \frac{1}{3}\right)\theta_{\xi\xi\xi} + 3\theta\theta_\xi\right)_\xi + \theta_{\eta\eta} = 0.$$

KP equation is also a nonlinear shallow water waves model which is a maximally balanced (maximal balance occurs when all parameters of the model have a relation between them) multidimensional approximation of the Euler water wave equations [2, 4]. Depending on surface tension, there are two types of KP equations; large surface tension yields KP-I, and small surface tension yields KP-II. KP equation is the unique model that has many closed form solutions and a Lax Pair. Analyzing and computing the solutions of both the KP and the Benney-Luke equations are easier than the governing equations of water waves (i.e. Euler equations). Moreover, the Benney-Luke equation allows for two-directional waves, while the KP and Korteweg-de Vries equations model one-directional waves (see also [3, 22]).

In this paper, we concern with the existence of global solutions of the following problem

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \mu\Delta u + \varepsilon\left(u_t \Delta u + 2\nabla u \cdot \nabla u_t\right) + \beta\nabla\left(|\nabla u|^p \nabla u\right) = 0, x \in R^n, t > 0, \tag{2}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in R^n, \tag{3}$$

where μ, $\beta > 0$, ε is constant, φ and ψ are the initial value functions and

$$\begin{aligned} &0 < p < \infty,\ n = 1, 2, \\ &0 < p \leq \frac{4}{n-2},\ n = 3, 4. \end{aligned} \tag{4}$$

The energy functional related with (2), (3) is

$$E(t) = \frac{1}{2}\left[\|u_t\|^2 + \|\nabla u_t\|^2 + \mu\|\nabla u\|^2 + \|\Delta u\|^2\right] - \frac{\beta}{p+2}\|\nabla u\|_{p+2}^{p+2}. \tag{5}$$

The potential energy, a Nehari functional and the depth of potential well for (2) are respectively given as

$$\begin{aligned} J(u) &= \frac{1}{2}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) - \frac{\beta}{p+2}\|\nabla u\|_{p+2}^{p+2}, \\ I(u) &= \mu\|\nabla u\|^2 + \|\Delta u\|^2 - \beta\|\nabla u\|_{p+2}^{p+2}, \\ d &= \inf_{u\in\mathcal{N}} J(u), \end{aligned} \tag{6}$$

where $\mathcal{N} = \{u \in H^2(R^n) \mid I(u) = 0, \nabla u \neq 0\}$.

There is a considerable number of previous works investigating Benney-Luke equation from different perspectives (especially see [3, 7, 12, 10, 17, 13, 14, 6, 22] and references therein). Nieva [7] examined various Benney-Luke equations. She investigated Cauchy problem of Benney-Luke eqautions and gave some results about local and global well-posedness of the problems in some Sobolev spaces by using fixed point argument and Strichartz inequalities. In [14], the authors proved the existence of solitary waves for isotropic Benney–Luke equations with finite-energy via the concentration-compactness method.

Depending on the initial energy ($E(0)$), there are three cases for global existence of solutions of (2), (3):

- Sub-critical case ($E(0) < d$),
- Critical case ($E(0) = d$),
- Super-critical case (sometimes called as high initial energy case($E(0) > d$)).

The problem of the existence of global solutions of problem (2), (3) in the subcritical case was previously considered in [22] by the potential well method. But global existence of solutions for problem (2), (3) in critical and super-critical cases has not been investigated yet. The aim of the present paper is to extend the global existence results of [22] to the critical and supercritical cases. To prove the global existence in the supercritical case, we use the potential well method [18, 24] with a functional that includes the initial velocity $\psi(x)$ and initial displacement $\varphi(x)$ together. Such a functional is first introduced in [8] to obtain the existence of global solutions of a Boussinesq equation

$$u_{tt} - \Delta u - \beta_1\Delta u_{tt} + \beta_2\Delta^2 u = \Delta f(u)$$

with sign preserving nonlinear term ($f(u) = \alpha |u|^p, \quad \alpha > 0$) in the case of high initial energy. The existence of global solutions was previously proved for some evolution equations with particular types of nonlinear terms, but not for all types of nonlinear terms, in the high initial energy case [15, 21]. The method of [8] filled this gap in the literature and was used by some authors [19, 16, 23, 9, 20] for proving the global existence of some evolution equations with different types of nonlinear terms and high initial energy.

Our paper consists of three more sections besides the introduction. In the second section, we introduce some functionals related to potential well for problem (2), (3), and give their properties. Moreover, we prove the global existence for the critical case and also give the sign invariance of $I(u)$ which provides the nonexistence of solutions for the critical case. In the third section, we construct some new functionals and prove the sign preserving property of this functional. Existence theorem of the global weak solution is discussed. The last section provides construction of initial data satisfying initial conditions.

The notation to be used is mostly standard. $H^s = H^s(R^n)$ denotes the Sobolev space on R^n with norm $\|f\|_{H^s} = \left\|(I-\Delta)^{\frac{s}{2}} f\right\| = \left\|(1+k^2)^{\frac{s}{2}} \widehat{f}\right\|$, where I is unitary operator, and s is a real number. $L^p(R^n)$ $(1 \le p < \infty)$ is the Lebesque space, and the norm on $L^p(R^n)$ and $L^2(R^n)$ are denoted by $\|f\|_p$ and $\|f\|$, respectively.

2 Preliminaries

In this section, we define some quantities and give their properties, and a local existence theorem . We also mention some results of [22] that are about existence of global solutions for problem (2), (3) in the case of $E(0) < d$. Furthermore, we will give the sign invariance of $I(u)$, which will be used in the proof of global existence and nonexistence (blow up) of solutions for problem (2), (3) in the case of critical initial energy, i.e., $E(0) = d$.

