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Certain Notions of Continuity in Bitopological Spaces

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Abstract

In this paper, we characterize the notion of ij-continuity. We show that a function mapping one bitopological space X to another bitopological space Y is continuous if for every open subset of Y and the inverse of that subset is in open set X. Moreover, a function $\chi : X \to Y$ is π_λ-continuous if and only if for every open subset H of Y, $\chi^{-1}(H)$ is π_λ-open in X. Furthermore, we give other new notions of continuity in bitopological spaces.

Keywords: *Bitopological space, Continuous function, Continuity, $ij-$Continuity.*
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1 Introduction

Studies have been conducted by different authors on continuity and its aspects. Bitopological spaces are seen to be continuous as described by different authors. Duszynski [23] defines continuity of bitopological space as a rigorous formulation of intuitive concept of a function that varies with no abrupt breaks or jumps. Nada [53] also describes that for a bitopological space to be continuous that is from one bitopological space to another there is a function f that maps one bitopological space to another bitopological space. A function is therefore defined as a relation that is between sets that associate to every element of a first set exactly to another one element of the second set.

A function f maps one bitopological space to another bitopological space if it can map each closed sets which are members of a bitopological space then that function is also said to be continuous. This shows clearly that bitopological spaces exhibit closedness and open properties. Since bitopological spaces are equipped and endowed by two independent topologies or topological structures as a result of this two topologies exhibit many properties to the space such as closeness, openness, normality, compactness, continuity among others. Continuity of bitopological spaces exhibit some forms and aspects of continuity which may include weak continuity, strong continuity, semi continuity, global continuity, almost continuity among others. Some of the literary work that have been done on these aspects of continuity by different authors are given by the following algebraic obstructions. In this present work, we have tried to focus more on some particular aspects of continuity that are exhibited by bitopological spaces. Studies which were done by some authors such as Birhman [16] and Abu-Donia [6] showed that these aspects of continuity can as well be extended from topological spaces to bitopological spaces. Hence bitopological spaces exhibit these aspects of continuity which include: Weak continuity, strong continuity, Semi continuity, global continuity and local continuity. Kohli and Singh [44] in the study of strong continuity and almost continuity stated that several weak, strong and other invariants of continuity occur and arise in very many ways in the field of mathematics. The notion of strong continuity was introduced by Levine [52]. Later the study of strong continuity was studied by very many authors. For instance, Noiri [55] initiated the σ-continuity. Noiri Coy [55] also recognizes the notion of weakly continuous functions to have been introduced by Levine [52] in the study of decomposition of continuity in topological spaces, states that a function $f : X \to Y$ is said to be weakly continuous if and only if $f^{-1}(V)Int(f^{-1}(Cl(V)))$ for every open set V of Y. The concept of weak openness is a natural dual to that of weak continuity. Let (X, τ) be a topological space and A be a subset of X, then the closure and interior of A are denoted by $Cl(A)$ respectively Van [84]. A function $f : X \longrightarrow Y$ from a topological space X into a topological space Y is said to be strongly continuous if $f(A) \subset A$ for all $A \subset X$, Kohli [44]. A bitopological space exhibits the global continuity for instance Let X and Y be bitopological spaces and let $f : X \longrightarrow Y$ be a function therefore f is called a continuous function if and only if for every open $U \subseteq Y$ the set $f^{-1}(U)$ is open. Continuity is expressed in terms of neighborhoods. A function f is continuous at some point $x \in X$ if and only if for any neighborhood V of $f(X)$, there exists a neighborhood U of x such that $f(U) \subseteq V$. Intuitively, continuity means that no matter how "small" V becomes, there is always a U containing x that map inside V and whose image under f contains $f(X)$. This is equivalent to the condition that the pre-images of the open (closed) in X. Suppose that X is discrete topology then all functions $f : X \to T$ to any topological space T are continuous. If given that (X, τ_1, τ_2) and (Y, τ_1, τ_2) are bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ is said to be continuous if the functions $f : (X, \tau_1) \to (Y, \tau_1)$ and $(X, \tau_2) \to (Y, \tau_2)$ are both continuous. Equivalently $(X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ it is called i-continuous if the function $f : (X, \tau_1) \to (Y, \tau_1)$ is continuous then f is also said to be continuous if it is i-continuous for each $i = 1, 2$. Suppose that (X, τ_1, τ_2) is a bitopological space and A is a subset of X then A is said to be regular open if it is the interior of its closure $A = \overline{A}$. The complement of a regular open set is regular closed. The union of regular open sets is called σ-open set and similarly the complement of a σ-open set is σ-closed set.

2 Preliminaries

In this section, we outline the basic concepts which are useful in the sequel.

Definition 2.1 *[46, Definition 2.4] A mapping $f : (X, \tau_1, \tau_2) \to (X_1, \tau_3, \tau_4)$ is called P-continuous (respectively open, P-closed)if the induced mapping $f : (X, \tau_1) \to (X_1, \tau_3)$ and $f : (X, \tau_2) \to (X_1, \tau_4)$ are continuous (respectively open, closed).*

Definition 2.2 *[51, Definition 4.1] Let X and Y be N-topological space. A function $f : X \to Y$ is said to be N^*-continuous on X if the inverse image of every $N\tau$-open set in Y is a $N\tau$-open set in X.*

Definition 2.3 *[18, Definition 3.14] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(1, 2) - \gamma$ semi-continuous if the inverse image of each σ_1-open set in Y is $(1, 2) - \gamma$ -semi-open set in X.*

Remark 2.4 *[42, Remark 5] Each of ij-irresoluteness ij-almost s-continuity and i-continuity is independent of one another and ij-almost s-continuity is not a generalization of i-continuity. Each of ij-quasi irresoluteness ij-semi-continuity and ij-almost continuity is independent of one another.*

Definition 2.5 *[65, Definition 2.1] A function $f : X \to Y$ from a topological space X to A topological space Y is said to be:*

(i). strongly continuous if $f(\overline{A}) \subset A$ for all $A \subset X$.

(ii). Perfectly continuous if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.

(iii). σ-perfectly continuous if for each σ-open set V in Y, $f^{-1}(V)$ is a clopen set.

Definition 2.6 *[64, Definition 1.4.1] A function f is continuous at some point $x \in X$ if and only if for any neighborhood V of $f(X)$ hence there is a neighborhood U of x such that $f(U) \subseteq V$.*

Definition 2.7 *[44, Definition 3.2] Let (X, τ_1, τ_2) be a topological space and N be a subset of X and p a point in N. Then N is said to be a neighborhood of the point p if there exists an open set U such that $p \in U \subseteq N$.*

Example 2.8 *[49, Example 2.3] Let $X = \{1, 2, 3\}$, $P = \{\emptyset, X, \{1, 2\}, \{3\}\}$ and $Q = \{, X, \{1\}, \{2, 3\}\}$. It is easy to check that (X, P, Q) is pairwise T_0 but not weak pairwise T_1 considering the points 1, 2.*

Definition 2.9 *[70, Definition 2] The intersection (resp. union) of all (i, j) -δ-closed (resp. (i, j)-δ-open sets X containing (resp. contained in) $A \subset X$ is called the $(i, j) - \delta$-closure (resp. $(i, j) - \delta$-interior of an A.*

Definition 2.10 *[43, Definition 3] A subset of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-δ-b-open. If $A \subset jCl(iInt - \delta(A)) \cup iInt(jCl_\delta(A))$, where $i \neq ji, j = 1, 2$. The complement of an $(i, j) - \delta - b$-open set is called an $(i, j) - \delta - b$-closed set.*

Example 2.11 *[76, Example 7] Suppose that intersection (resp.union) for all $(i, j) - \delta - b$-closed $(resp.(i, j) - \delta - b$-open) sets of X containing (resp. contained in) $A \subset X$ is called the $(i, j) - \delta - b$-closure $(resp.(i, j) - \delta - b$-interior) of A and is denoted by $(i, j) - bCl_\delta(A)(resp.(i, j) - bInt_\delta(A))$ The intersection of all $(i, j) - b - \delta$-open sets of X containing A is called $(i, j) - b - \delta$-kernel of A and is denoted by $(i, j) - bKer_\delta(A)$.*

Example 2.12 *[2, Example 2] Consider $X = \{a, b, c, d\}$ with topologies $P = \{\emptyset, \{a, b\}, X\}$ and $Q = \{\emptyset, \{a\}, \{b, c, d\}, X\}$ defined on X. Observe that P-closed subsets of X are $\emptyset, \{c, d\}$ and X. Q-closed subsets of X are $\emptyset, \{b, c, d\}, \{a\}$ and X. Hence (X, P, Q) is $P1$-normal as we can check since the only pairwise closed sets of X are $\emptyset$ and X. However, (X, P, Q) is not P-normal since the P-closed set $A = \{c, d\}$ and Q-closed set $B = \{a\}$ satisfy $A \cap B = \emptyset$ but do not exists the Q-open set U and P-open set V such that $A \subseteq U, B \subseteq V$ and $U \cup V = \emptyset$.*

Definition 2.13 *[13, Definition 4.2] A space (X, τ_1, τ_2) is said to be pairwise R_0 if for every T_i-open set G, $x \in G$ implies $T_i - cl\{x\} \subset G$ i,j = 1, 2, $i \neq j$.*

Remark 2.14 *[42, Remark 1] Every ij-regular open set is ji semiregular.*

Definition 2.15 *[45, Definition 4.3] A space (X, τ_1, τ_2) is said to be pairwise R_0 if for every T_i-open set G, $x \in G$ implies that $T_j - \delta cl\{x\} \subset G$, i, j = 1, 2, $i \neq j$.*

Example 2.16 *[80, Example 3.12] Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Clearly, $(X, \tau_1) = (X, \tau_2) = (X, \tau_1, \tau_2) = \tau_2$. Thus (X, τ_1, τ_2) is a a quasi-$b - T_{\frac{1}{2}}$ space that is not quasi-$b - T_1$.*

Definition 2.17 *[46, Definition 3.6] A space (X, τ_1, τ_2) is called:*

(i). Pairwise pre-T_0 (resp. pairwise pre-T_1) if for any pair of distinct points x and y in X, there exists a T_i-preopen set which contains one of them but not the other $i = 1or2$ (resp. there exists T_i-preopen set U and T_j-preopen set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset i, j = 1, 2, i \neq j$.