For the sake of completeness, let us firstly state existence theorems of [22] for finite and infinite time intervals in case of $E(0) < d$. Then we quote conservation of energy from [22].

Theorem 2.1. *[22] Let $\varphi \in H^2$, $\psi \in H^1$ and p fulfills the conditions (4). Problem (2), (3) has a unique maximal solution $u \in C([0,T_0); H^2) \cap C^1([0,T_0); H^1)$, where T_0 maximal time depends only initial data on $\|\varphi\|_{H^2} + \|\psi\|_{H^1}$. Furthermore, if*

$$\sup_{t\in[0,T_0)} \left(\|u(.,t)\|_{H^2} + \|u_t(.,t)\|_{H^1}\right) < \infty$$

then $T_0 = \infty$.

The above theorem was also written for solutions $u \in C([0,T_0);H^2) \cap C^1([0,T_0);H^1) \cap C^2([0,T_0);L^2)$ of problem (2), (3) in Theorem 2.2 of [22].

Theorem 2.2. *[22] Let $\varphi \in H^2$, $\psi \in H^1$. Suppose that $E(0) < d$ and $I(\varphi) > 0$ or $\nabla\varphi = 0$. Then problem (2), (3) admits a unique global weak solution $u \in C([0,\infty);H^2) \cap C^1([0,\infty);H^1) \cap C^2([0,\infty);L^2)$.*

Lemma 2.3. *[22] Assume that $\varphi \in H^2$, $\psi \in H^1$ and the solution $u(x,t)$ of (2), (3) is in $C([0,T_0);H^2) \cap C^1([0,T_0);H^1) \cap C^2([0,T_0);L^2)$. Then the energy of the system is conserved, i.e, for $\forall t \in [0,T_0)$*

$$E(t) = \frac{1}{2}\left[\|u_t\|^2 + \|\nabla u_t\|^2 + \mu\|\nabla u\|^2 + \|\Delta u\|^2\right] - \frac{\beta}{p+2}\|\nabla u\|_{p+2}^{p+2} = E(0).$$

By the Sobolev imbedding theorem and the assumptions on p in (4), we have

$$\|\nabla u\|_{p+2} \le C_*\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)^{1/2}. \tag{7}$$

Here, the imbedding constant is

$$C_* = \sup_{u \in H^2, \nabla u \neq 0} \frac{\|\nabla u\|_{p+2}}{\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)^{1/2}}. \tag{8}$$

The potential well depth can also be written with the aid of imbedding constant, the proof of which was given in Lemma 3.5 of [22], as

$$d = \left[\frac{p}{2(p+2)}\right]\left(\beta C_*^{p+2}\right)^{-2/p} > 0. \tag{9}$$

For global existence in the case $E(0) = d$, we need some properties of the functional $J(u)$.

Lemma 2.4. *[22] Let $u \in H^2$ and $\nabla u \neq 0$. Then $J(\lambda u) = \frac{1}{2}\lambda^2\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) - \frac{\beta}{p+2}\lambda^{p+2}\|\nabla u\|_{p+2}^{p+2}$ satisfies the followings:*

(i) $\lim\limits_{\lambda\to 0} J(\lambda u) = 0$, $\lim\limits_{\lambda\to\infty} J(\lambda u) = -\infty$.

(ii) For $0 < \lambda < \infty$, there exists a unique extreme point

$$\overline{\lambda} = \left(\mu\|\nabla u\|^2 + \|\Delta u\|^2 / \beta\|\nabla u\|_{p+2}^{p+2}\right)^{1/p}$$

of $J(\lambda u)$, and $J(\lambda u)$ takes its maximum at that point (i.e. $\sup\limits_{\lambda\ge 0} J(\lambda u) = J(\overline{\lambda}u)$).

Remark 2.1. *Note that from* $\frac{d}{d\lambda}J(\lambda u) = \lambda\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) - \beta\lambda^{p+1}\|\nabla u\|_{p+2}^{p+2}$ *one can conclude* $J(\lambda u)$ *is increasing on* $0 \le \lambda \le \overline{\lambda}$ *and decreasing on* $\overline{\lambda} < \lambda < \infty$*. This lemma also implies that* $I(\lambda u) > 0$ *for* $0 \le \lambda \le \overline{\lambda}$, $I(\lambda u) < 0$ *for* $\overline{\lambda} < \lambda < \infty$, *and* $I\left(\overline{\lambda}u\right) = 0$ *due to the fact that* $\lambda\frac{d}{d\lambda}J(\lambda u) = I(\lambda u)$.

Lemma 2.5. *[22]*

(i) If $0 < \mu\|\nabla u\|^2 + \|\Delta u\|^2 < \frac{2(p+2)}{p}d$, *then* $I(u) > 0$.

(ii) If $I(u) = 0$, *then either* $\|\nabla u\| = 0$ *or* $\mu\|\nabla u\|^2 + \|\Delta u\|^2 \ge \frac{2(p+2)}{p}d$.

(iii) If $I(u) < 0$, *then* $\mu\|\nabla u\|^2 + \|\Delta u\|^2 > \frac{2(p+2)}{p}d$.

Proof. We only give the proof of ii), the proof of i) and iii) can be found in [22]. If $\|\nabla u\| = 0$, then $I(u) = 0$. If $I(u) = 0$ and $\|\nabla u\| \neq 0$, then from

$$\mu\|\nabla u\|^2 + \|\Delta u\|^2 = \beta\|\nabla u\|_{p+2}^{p+2} \le \left(\beta C_*^{p+2}\right)\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)^{(p+2)/2}$$

we have

$$\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)^{p/2} \ge \left(\beta C_*^{p+2}\right)^{-1}.$$

With the aid of (9) the proof is completed. □

Theorem 2.6. *Let* $\varphi \in H^2$, $\psi \in H^1$. *Suppose that* $I(\varphi) \ge 0$ *and* $E(0) = d$. *Then problem (2), (3) has a global weak solution* $u \in C([0,\infty); H^2) \cap C^1([0,\infty); H^1)$.

Proof. We split the proof of the theorem into two parts:

1. $\|\nabla\varphi\| \neq 0$

i) In $I(\varphi) > 0$ case, $\frac{d}{d\lambda}J(\lambda\varphi)|_{\lambda=1} = \frac{1}{\lambda}I(\lambda\varphi)|_{\lambda=1} > 0$. Thus for some $\lambda \in (\lambda_1, \lambda_2)$, we have $\frac{d}{d\lambda}J(\lambda\varphi) > 0$ and $I(\lambda\varphi) > 0$, where $\lambda_1 < 1 < \lambda_2$. Let us take the sequence $\{\lambda_k\}$ such that $\lambda_1 < \lambda_k < 1$, $\lambda_k = 1 - \frac{1}{k}$ and $\varphi_k = \lambda_k\varphi$, $\psi_k = \lambda_k\psi$, $k = 2, 3, ...$.Consider problem (2) with the initial conditions

$$u(x,0) = \varphi_k(x), \quad u_t(x,0) = \psi_k(x). \tag{10}$$

Then

$$I(\varphi_k) = I(\lambda_k\varphi) > 0 \tag{11}$$

and

$$\begin{aligned} E_k(0) &= \frac{1}{2}\left[\|\psi_k\|^2 + \|\nabla\psi_k\|^2 + \mu\|\nabla\varphi_k\|^2 + \|\Delta\varphi_k\|^2\right] - \frac{\beta}{p+2}\|\nabla\varphi_k\|_{p+2}^{p+2} \\ &= \frac{1}{2}\left[\|\psi_k\|^2 + \|\nabla\psi_k\|^2\right] + J(\varphi_k) \\ &= \frac{1}{2}\left[\|\psi_k\|^2 + \|\nabla\psi_k\|^2\right] + J(\lambda_k\varphi) \\ &< \frac{1}{2}\left[\|\psi\|^2 + \|\nabla\psi\|^2\right] + J(\varphi) = E(0) = d \end{aligned} \tag{12}$$

ii) If $I(\varphi) = 0$, then $\varphi \in \mathcal{N}$ and definition of d implies $J(\varphi) \geq d$, which together with

$$\frac{1}{2}\left[\|\psi\|^2 + \|\nabla\psi\|^2\right] + J(\varphi) = E(0) = d$$

yields $J(\varphi) = d$. Then it follows from Lemma 2.4 that $\overline{\lambda} = \overline{\lambda}(\varphi) = 1$, $J(\lambda\varphi)$ is increasing and $I(\lambda\varphi) > 0$ for $0 < \lambda < 1$. Let us take a sequence $\{\lambda_k\}$ such that $0 < \lambda_k < 1$, $\lambda_k = 1 - \frac{1}{k}$, $k = 2, 3, \ldots$. Let $\varphi_k = \lambda_k\varphi$, $\psi_k = \lambda_k\psi$ and consider problem (2), (10). Then (11) and (12) also hold.

From Theorem 2.2, for each k, problem (2), (10) has a global weak solution $u_k(t) \in C([0,\infty); H^2)$ and $u_{kt}(t) \in C^1([0,\infty); H^1)$ for $t \in [0,\infty)$ which satisfies

$$\begin{aligned}&(u_{kt}, \upsilon) + (\nabla u_{kt}, \nabla\upsilon) + (\Delta u_k, \Delta\upsilon) + \mu(\nabla u_k, \nabla\upsilon) - \beta\left(|\nabla u_k|^p \nabla u_k, \nabla\upsilon\right)\\ &= (\psi_k, \upsilon) + (\nabla\psi_k, \nabla\upsilon), \quad \forall\upsilon \in H^2, \quad t \in [0,\infty)\end{aligned} \tag{13}$$

and

$$\frac{1}{2}\left[\|u_{kt}\|^2 + \|\nabla u_{kt}\|^2\right] + J(u_k) = E_k(0) < d, \tag{14}$$

From (14) and

$$\begin{aligned}J(u_k) &= \frac{p}{2(p+2)}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right) + \frac{1}{p+2}I(u_k)\\ &\geq \frac{p}{2(p+2)}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right)\end{aligned}$$

we have

$$\frac{1}{2}\left[\|u_{kt}\|^2 + \|\nabla u_{kt}\|^2\right] + \frac{p}{2(p+2)}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right) < d, \quad t \in [0,\infty). \tag{15}$$

The above inequality gives

$$\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2 < \frac{2(p+2)}{p}d, \tag{16}$$

$$\left[\|u_{kt}\|^2 + \|\nabla u_{kt}\|^2\right] \leq 2d, \tag{17}$$

$$\beta\|\nabla u_k\|_{p+2}^{p+2} \leq C_*^{p+2}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right)^{(p+2)/2} \leq |\beta|^{-1}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right), \tag{18}$$

$$\begin{aligned}J(u_k) &\geq \frac{1}{2}\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2 - \frac{1}{p+2}\int_{R^n}|\nabla u_k|^{p+2}\,dx\\ &\geq \frac{p}{2(p+2)}\left(\mu\|\nabla u_k\|^2 + \|\Delta u_k\|^2\right) \geq 0.\end{aligned} \tag{19}$$