(ii). A space (X, τ_1, τ_2) is said to be pairwise pre-T_2 if for any pair of distinct points x and y in X, there exists τ_i-preopen set U and τ_j-preopen set V such that $x \in U$, $y \in V$ and $U \cap V \neq \emptyset$ i, j= 1, 2, where $i \neq j$.

Example 2.18 *[4, Example 2] Let $X = \{a, b, c, d, e\}, \tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, X, \{b, e\}\}$, $Y = \{a, b, c, d\}, \sigma_1 = \{\emptyset, Y, \{c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b, d\}\}$. Suppose $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \tau_2)$ is defined by $f(a) = f(c) = f(d), f(b) = f(c), f(b) = f(e) = b$.Then the map f is 12-preweak continuous but not 12-preweak semicontinuous since $f^{-1}(\{c\}) = \{a, c, d\}$ which is not 1-open set in the subspace $2 - cl(1 - int(2 - cl(f^{-1}(2 - cl(\{c\}))$.*

3 Main results

In this section, we give an in depth characterization of various notions of continuity in bitopological spaces. We begin with the following proposition.

Proposition 3.1 *Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ be an open function. A subset W of X is π_λ-open if and only if it is semi-closed and an intersection of π_λ-open sets in X. Moreover, χ is π_λ-continuous.*

Proof. To prove the first part, let W of X be π_λ-open. We prove that it is semi-closed and also an intersection of open sets in X. Let U be an open set in X and V be open set in Y containing $\chi(x)$ for some $x \in X$. By hypothesis, there exists a π_λ-open set U of X which is containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set then $x \in U$. Since x belongs to U of X, then there exists a subset W of X that is semi-closed. By criterion for continuity, λ is closed then closure interior of W is a subset of W, that is $int(clW) \subseteq W$. Since W is a semi-closed subset of X it follows that it is π_λ-open set and hence

$x \in W \subseteq U$. Therefore, we have $\chi(W) \subseteq V$. Now, U and V are open sets in X and Y respectively which implies that $W = U \cap V$ is semi-closed set and π_λ-open in X. Therefore, V is an open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Hence χ is π_λ-continuous at every point $x \in X$. This completes the proof.

A function χ is also π_λ-continuous if the inverse of a subset in a nonempty set is π_λ-open. This is illustrated in the next result.

Proposition 3.2 *A function $\chi : X \to Y$ is π_λ-continuous if and only if for every open subset H of Y and $\chi^{-1}(H)$ is π_λ-open in X.*

Proof. Let χ be a π_λ-continuous function and B be any set in Y. To show that $\chi^{-1}(B)$ is a π_λ-open set in X, it is enough that $\chi^{-1}(B) = \emptyset$ in X hence this follows $\chi^{-1}(B)$ is a π_λ-open set in X. However, if $\chi^{-1}(B) \neq \emptyset$ then for every $x \in \chi^{-1}(B)$, we have $\chi(x) \in B$. Since χ is π_λ-continuous then there exists a π_λ-open set H_x in X such that $x \in H_x$ and $\chi(H_x) \subseteq B$. By criteria for inverse continuity, it implies that $x \in H_x \subseteq \chi^{-1}(B)$. So this implies that $\chi^{-1}(B)$ is π_λ-open in X. Conversely, if $x \in X$ and we let V to be an open set in Y containing $\chi(x)$, then $x \in \chi^{-1}(V)$ by criterion of continuity it implies that $\chi^{-1}(V)$ is π_λ-open in X containing x. Therefore, $\chi(\chi^{-1}(V)) \subseteq V$. Hence χ is π_λ-continuous.

Next, we show that every π_λ-continuous function is semi-continuous. However, a semi-continuous function is not necessarily π_λ-continuous.

Lemma 3.3 *Every π_λ-continuous function $\chi : X \to Y$ is semi-continuous but the converse is not true in general.*

Proof. By hypothesis, there exists a π_λ-open set H of X having x as one of its element and so it implies that $\chi(H) \subseteq V$. By Proposition 3.2, we see that H is a π_λ-open set and $x \in H$. It therefore implies that there exists a π_λ-closed set F of X such that $x \in F \subseteq V$. By criterion for continuity, it follows that χ is a π_λ-continuous function and so it follows that χ is semi-continuous. However, the converse is not true in general. This can be illustrate as follows: If we have two topological spaces as (X, τ) and (X, τ_1), then a function $\chi : (X, \tau) \to (X, \tau_1)$ is continuous. Then given the cardinalities as $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}\}$ and τ_1 are $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ then $\pi B(X, \tau) = \{\emptyset, X, \{b\}, \{b, c\}, \{a, b\}\}$.
Therefore, by criterion for continuity, $\pi_\lambda B(X, \tau) = \{\emptyset, X, \{a, b\}, \{a, c\}\}$.
Similarly, $\pi B(X, \tau_1) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
So $\pi_\lambda B(X, \tau) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. Let $\chi : (X, \tau) \to (X, \tau_1)$ be defined by $\chi(a) = \chi(b) = b$. If $\chi(a) = \chi(b) = b$, then it applies that $\chi(c) = c$. Therefore, χ is semi-continuous but not π_λ-continuous.