From the above inequalities we get

$$\|\nabla u_k\|_{p+2}^{p+2} \leq |\beta|^{-1} \frac{2(p+2)}{p} d.$$

It follows from (16)-(18) that there exists a $\widetilde{u}$ such that $I(\widetilde{u}) > 0$ and a subsequence $\{u_\eta\}$ of $\{u_k\}$ such that as $\eta \to \infty$
$u_\eta \to \widetilde{u}$ in $L^\infty([0,\infty); H^2)$ weakly star and a.e. in $R^n \times [0,\infty)$,
$u_{\eta t} \to \widetilde{u}_t$ in $L^\infty([0,\infty); H^1)$ weakly star,
$|\nabla u_\eta|^p \nabla u_\eta \to |\nabla \widetilde{u}|^p \nabla \widetilde{u}_\eta$ in $L^\infty\left([0,\infty); L^{(p+2)/p}\right)$ weakly star.
Substituting $k = \eta \to \infty$ in (13) yields

$$\begin{aligned}
&(\widetilde{u}_{kt}, \upsilon) + (\nabla \widetilde{u}_{kt}, \nabla \upsilon) + (\Delta \widetilde{u}_k, \Delta \upsilon) + \mu(\nabla \widetilde{u}_k, \nabla \upsilon) - \beta\left(|\nabla \widetilde{u}_k|^p \nabla \widetilde{u}_k, \nabla \upsilon\right) \\
&= (\psi_k, \upsilon) + (\nabla \psi_k, \nabla \upsilon), \quad t \in [0,\infty)
\end{aligned}$$

for any $\upsilon \in H^2$ from which we infer that $\widetilde{u}$ satisfies (2). Moreover, we have

$$\widetilde{u}(x,0) = \varphi(x), \quad \widetilde{u}_t(x,0) = \psi(x).$$

So, $\widetilde{u}$ is a global solution for problem (2), (3). By the uniqueness of the solution, we get $\widetilde{u} = u$ on $R^n \times [0,T)$ and $I(u) = I(\widetilde{u}) > 0$. By Theorem 2.2 we conclude that $T = \infty$. This completes the proof.

2. $\|\nabla \varphi\| = 0$

Let $\lambda_k = 1 - \frac{1}{k}$ and $\varphi_k = \lambda_k \varphi$, $\psi_k = \lambda_k \psi$, $k = 2, 3, ...$Consider problem (2) with the initial conditions (10). Note that $\|\nabla \varphi\| = 0$ implies that $J(\varphi) = 0$ and

$$\frac{1}{2}\left[\|\psi\|^2 + \|\nabla \psi\|^2\right] = E(0) = d.$$

By using the following inequality

$$\begin{aligned}
E_k(0) &= \frac{1}{2}\left[\|\psi_k\|^2 + \|\nabla \psi_k\|^2\right] + J(\varphi) \\
&= \frac{1}{2}\left[\|\psi_k\|^2 + \|\nabla \psi_k\|^2\right] = \frac{1}{2}\lambda_k^2\left[\|\psi\|^2 + \|\nabla \psi\|^2\right] \\
&< \frac{1}{2}\left[\|\psi\|^2 + \|\nabla \psi\|^2\right] = E(0) = d
\end{aligned}$$

and Theorem 2.2, we conclude that for sufficiently large k, problem (2), (10) has a global weak solution $u_k(t) \in C([0,\infty); H^2)$ and $u_{kt}(t) \in C^1([0,\infty); H^1)$ and $I(u_k) = 0$ for $t \in [0,\infty)$. The rest of the proof can be made in the same way as the proof of case ii) of this theorem. □

Lemma 2.7. *Assume that* $\varphi \in H^2$, $\psi \in H^1$, $\varepsilon = 0$, $E(0) = d$, *and* $(\psi, \varphi) + (\nabla \psi, \nabla \varphi) \geq 0$. *If* $I(\varphi) < 0$, *then* $I(u(t)) < 0$ *for* $\forall t \in [0,\infty)$.

Proof. Let $u(t)$ be any weak solution of problem (2), (3) having the critical initial energy. If the statement fails, then there is a time $0 < t^* < T$ such that $I(u(t^*)) = 0$ and $I(u(t)) < 0$ for $0 < t < t^*$ (i.e. t^* is the first time with this property). By continuity of $\mu \|\nabla u(t)\|^2 + \|\Delta u(t)\|^2$ in time t and Lemma 2.5 (i) there exists a time $t^* \in (0, T)$ such that $\mu \|\nabla u(t^*)\|^2 + \|\Delta u(t^*)\|^2 = \frac{2(p+2)}{p} d$. By the inequality (7), Lemma 2.5 (i) and definition of d in terms of Sobolev constant, namely (9), we get

$$\begin{aligned} J(u(t^*)) &= \frac{1}{2}\left(\mu \|\nabla u(t^*)\|^2 + \|\Delta u(t^*)\|^2\right) - \frac{\beta}{p+2} \|\nabla u(t^*)\|_{p+2}^{p+2} \\ &\geq \frac{1}{2}\left(\mu \|\nabla u(t^*)\|^2 + \|\Delta u(t^*)\|^2\right) \\ &- \frac{\beta}{p+2} C_*^{p+2} \left(\mu \|\nabla u(t^*)\|^2 + \|\Delta u(t^*)\|^2\right)^{(p+2)/2} \\ &\geq d \end{aligned}$$

Combining the above inequality with

$$J(u(t^*)) \leq E(t^*) = E(0) = d$$

we get $J(u(t^*)) = d$. Hence

$$J(u(t^*)) - d = E(t^*) - J(u(t^*)) = 0$$

yields $\|u_t(t^*)\|^2 + \|\nabla u_t(t^*)\|^2 = 0$, meanwhile $\|u_t(t^*)\| = \|\nabla u_t(t^*)\| = 0$. Let us define

$$\theta(t) = \|u\|^2 + \|\nabla u\|^2.$$

Then

$$\begin{aligned} \theta'(t) &= 2(u_t, u) + 2(\nabla u_t, \nabla u), \\ \theta'(t^*) &= 0, \\ \theta'(0) &= 2(\psi, \varphi) + 2(\nabla \psi, \nabla \varphi) \geq 0, \end{aligned} \tag{20}$$

and

$$\theta''(t) = 2\|u_t\|^2 + 2(u_{tt}, u) + 2\|\nabla u_t\|^2 + 2(\nabla u_{tt}, \nabla u).$$

Multiplying (2) by u, we have

$$\langle u_{tt} - \Delta u_{tt}, u\rangle_{X^*X} = -(\Delta u, \Delta u) - \mu(\nabla u, \nabla u) + \beta(|\nabla u|^p \nabla u, \nabla u),$$

where $\langle \cdot, \cdot \rangle_{X^*X}$ denotes duality between X^* and X with $X = H^2$. Using the above equality, (6) and the assumption that $I(u) < 0$ in $(0, t^*)$, we get the following inequality

$$\theta''(t) = 2\|u_t\|^2 + 2\|\nabla u_t\|^2 - 2I(u) > 0, \quad t \in [0, t^*).$$

which yields $\theta'(t)$ is strictly increasing in accordance with $t \in [0, t^*]$. Combining this fact with (20) yields

$$\theta'(t^*) = 2(u_t(t^*), u(t^*)) + 2(\nabla u_t(t^*), \nabla u(t^*)) > 0.$$

It is not able to occur, because it is against the fact that $\theta'(t^*) = 0$. This finishes the proof. □

In the following, we give a theorem on the nonexistence (blow up) of solutions of problem (2), (3), the proof of which can be established by aid of Lemma 2.5 iii), Lemma 2.7 and a similar argument as in [21].

Theorem 2.8. *Let $\varphi \in H^2$, $\psi \in H^1$, $E(0) = d$, $I(u) < 0$, $\varepsilon = 0$ and $(\psi, \varphi) + (\nabla\psi, \nabla\varphi) \geq 0$. Then the solution of the Cauchy problem (2), (3) ceases to exist in finite time.*

3 Global Weak Solutions for the Case of High Initial Energy

This section is devoted to the existence of high initial energy solutions. For $E(0) < d$, the existence of global solutions of (2), (3) was proven by the sign invariance of the functional $I(u)$ in [22], and for the case of $E(0) = d$ global existence was given in the previous section by aid of sign invariance of the same functional. But for $E(0) > d$, $I(u(t))$ is not always positive, therefore, we cannot prove the global existence with sign invariance of this functional. Some additional conditions should be imposed on initial data and a new functional should be constructed according to these conditions. Blow up of solutions for problem (2), (3) with high initial energy is an open question because of the lack of Poincaré inequality.

In the present section, we need to construct a more general functional

$$I_\delta(u) = (1-\delta)\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) - \beta\|\nabla u\|_{p+2}^{p+2}, \quad \delta > -\frac{p}{2}.$$

Considering the above functional, the new potential well depth D_δ can be defined as follows

$$D_\delta = \inf_{u \in \mathcal{N}_\delta} J(u), \qquad \mathcal{N}_\delta(R^n) = \left\{u \in H^2 : I_\delta(u) = 0, \|\nabla u\| \neq 0\right\}.$$

Lemma 3.1. *Let $\delta > -\frac{p}{2}$. Then*

$$D_\delta = \frac{p+2\delta}{2(p+2)}\left(\frac{|1-\delta|}{\beta C_*^{p+2}}\right)^{2/p} = \frac{p+2\delta}{p} d\,|1-\delta|^{2/p}.$$

Proof. For every $u \in \mathcal{N}_\delta$, we have

$$(1-\delta)\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) = \beta\|\nabla u\|_{p+2}^{p+2} \leq \beta C_*^{p+2}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)^{(p+2)/2}$$

from which it follows that

$$\mu\|\nabla u\|^2 + \|\Delta u\|^2 \geq \left(\frac{1-\delta}{\beta C_*^{p+2}}\right)^{2/p}.$$

The above inequality is an equality iff u is a minimizer of the imbedding $H^2 \hookrightarrow L^{p+2}$ and the equality is attained only for ground state solution of (2) [11], so we have

$$\inf_{u\in\mathcal{N}_\delta}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) = \left(\frac{1-\delta}{\beta C_*^{p+2}}\right)^{2/p}.$$

By definition of D_δ, we obtain

$$\begin{aligned} D_\delta = \inf_{u\in\mathcal{N}_\delta} J(u) &= \inf_{u\in\mathcal{N}_\delta}\left(\frac{1}{p+2} I_\delta(u) + \frac{p+2\delta}{2(p+2)}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right)\right) \\ &= \frac{p+2\delta}{2(p+2)} \inf_{u\in\mathcal{N}_\delta}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) = \frac{p+2\delta}{2(p+2)}\left(\frac{1-\delta}{\beta C_*^{p+2}}\right)^{2/p} \end{aligned}$$

which completes the proof. □

Remark 3.1. *We should note that*

(i) *For* $\delta = 0$, $I_\delta(u)$ *corresponds to* $I(u)$,

(ii) *From Lemma 3.1, we conclude that* D_δ *has a local maximum at* $\delta = 0$ *and a local minimum at* $\delta = 1$,

(iii) *For* $\delta > -\frac{p}{2}$, D_δ *is strictly increasing for* $\left(-\frac{p}{2}, 0\right) \cup (1, \infty)$ *and strictly increasing for* $(0, 1)$,

(iv) $D_\delta < 0$ *for* $\delta < -\frac{p}{2}$.