Theorem 3.1 *Every θ_η-continuous function $\chi : X \to Y$ is π_λ-continuous but the converse is not true in general.*

Proof. Let θ_η-continuous function $\chi : X \to Y$ be π_λ-continuous. Let $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ be π_λ-continuous at a point $x \in X$, if for each U of Y containing $\chi(x)$ there exists π_λ-open G in X that is containing x such that $\chi(G) \subseteq V$. By hypothesis, if G is a π_λ-open set then it implies that there exists a π_λ-closed set F of X such that $x \in F \subseteq V$. By Lemma 3.3, if there is an open set V in X which contains x such that $\chi(G) = U$, by criterion for continuity a function χ is π_λ-continuous at every point x of X then it is π_λ-continuous and θ_η-continuous. However, the converse need not to be true in general. For instance, if the cardinalities of $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then we have $\pi B(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. It suffices that $\pi_\lambda B(X) = \{\emptyset, X, \{a, c\}\}$ and $\theta_\eta B(X) = \{\emptyset, X\}$ if a function $\chi : X \to Y$ is defined by $\chi(a) = \chi(c) = a$ and also $\chi(b) = b$ hence χ is a π_λ-continuous function since $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda B(X)$ but $\{a, c\}$ does not exists in $\theta\pi B(X)$.

The following consequence follows immediately.

Corollary 3.4 *Every π_d-continuous and $\delta\pi_d$-continuous functions $\chi : X \to Y$ is π_λ-continuous but the converse are not true in general.*

Proof. Since π_d-continuous and $\delta\pi_d$-continuous functions are π_λ-continuous then let U be any open set in X containing $\chi(x)$ and V be any open set in Y containing $\chi(y)$. Suppose that G is a π_λ-open set in X then this implies that $\chi(G) = U$. By Theorem 3.1, since $\chi(G) = U$ then a function χ is π_λ-continuous at every point x of X, then a function $\chi : X \to Y$ is also π_d-continuous and $\delta\pi_d$-continuous function. However, the converse is not true in general. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then it follows that $\pi B(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $GD(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$. Hence $\pi_\lambda(X) = \{\emptyset, \{a, c\}, \{a, b\}, X\}$ and also $\pi_d B(X) = \delta\pi_d(X) = \{\emptyset, X\}$. Therefore, if a function $\chi : X \to Y$ can be defined by $\chi(a) = \chi(c) = a$ and $\chi(b) = b$, then χ is π_λ-continuous, as $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda(X)$ but neither π_d-continuous nor $\delta\pi_d$-continuous $\{a, c\}$ does not exists in $\pi_d B(X) = \delta\pi_d B(X)$. Therefore, every π_d-continuous and $\delta\pi_d$-continuous functions are π_λ-continuous but the converse does not imply.

We use proposition 3.5 to illustrate that functions which are perfectly continuous are also π_λ-continuous.

Proposition 3.5 *Let a function $\chi : X \to Y$ be perfectly continuous. Then χ is π_λ-continuous.*

Proof. Let A be any open set in Y. Then $\chi^{-1}(A)$ is clopen in X. Hence it implies that $\chi^{-1}(A) \in \pi_\lambda B(X)$, then by Proposition 3.2, a function χ is π_λ-continuous since A is an open set in Y. Then we can show that $\chi^{-1}(A)$ is a π_λ-open set in X, suppose that $\chi^{-1}(A) \neq \emptyset$ then it therefore implies that $\chi^{-1}(A)$ is a π_λ-open set in X, and if $\chi^{-1}(A) = \emptyset$, then for each $x \in \chi^{-1}(A)$, we have $\chi(x) \in A$. Since χ is π_λ-continuous then it implies that there exists a π_λ-open set B_x in X such that $x \in B_x$ and $\chi(B_x) \subseteq A$. This implies that $x \in B_x \subseteq \chi^{-1}(A)$. This therefore shows that $\chi^{-1}(A)$ is π_λ-open in X. Conversely, let $x \in X$ and A be an open set in Y containing $\chi(x)$. Then it follows that $x \in \chi^{-1}(A)$. By hypothesis, $\chi^{-1}(A)$ is π_λ-open in X containing x, hence it suffices that $\chi(\chi^{-1}(A)) \subseteq A$. Therefore, χ is π_λ-continuous and perfectly continuous.

In our subsequent result we illustrate how globally indiscrete mappings exhibit characteristics of semi-continuous functions.

Lemma 3.6 *Let $\chi : X \to Y$ be globally indiscrete. Then a function χ is π_λ-continuous if and only if it is semi continuous.*

Proof. Let χ be semi-continuous and X be globally indiscrete. Let U be any open subset in Y. Then it follows that $\chi^{-1}(U)$ is also semi-open in X. Since X is globally indiscrete, then U is any open set in X and V be an open set in Y containing $\chi(x)$ for some $x \in X$. By hypothesis, there exists a π_λ-open set U of X which is containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set, then we can say that $x \in U$. Since x belongs to U of X then there exists a subset W of X that is semi-closed. By criterion for continuity, the interior closure of W is a subset of W, that is $int(clW) \subseteq W$. Since W is a semi-closed subset of X it implies that it is π_λ-open set and hence $x \in W \subseteq U$. Therefore, we have $\chi(W) \subseteq V$. Now, since U and V are open sets in X and Y respectively then it follows that $W = U \cap V$ is a semi-closed set and π_λ-open in X. Therefore, V is π_λ-open set in Y containing y and U is a π_λ-open set in X containing x such that $\chi(U) \subseteq V$. Therefore, χ is π_λ-continuous at every point $x \in X$. Conversely, let χ be π_λ-continuous. By hypothesis, there exists a π_λ-open set U of X containing x such that $\chi(U) \subseteq V$. Since U is a π_λ-open set and $x \in U$, then there exists a g-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $\chi(F) \subseteq V$. Since χ is π_λ-continuous, then by Lemma 3.3, χ is semi-continuous. Therefore, since χ is π_λ-continuous then it is also semi-continuous.