In the following lemma, we state some properties of $I_\delta(u)$, which can be proved in a similar way as in [**?**].

Lemma 3.2. *For* $-\frac{p}{2} < \delta < 1$,

(i) *If* $\|u\|_{H^1} < f(\delta)$, *then* $I_\delta(u) \geq 0$ *(the equality is satisfied iff* $\|u\|_{H^1} = 0$*)*

(ii) *If* $I_\delta(u) = 0$, *then* $\|u\|_{H^1} \geq f(\delta)$ *or* $\|u\|_{H^1} = 0$

(iii) *If* $I_\delta(u) < 0$, *then* $\|u\|_{H^1} > f(\delta)$,
where $f(\delta) = \left(\frac{1-\delta}{\beta C_*^{p+2}}\right)^{2/p}$.
For $\delta > 1$,

(iv) *If* $\|u\|_{H^1} < g(\delta)$, *then* $I_\delta(u) \leq 0$ *(the equality is satisfied iff* $\|u\|_{H^1} = 0$*)*

(v) *If* $I_\delta(u) = 0$, *then* $\|u\|_{H^1} \geq g(\delta)$ *or* $\|u\|_{H^1} = 0$

(vi) *If* $I_\delta(u) > 0$, *then* $\|u\|_{H^1} > g(\delta)$,
where $g(\delta) = \left(\frac{\delta-1}{\beta C_*^{p+2}}\right)^{2/p}$.

Theorem 3.3. *Assume that* $\varphi \in H^2$, $\psi \in H^1$. *Let* δ_k *be the maximal positive root of* $E(0) = D_\delta$. *If* $E(0) > 0$, *then* $I_\delta(u(t)) \leq 0$ *for every* $t \geq 0$ *and* $\delta \geq \delta_k$.

Proof. (i) For $\delta = \delta_k$, let us proceed by contradiction and, presume that there exists some $t' > 0$ satisfying $I_{\delta_k}(u(t')) > 0$. From Lemma 3.2 iv), we have $\|u\|_{H^1} > 0$ and for some δ, $\delta > \delta_k$, we have $I_\delta(u(t')) = 0$. Then by energy identity, $D_{\delta_k} = E(0) \geq J(u(t')) \geq \inf_{u \in N_\delta} J(u) = D_\delta$. From definition of D_δ, for $\delta > \delta_k > 1$ we have $D_\delta > D_{\delta_k}$. Then we come into a contradiction that proves the theorem for $\delta = \delta_k$.

(ii) For $\delta \geq \delta_k$, from $I_{\delta_k}(u(t)) \geq I_\delta(u(t))$ we infer that the theorem is true for every $\delta \geq \delta_k$.

□

Now, we construct the second functional that will be needed for global existence of high initial energy solutions. For the proof of the global existence theorem, the sign invariance of this functional has an important role. Let us define this functional

$$\begin{aligned} K(u(t)) &= \left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) - \beta\|\nabla u\|_{p+2}^{p+2} - \|u_t\|_{H^1}^2 \\ &= I_0(u) - \|u_t\|_{H^1}^2. \end{aligned} \tag{21}$$

Theorem 3.4. *Let* $\varphi \in H^2$, $\psi \in H^1$, $\varepsilon = 0$ *and* $E(0) > 0$. *Assume that the following inequality is satisfied*

$$(\psi, \varphi) + (\nabla\psi, \nabla\varphi) + \frac{1}{2}\|\varphi\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 \leq -\frac{(p+2)\delta}{p+(p+4)\delta}E(0). \tag{22}$$

Then $K(u(t))$ *does not change its sign (i.e. if* $K(u(0)) > 0$, *then* $K(u(t)) > 0$*) under the flow of problem (2), (3) for every* $t \in [0, \infty)$.

Proof. The theorem will be proved by contradiction. Recall that for $\theta(t)$, we have

$$\theta'(t) = 2(u_t, u) + 2(\nabla u_t, \nabla u),$$

$$\theta''(t) = 2\|u_t\|^2 + 2(u_{tt}, u) + 2\|\nabla u_t\|^2 + 2(\nabla u_{tt}, \nabla u).$$

Multiplying (2) by u and using the definition of $K(u(t))$, we get

$$\theta''(t) = -2K(u(t)).$$

For a contradiction, assume the existence of a time $t' > 0$ such that $K(u(t')) = 0$, and t' be the first time this property is satisfied. Since $\theta''(t) < 0$, we infer that $\theta'(t)$ is a strictly decreasing function in $[0, t')$. It follows from (22) that $\theta'(0) < 0$ and thus $\theta'(t) < 0$ in $[0, t']$, which means that $\theta(t)$ is a strictly decreasing function on $[0, t']$. By (22), we have

$$\theta(t) < \|\varphi\|^2 + \|\nabla\varphi\|^2 < -2(\varphi, \psi) - 2(\nabla\varphi, \nabla\psi) - \frac{2(p+2)\delta}{p + (p+4)\delta} E(0),$$

for all $t \in [0, t')$. Furthermore, the continuity of θ in t yields

$$\theta(t') \leq -2(\varphi, \psi) - 2(\nabla\varphi, \nabla\psi) - \frac{2(p+2)\delta}{p + (p+4)\delta} E(0).$$

Due to $K(u(t')) = 0$ and the conservation of energy, we get

$$\begin{aligned} E(0) &= \frac{1}{2}\left(\|u_t(t')\|^2 + \|\nabla u_t(t')\|^2\right) + \frac{p}{2(p+2)}\left(\mu\|\nabla u(t')\|^2 + \|\Delta u(t')\|^2\right) \\ &\quad + \frac{1}{p+2} I(u(t')) \\ &= \left(\frac{1}{2} + \frac{1}{p+2}\right)\left(\|u_t(t')\|^2 + \|\nabla u_t(t')\|^2\right) \\ &\quad + \frac{p}{2(p+2)}\left(\mu\|\nabla u(t')\|^2 + \|\Delta u(t')\|^2\right). \end{aligned} \tag{23}$$

Then from Theorem 3.3 and $K(u(t')) = 0$, we obtain

$$\mu\|\nabla u(t')\|^2 + \|\Delta u(t')\|^2 \geq \frac{1}{\delta_k} I_0(u(t')) \geq \frac{1}{\delta}\left(\|u_t(t')\|^2 + \|\nabla u_t(t')\|^2\right).$$

Using the above inequality, Eq. (23) becomes

$$E(0) \geq \left(\frac{1}{2} + \frac{1}{p+2} + \frac{p}{2(p+2)\delta}\right)\left(\|u_t(t')\|^2 + \|\nabla u_t(t')\|^2\right)$$

from which we conclude that

$$E(0) \geq \frac{p+(p+4)\delta}{2(p+2)\delta}\Big[\|(u_t(t')+u(t'))\|^2 + \|\nabla u_t(t') + \nabla u(t')\|^2$$
$$-2(u_t(t'), u(t')) - 2(\nabla u_t(t'), \nabla u(t')) - \|u(t')\|^2 - \|\nabla u(t')\|^2\Big].$$

Here we used the following equalities

$$\|u_t(t')\|^2 = \|u_t(t') + u(t')\|^2 - \|u(t')\|^2 - 2(u_t(t'), u(t')),$$
$$\|\nabla u_t(t')\|^2 = \|\nabla u_t(t') + \nabla u(t')\|^2 - \|\nabla u(t')\|^2 - 2(\nabla u_t(t'), \nabla u(t')).$$

Since $\theta(t)$ and $\theta'(t)$ are monotone, we get

$$E(0) > \frac{p+(p+4)\delta}{(p+2)\delta}\left[-(\psi,\varphi) - (\nabla\psi,\nabla\varphi) - \frac{1}{2}\|\varphi\|^2 - \frac{1}{2}\|\nabla\varphi\|^2\right].$$

This contradicts (22). So the proof is complete. □

Now, we can state and prove global existence of problem (2), (3).

Theorem 3.5. *Let $n \leq 4$, $0 < p < \infty$ for $n = 1,2$; $0 < p \leq \frac{2}{n-2}$ for $n = 3,4$, $\varepsilon = 0$ and $\varphi \in H^2$, $\psi \in H^1$. Assume that condition (22) holds, $E(0) > 0$, $K(u(0)) > 0$. Then, for problem (2), (3) u is a global weak solution.*

Proof. The proof will be made with the aid of the local existence result given in preliminaries. Since $K(u(t))$ is invariant under the flow of (2), (3), we have $I(u(t)) > 0$ for every $t > 0$. By (7), we have

$$E(0) = \frac{1}{2}\left(\|u_t\|^2 + \|\nabla u_t\|^2\right) + \frac{p}{2(p+2)}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right) + \frac{1}{p+2}I(u)$$
$$\geq \frac{1}{2}\left(\|u_t\|^2 + \frac{1}{2}\|\nabla u_t\|^2\right) + \frac{p}{2(p+2)}\left(\mu\|\nabla u\|^2 + \|\Delta u\|^2\right).$$

This yields the boundedness of $\|u\|_{H^2}$ and $\|u_t\|_{H^1}$ for every $0 \leq t < T_0$. Combining the above estimate with the local existence theorem of Section 2 gives the existence of the global solution. □

4 Initial Data Satisfying Conditions of Theorems 2.6 and 3.5

In this section, we give sets of initial data (φ, ψ) that satisfy conditions of Theorems 2.6 and 3.5. Let us choose the initial displacement $\varphi \in H^2$ and initial velocity $\psi \in H^1$ as

$$\varphi(x) = a\omega(\gamma x), \quad \psi = ab\omega(\gamma x), \tag{24}$$

where ω is a fixed function, $\gamma > 0$, $a \neq 0$, $b > 0$ are constants. Then by straightforward calculations, we have

$$\|\varphi\|^2 = \frac{a^2}{\gamma^n}\|\omega\|^2, \quad \|\psi\|^2 = \frac{a^2b^2}{\gamma^n}\|\omega\|^2, \quad (\varphi,\psi) = \frac{a^2b}{\gamma^n}\|\omega\|^2, \tag{25}$$

$$(\nabla\varphi,\nabla\psi) = \frac{a^2b}{\gamma^{n-2}}\|\nabla\omega\|^2, \quad \|\nabla\varphi\|^2 = \frac{a^2}{\gamma^{n-2}}\|\nabla\omega\|^2, \quad \|\nabla\psi\|^2 = \frac{a^2b^2}{\gamma^{n-2}}\|\nabla\omega\|^2, \tag{26}$$

$$\|\Delta\varphi\|^2 = \frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2, \quad \|\nabla\varphi\|_{p+2}^{p+2} = \frac{a^{p+2}}{\gamma^{n-2}}\int_{R^n}|\nabla\omega|^{p+2}\,dx. \tag{27}$$

For simplicity, we take $p = 2$.