We use theorem 3.2 to show that a function χ is π_λ-continuous if it is semi-continuous and there exists a closed set that is a subset of an open set H. This is stated in the next result.

Theorem 3.2 *Let $\chi : X \to Y$ be π_λ-open. Then χ is π_λ-continuous if and only if it is semi continuous and for every $p \in X$ and each open set H of Y containing $\chi(x)$ there exists λ-closed set P of X containing x such that $\chi(P) \subseteq H$.*

Proof. Let $\chi : X \to Y$ be π_λ-open. Since χ is π_λ-continuous then it implies that it is semi-continuous. Let $p \in X$ and H be any open set in Y containing $\chi(x)$. By hypothesis, there exists a π_λ-open set G of X containing x such that $\chi(G) \subseteq H$. Since G is a π_λ-open set and $x \in G$, then it shows that there exists λ-closed set P of X such that $p \in P \subseteq G$. Then it follows that $\chi(P) \subseteq H$. Since χ is π_λ-continuous then by Lemma 3.3, χ is semi-continuous. By hypothesis, χ is π_λ-continuous so it implies that it is also semi-continuous. Since $A \subseteq X$ and $p \in G$ of X, then a subset A of X is semi-closed. By criterion for continuity, the interior closure of A is a subset of A that is, $int(clA) \subseteq A$. Then since A is a semi-closed subset of X it is π_λ-open set and so $p \in A \subseteq G$, then $\chi(A) \subseteq H$. Now, G and H are open sets in X and Y respectively so it implies that the intersection of G and H is in A that is $G \cap H \in A$ which is π_λ-open. Therefore, H is open set in Y containing y and G is a π_λ-open set in X containing p such that $\chi(G) \subseteq H$. Hence χ is π_λ-continuous at every point $p \in X$ and so χ is π_λ-continuous.

For a function that maps a Housdorff space to a bitopological space is is both semi-continuous and π_λ-continuous. We state the result as follows.

Theorem 3.3 *Let X be a Hausdorff space and Y be any bitopological space. For $\chi : X \to Y$, then the following are equivalent:*

(i). χ is semi-continuous.

(ii). χ is π_λ-continuous.

Proof. Let $\chi : X \to Y$ and also $\omega : X \to Y$ then $\chi, \omega : X \to Y$ are π_λ-continuous functions. Since X is a Hausdorff space, therefore there is set $M = \{x \in X : \chi(x) = \omega(x)\}$ which is π_λ-closed in X. So x does not exists in H, this follows that $\chi(x) \neq \omega(x)$. Since X is a Hausdorff space then there exist open sets V_1 and V_2 of X such that $\chi(x) \subseteq V_1$ and $\omega(x) \subseteq V_2$. Then it implies that $V_1 \cap V_2 \neq \emptyset$. Since χ and ω are π_λ-continuous functions then there exist π_λ-open sets U_1 and U_2 in Y containing y such that $\chi(U_1) \subseteq V_1$ and $\omega(U_2) \subseteq V_2$. By criterion for continuity, the intersection of U_1 and U_2 is a proper subset of W that is, $W = (U_1) \cap (U_2)$. Then it is π_λ-open in Y since $M \in Y$ then $U \cap M = \emptyset$. Hence it follows that $x \in \pi_\lambda cl(H)$, this implies that H is π_λ-closed in X. Since V is any open set in Y then $\chi^{-1}(V)$ is clopen in X, and so $\chi^{-1}(V) \in \pi_\lambda(X)$. Therefore, a function χ is π_λ-continuous. Since V and U are open sets in Y and X respectively then we have $x \in \chi^{-1}(V)$ with x

being closed hence $x \in \{x\} \subseteq \chi^{-1}(V)$. Therefore, $\chi^{-1}(U)$ is semi-open in X. By criterion for inverse continuity, $\chi^{-1}(V)$ is a π_λ-open set in X. Hence χ is a semi-continuous function.

We use theorem 3.4 to illustrate that a function χ is ij-continuous if and only if there exists an open subset. This is indicated in the result that follows.

Theorem 3.4 *Let $\chi : X \to Y$ be π_λ-continuous. Then χ is ij-continuous if X_0 is an open subset of X. Moreover, χ is an $ij - \pi_\lambda$-continuous if $\chi\mid_{X_0}: X_0 \to Y$ is π_λ-continuous.*

Proof. Let $\chi : X \to Y$ be π_λ-continuous then it implies that it is continuous at every point $x \in X$. Suppose that we have open set V of Y which contains $\chi(x)$ then by hypothesis we say that there exists a π_λ-open set U. If $X_0 \subseteq U$ then X_0 also exists in X and contains x. Then it follows that $\chi(X_0) \subseteq V$. Hence χ is π_λ-continuous at every point x in X. Since V is an open set in Y then $\chi^{-1}(V)$ is π_λ-open in X, since $\chi^{-1}(V) = \emptyset$. So $\chi^{-1}(V)$ is also π_λ-open in X. Suppose that $\chi^{-1}(V) = \emptyset$ then it implies that for every $x \in \chi^{-1}(V)$ we have $\chi(x) \in V$. Then by criterion for continuity, there exists a π_λ-open set X_0 in X such that $x \in X_0$ and $\chi(X_0) \subseteq V$. Therefore, $x \in X_0 \subseteq \chi^{-1}(V)$. Then it shows that $V \in i - X_0(Y)$ and $\chi^{-1}(V)$ are members of $ij\pi X_0 X$. So χ is ij-continuous since X_0 is an open subset of X. Moreover, since sets X and Y have open sets U and V respectively and that V of Y contains $\chi(x)$. This follows that a π_λ-open set X_0 in X also contains x. Then since X_0 is a subset of X and $\chi(x)$ is a proper subset V of Y, then it implies that U is also a subset of V. Therefore, since a function $\chi : X \to Y$ it shows that X_0 then $\chi\mid_{X_0}: X_0 \to Y$. This follows that for all V in Y there exists $j - X_0(Y)$ such that $\chi^{-1}(V)$ exists in $ij - \pi U X_0(X)$. Since $j - X_0$ is open in Y and $\chi^{-1}(V)$ is an element of $ij - \pi U X_0(X)$ then $x \in \chi^{-1}(V)$ where $\chi(x) \in V$. By hypothesis, χ is π_λ-continuous hence it is also $ij - \pi_\lambda$-continuous since $\chi\mid_{X_0}: X_0 \to Y$ is π_λ-continuous.

Theorem 3.5 *$\chi : X \to Y$ is $ij - \pi_\lambda$-continuous if for each open set X_0 of X we have $\eta \in X$ in X_0. Such that $\chi\mid_{X_0}: X_0 \to Y$ is π_d-continuous.*

Proof. Let V be any open set in Y and X_0 be any open set in X, then there exists η which is an element of X. Hence this implies that X is a subset of X_0. By Theorem 3.4, $\chi : X \to Y$ is π_λ-continuous hence there exists $\eta \in X$ and open set V of Y such that it contains $\chi(x)$. Therefore, it implies that there is a π_λ-open set X_0 in X containing η such that $\chi(X_0) \subseteq V$. Hence χ is π_λ-continuous if and only if it is continuous at every point η of X. Since there is an open set V of Y such that $V \in j - X_0(Y)$ and $\chi^{-1}(V) \in ij - \pi\eta X_0(X)$, then it suffices that $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous. Since V is an open set in Y then $\chi^{-1}(V)$ is an element of X, then by criteria for inverse continuity it implies that $\chi^{-1}(V) = \emptyset$. Suppose that $\chi^{-1}(V) \neq \emptyset$ and every $\eta \in \chi^{-1}(V)$ then it shows that $\chi(x) \in V$. Then it implies that X_η exists in X where $\eta \in X$, hence $\eta \in X_\eta \subseteq \chi^{-1}(V)$. Therefore, $\chi^{-1}(V)$ is π_λ-open in X and so it implies that it is π_λ-continuous since $\chi\mid_{X_0}: X_0 \to Y$ with X_0 having induced properties from χ. Since $\chi\mid_{X_0}: X_0 \to Y$ then it is also π_d-continuous.

This leads to the following consequence.

Corollary 3.7 *Every $ij - \pi_\lambda$-continuous function is ij-continuous but the converse is not true in general.*

Proof. By hypothesis, χ is π_λ-continuous if and only if it is continuous at every point of $x \in X$. Then suppose that there is any open set V of Y which contains $\chi(x)$ then χ is π_λ-continuous. This implies that $\chi^{-1}(V)$ is π_λ-open in X and so if $\chi^{-1}(V)$ is an empty set then $\chi^{-1}(V)$ is also a π_λ-open set in X. Hence suppose that $\chi^{-1}(V) = \emptyset$ then it implies that each $x \in \chi^{-1}(V)$, therefore $\chi(x) \in V$. By Theorem 3.4, χ is π_λ-continuous and also there exists a π_λ-open set U in X such that $x \in U$ and hence $\chi(U) \subseteq V$, by extensions $x \in U \subseteq \chi^{-1}(V)$. This implies that we have V in $j - V(Y)$ and $\chi^{-1}(V) \in ij - U\pi V(X)$ then χ is said to be $ij - \pi_\lambda$-continuous. For ij-continuous we have $V \in i - V(Y)$ and $\chi^{-1}(V) \in \pi V(X)$. By criterion for continuity, it implies that every $ij - \pi_\lambda$-continuous function is also ij-continuous. However, not every ij-continuous is $ij - \pi_\lambda$-continuous. Let V be an open set in Y and U open set in X. Then $\chi^{-1}(V)$ is π_λ-open in X. For open set U of X we have $\chi(U) \subseteq V$, therefore it follows that $x \in U \subseteq \chi^{-1}(V)$. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then $\pi U(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Similarly, $VU(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$ and so $\pi_\lambda(X) = \{\emptyset, \{a, c\}, \{a, b\}, X\}$. Therefore, $\pi_d U(X) = \delta\pi_d(X) = \{\emptyset, X\}$. Suppose that a function $\chi : X \to X$ is defined by $\chi(a) = \chi(c) = a$ then it shows that χ is π_λ-continuous. Since $\{a\} \in \tau$ and $\{a, c\} \in \pi_\lambda(X)$, it implies that neither π_d-continuous nor $\delta\pi_d$-continuous is π_λ-continuous. So $\{a, c\}$ does not exists in $\pi_d U(X) = \delta\pi_d U(X)$ therefore, since χ is π_d-continuous then it implies that it is also π_λ-continuous. For ij-continuity with open set V of Y we have $V \in i - V(Y)$ this implies that $\chi^{-1}(V) \in ij - \pi V(X)$. Hence for $ij - \pi_\lambda$-continuous there exists an open set $V \in jV(Y)$ hence $\chi^{-1}(V) \in ij - U\pi V(X)$. Therefore, every $ij - \pi_\lambda$-continuous function is ij-continuous but its converse is not true in general.