4.1 Construction of initial data satisfying conditions of Theorem 2.6

Using the initial data (24), we can write conditions of Theorem 3 as

$$\begin{aligned} E(0) &= \frac{1}{2}\left[\frac{a^2b^2}{\gamma^n}\|\omega\|^2 + \left(\frac{a^2b^2}{\gamma^{n-2}} + \mu\frac{a^2}{\gamma^{n-2}}\right)\|\nabla\omega\|^2 + \frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2\right] \\ &\quad - \frac{\beta a^4}{4\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx \\ &= d, \end{aligned} \tag{28}$$

and

$$I(\varphi) = \mu\frac{a^2}{\gamma^{n-2}}\|\nabla\omega\|^2 + \frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2 - \frac{\beta a^4}{\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx \geq 0. \tag{29}$$

From inequality (29), we have

$$\frac{\beta a^4}{\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx \leq \mu\frac{a^2}{\gamma^{n-2}}\|\nabla\omega\|^2 + \frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2.$$

Using the above inequality in (28), we obtain

$$\begin{aligned} &\frac{1}{2}\left[\frac{a^2b^2}{\gamma^n}\|\omega\|^2 + \left(\frac{a^2b^2}{\gamma^{n-2}} + \mu\frac{a^2}{\gamma^{n-2}}\right)\|\nabla\omega\|^2 + \frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2\right] \\ &- \frac{\mu a^2}{4\gamma^{n-2}}\|\nabla\omega\|^2 - \frac{a^2}{4\gamma^{n-4}}\|\Delta\omega\|^2 \leq d, \end{aligned}$$

and

$$2b^2\|\omega\|^2 + \gamma^2\left(2b^2 + \mu\right)\|\nabla\omega\|^2 + \gamma^4\|\Delta\omega\|^2 \leq \frac{4\gamma^n}{a^2}d.$$

L.h.s. of the above inequality is positive. Considering $d = \left[\frac{p}{2(p+2)}\right]\left(\beta C_*^{p+2}\right)^{-2/p} > 0$, the constants a and b satisfying the above inequality can be obtained.

4.2 Construction of initial data satisfying conditions of Theorem 3.5

Using (24), condition (22) becomes

$$\left(\frac{a^2b}{\gamma^n}+\frac{1}{2}\frac{a^2}{\gamma^n}\right)\|\omega\|^2+\left(\frac{a^2b}{\gamma^{n-2}}+\frac{1}{2}\frac{a^2}{\gamma^{n-2}}\right)\|\nabla\omega\|^2+\frac{4\delta}{2+6\delta}E(0)\le 0, \tag{30}$$

where

$$\begin{aligned}E(0)=&\frac{1}{2}\left[\frac{a^2b^2}{\gamma^n}\|\omega\|^2+\left(\frac{a^2b^2}{\gamma^{n-2}}+\mu\frac{a^2}{\gamma^{n-2}}\right)\|\nabla\omega\|^2+\frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2\right]\\&-\frac{\beta a^4}{4\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx.\end{aligned}$$

Inserting $E(0)$ in (30), we get

$$\begin{aligned}&\left(\frac{a^2b}{\gamma^n}+\frac{1}{2}\frac{a^2}{\gamma^n}+\frac{2\delta}{1+3\delta}\frac{a^2b^2}{\gamma^n}\right)\|\omega\|^2+\left[\frac{a^2b}{\gamma^{n-2}}+\left(\frac{1}{2}+\frac{2\delta\mu}{1+3\delta}\right)\frac{a^2}{\gamma^{n-2}}+\frac{2\delta}{1+3\delta}\frac{a^2b^2}{\gamma^{n-2}}\right]\|\nabla\omega\|^2\\&+\frac{2\delta}{1+3\delta}\frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2-\frac{\beta\delta a^4}{(2+6\delta)\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx\le 0.\end{aligned} \tag{31}$$

On the other hand, from the the assumption $K(u(0))>0$ one can write

$$-\beta\frac{a^4}{\gamma^{n-2}}\int_{R^n}|\nabla\omega|^4\,dx>\frac{a^2b^2}{\gamma^n}\|\omega\|^2+\frac{a^2b^2}{\gamma^{n-2}}\|\nabla\omega\|^2-\mu\frac{a^2}{\gamma^{n-2}}\|\nabla\omega\|^2-\frac{a^2}{\gamma^{n-4}}\|\Delta\omega\|^2. \tag{32}$$

Using (32) in (31), we obtain

$$\begin{aligned}&\left(b+\frac{1}{2}+\frac{5\delta}{2+6\delta}b^2\right)\|\omega\|^2+\gamma^2\left(b+\frac{1}{2}+\frac{5\delta}{2+6\delta}b^2\right)\|\nabla\omega\|^2\\&+\frac{3\delta\mu}{2+6\delta}\|\nabla\omega\|^2+\frac{3\delta\gamma^4}{2+6\delta}\|\Delta\omega\|^2\le 0.\end{aligned} \tag{33}$$

From (33), we conclude the following necessary conditions for the parameter b and δ

$$b+\frac{1}{2}+\frac{5\delta}{2+6\delta}b^2<0, \tag{34}$$

$$\frac{3\delta}{2+6\delta}<0. \tag{35}$$

To ensure the second inequality of the above inequalities, δ must be in the interval $\left(-\frac{1}{2},0\right)$ and $\delta\neq-\frac{1}{3}$. For the first inequality of the above inequalities to be satisfied, we must take $b\in(-\infty,b_1)\cup(b_2,\infty)$, where

$$b_{1,2}=\frac{-(1+3\delta)\pm\sqrt{1+\delta-6\delta^2}}{5\delta}.$$

The conjectures of Theorem 3.5 are valid if 33 holds with 34, 35 and for some $a\neq 0$.

5 Open Problem

In this paper, we obtain existence of global weak solutions for an equation modeling shallow water waves. The subject of existence is discussed for two different initial energy states: Critical and high initial energy. A non-existence theorem is also given for critical initial energy. Initial data verifying main results are provided. But the problem of blow up of solutions for high initial energy remained open. This problem may be handled since the results will have great importance for the physical phenomena modeled by the equation.

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