Suppose independent functions mapping one bitopological space to another are $ij - \pi_\lambda$-continuous then their composition is π_λ-continuous as it is shown in the result that follows.

Proposition 3.8 *Let $\chi_1 : X \to Y$ be $ij - \pi_\lambda$-continuous and $\chi_2 : Y \to Z$ be $ij - \pi_d$-continuous. Then $\chi_2 \circ \chi_1$ is π_λ-continuous.*

Proof. Let $\chi_1 : X \to Y$ and $\chi_2 : Y \to Z$. Let C be any subset of X, then C is π_λ-open if and only if it is a semi-closed set. Therefore, if U is any open set in X containing $\chi(y)$ and V any open set in Y containing $\chi(x)$ then by hypothesis there is π_λ-open set U of X which contains x such that $\chi_1(U) \subseteq V$. Since U is π_λ-open then it implies that $x \in U$. There is a subset C that is semi-closed and hence the interior closure of C is a subset of C hence $int(clC) \subseteq C$. Then it implies that C is a subset of semi-closed set of X and π_λ-open. Hence $x \in C \subseteq U$ and $\chi(C) \subseteq V$. Therefore, it shows that for all V that exist in $j - C(Y)$ there is $\chi^{-1}(V) \in ij\pi DC(X)$. Therefore, $\chi_1 : X \to Y$ is said to be $ij - \pi_\lambda$-continuous. Similarly, $\chi_2 : Y \to Z$ is also π_λ-continuous. Let E be an open set in Z and D be open in Y and there exists $\eta \in Y$ then it implies that X is a subset of D. Since $\chi_2 : Y \to Z$ is π_λ-continuous and $\eta \in Y$ for each open set V of Y such that it contains $\chi_2(x)$ then it implies that there is a π_λ-open set D in X containing η such that $\chi_2(D) \subseteq V$ and hence χ_2 is π_λ-continuous if and only if χ_2 is continuous at each point η of X. Since there is an open set V of Y where $V \in j - D(Y)$ and also $\chi_2^{-1}(V) \in ij - \pi\eta D(X)$ then $\chi : X \to Y$ is $ij - \pi_\lambda$-continuous. This shows that $\chi_2 : Y \to Z$ is also $ij - \pi_d$-continuous. Since $\chi_1 : X \to Y$ is $ij - \pi_\lambda$-continuous then $\chi_2 : Y \to Z$ is $ij - \pi_d$-continuous. Hence a function $\chi : X \to Z$ is also $ij - \pi_d$-continuous. By hypothesis, $\chi_2 \circ \chi_1$ is also π_λ-continuous.

The next result follows closely.

Proposition 3.9 *Let $\chi_1 : X \to Y$ be π_λ-continuous and $\chi_2 : Y \to Z$ be π_d-continuous. Then $\chi_2 \circ \chi_1$ is $ij - \pi_d$ continuous.*

Proof. Let $\chi_1 : X \to Y$ be π_λ-continuous and $\chi_2 : Y \to Z$ be π_d-continuous. If χ_1 is a π_λ-continuous function then it implies that there exists any open set B in Y. Then it therefore implies that $\chi_1^{-1}(B)$ is a π_λ-open set in X. Suppose $\chi_1^{-1}(B) = \emptyset$ then it implies that $\chi_1^{-1}(B)$ is π_λ-open in X. However, if $\chi_1^{-1}(B) \neq \emptyset$ then $x \in \chi_1^{-1}(B)$ we have $\chi_1(x) \in B$. Therefore, since χ is π_λ-continuous, it implies that there exists a π_λ-open set H_x in X such that $x \in H_x$ and $\chi_1(H_x) \subseteq B$ hence $x \in H_x \subseteq \chi_1^{-1}(B)$. Therefore, $\chi_1^{-1}(B)$ is π_λ-open in X. Conversely, suppose that $x \in X$ and V be any open set in Y containing $\chi(x)$ then $x \in \chi^{-1}(V)$, by criterion of continuity $\chi_1^{-1}(V)$ is π_λ-open in X containing x. Therefore, $\chi_1(\chi^{-1}(V)) \subseteq V$ so this implies that χ_1 is a π_λ-continuous function. So $\chi_2 : Y \to Z$ where all $x \in \chi^{-1}(B)$ is closed and therefore $x \in \{x\} \subseteq \chi^{-1}(B)$. Then it implies that $\chi^{1}(B) \in \pi_d C(X)$ and hence χ is a π_d-continuous function. Suppose that $\chi_1 : X \to Y$ is π_λ-continuous then $\chi_2 : Y \to Z$ is also π_d-continuous. Therefore, $\chi_2 \circ \chi_1$ is $ij - \pi_d$ continuous.

In our next result we show that ij-continuous function $\chi : X \to Y$ is continuous in bitopological spaces.

Theorem 3.6 *Every ij-continuous function $\chi : X \to Y$ is continuous.*

Proof. By hypothesis, every ij-continuous function is π_λ-continuous. By Corollary 3.7, χ is π_λ-continuous if it is continuous at every point of $x \in X$. Then there is an open set B of Y containing $\chi(x)$. This implies that $\chi^{-1}(B)$ is π_λ-open in X. Hence suppose that $\chi^{-1}(B) = \emptyset$ then it is π_λ-open in X while if $\chi^{-1}(B) \neq \emptyset$ then $x \in \chi^{-1}(V)$ and so $\chi(x) \in B$. By Theorem 3.4, χ is π_λ-continuous and we have a π_λ-open set A in X such that $x \in A$ hence $\chi(A) \subseteq B$, then $x \in A \subseteq \chi^{-1}(B)$. This implies that $B \in j - B(Y)$ and hence $\chi^{-1}(B) \in ij - A\pi B(X)$. Then χ is said to be $ij - \pi_\lambda$-continuous. Since $B \in i - B(Y)$ and $\chi^{-1}(B) \in \pi B(X)$ then it is ij-continuous. This implies that every $ij - \pi_\lambda$-continuous function is also ij-continuous. Similarly, let $X = \{a, b, c, d\}$, $Y = \{u, v, w\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, d\}\}$, and $\tau_2 = \{X, \emptyset, \{a, b\}, \{c, d\}\}$. We have $\delta_1 = \{Y, \emptyset, \{u\}, \{v\}, \{u, v\}\}$ and $\delta_2 = \{Y, \emptyset, \{u\}, \{u, v\}\}$. Therefore, by Proposition 3.9, $\chi : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is defined by $\chi(a) = \chi(d) = u$, $\chi(c) = v$. Therefore, χ is ij-continuous by hypothesis $\chi : X \to Y$ is also continuous.

Theorem 3.7 *Let $\chi : X \to Y$ be $ij - \pi_d$-continuous. Then χ is $ij - \Omega$-continuous.*

Proof. By hypothesis, if $\chi : X \to Y$ then it is $ij - \pi_d$-continuous if and only if χ is π_λ-continuous. By Proposition 3.1, χ is π_λ-continuous if there is any open set V in Y that is containing $\chi(x)$ and hence χ is continuous at every point $x \in X$. Then this follows that there exists a π_λ-open set U of X containing x such that $\chi(U) \subseteq V$. By Theorem 3.2, every π_λ-continuous function is also semi-continuous. Suppose that there exists a π_λ-open set P of X containing x then it follows that $\chi(P) \subseteq H$. Similarly, there exists a π_λ-closed set U of X such that $x \in U \subseteq A$ by criterion of continuity, χ is therefore semi-continuous. Let (X, τ_1, τ_2) and (Y, τ_1, τ_2) be bitopological spaces. Then $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2))$ then let $X = \{d, e, f\}$, $\tau = \{\emptyset, X, \{e\}\}$ and $\tau = \{\emptyset, X, \{d\}, \{e\}, \{d, e\}\}$. This implies that $\pi V(X, \tau) = \{\emptyset, X, \{e\}, \{e, f\}, \{d, e\}\}$ by criterion for continuity $\pi_\lambda V(X, \tau) = \{\emptyset, X, \{d, e\}, \{d, e\}\}$.
Therefore, it follows that $\pi V(X, \tau_1) = \{\emptyset, X, \{d\}, \{e\}, \{d, e\}, \{d, f\}, \{d, e\}\}$.
Hence $\pi_\lambda V(X, \tau) = \{\emptyset, X, \{d, f\}, \{e, f\}\}$.
A function $\chi : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ can be defined by $\chi(a) = \chi(e) = e$. If $\chi(d) = \chi(e) = e$ then it implies that $\chi(f) = f$ hence χ is semi-continuous and $ij - \pi_d$-continuous but not π_λ-continuous. Therefore, $\chi^{-1}(V) \in ij - \Omega V(X)$ then χ is $ij - \Omega$-continuous.

4 Conclusion

In this paper, we have shown that a function mapping one space X to another bitopological space Y is continuous if for every open subset of Y and the inverse of that subset is in open set X. Moreover, a function $\chi : X \to Y$ is π_λ-continuous if and only if for every open subset H of Y, $\chi^{-1}(H)$ is an π_λ-open X. Next, we have shown that if a function $\chi : X \to Y$ is π_λ-continuous. Then χ is ij-continuous if X_0 is an open subset of X. Moreover, χ is $ij - \pi_\lambda$-continuous if $\chi\ |_{X_0}$: $X_0 \to Y$ is π_λ-continuous. Furthermore, we have shown that every $ij - \pi_\lambda$-continuous function is ij-continuous but the converse does not need to be true in general.

5 Open Problem

These notions of continuity are presented in bitopological space. Can one generalize these notions in fuzzy N-topological spaces?

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