Chlodowsky type (λ, q)-Bernstein Stancu operator of rough fuzzy Borel summability of triple sequences

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Abstract

We define the concept of rough limit set of a triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator of Borel summability of fuzzy numbers and obtain the relation between the set of rough limit and the extreme limit points of a triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu operator of Borel summability method of fuzzy numbers. Finally, we investigate some properties of the rough limit set of Chlodowsky type (λ, q)-Bernstein Stancu operator under which Borel summable sequence of fuzzy numbers are convergent. Also, we give the results for Borel summability method of series of fuzzy numbers.

1 Introduction

A triple sequence (S_{mnk}) of complex numbers is said to be Borel summable to S if the series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} S_{mnk}$ converges for all $x \in \mathbb{R}$ and

$$e^{-x} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} S_{mnk} \to S, x \to \infty.$$

In this we define Borel summability method for sequences and series of fuzzy numbers.

Definition 1.1. *Let (u_{mnk}) be a triple sequence of fuzzy numbers. Then the expression $\sum \sum \sum u_{mnk}$ is called a series of fuzzy numbers. Denote*

$$S_{rst} = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} u_{mnk} \text{ for all } r, s, t \in \mathbb{N}.$$

If the sequence (S_{rst}) converges to a fuzzy number u, then we say that the series $\sum \sum \sum u_{mnk}$ of fuzzy numbers converges to u and write $\sum \sum \sum u_{mnk} = u$ which implies as $r, s, t \to \infty$ that $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} u_{mnk}^{-}(\lambda) \to u^{-}(\lambda)$ and $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} u_{mnk}^{+}(\lambda) \to u^{+}(\lambda)$ uniformly in $\lambda \in [0,1]$.

Conversely, if the fuzzy numbers $u_{mnk} = \left\{ \left(u_{mnk}^{-}(\lambda), u_{mnk}^{+}(\lambda) \right) : \lambda \in [0,1] \right\}$, $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} u_{mnk}^{-}(\lambda) \to u^{-}(\lambda)$ and $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} u_{mnk}^{+}(\lambda) \to u^{+}(\lambda)$ converge uniformly in λ, then $u = \{(u^{-}(\lambda), u^{+}(\lambda)) : \lambda \in [0,1]\}$ defines a fuzzy number such that $u = \sum \sum \sum u_{mnk}$.

We say other wise the series of fuzzy numbers diverges. Additionally, if the triple sequence (S_{rst}) is bounded then we say that the series $\sum \sum \sum u_{mnk}$ of fuzzy numbers is bounded. We denote the set of all bounded series of fuzzy numbers by $bs(F)$.

Definition 1.2. *A triple sequence (u_{mnk}) of fuzzy numbers is said to be Borel summable to $\zeta \in E^{\cdot}$ if the series*

$$f(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} u_{mnk}$$

converges for $x \in (0, \infty)$ and $\lim_{x \to \infty} e^{-x} f(x) = \zeta$.

The idea of rough convergence was first introduced by Phu [15, 16, 17] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dundar and Cakan [11] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of I-convergence of a triple sequence spaces which is based on the structure of the

ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First applied the concept of (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators. Later, based on (p, q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p, q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p, q)-Bezier curves and surfaces based on (p, q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p, q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu operators based on (p, q)-integers.

Now we recall some basic definitions about (p, q)-integers. For any $u, v, w \in \mathbb{N}^3$, the (p, q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p - q} \text{ if } u, v, w \geq 1,$$

where $0 < q < p \leq 1$. The (p, q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [uvw]_{p,q}$$
$$\text{if } u, v, w \geq 1 \text{ and } u, v, w, m, n, k \in \mathbb{N}.$$

Also the (p, q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[u]!_{p,q}}{[m]!_{p,q}[u-m]!_{p,q}} \frac{[v]!_{p,q}}{[n]!_{p,q}[v-n]!_{p,q}} \frac{[w]!_{p,q}}{[k]!_{p,q}[w-k]!_{p,q}}$$

for all $u, v, w, m, n, k \in \mathbb{N}$ with $(u, v, w) \geq (m, n, k)$.

The formula for (p, q)-binomial expansion is as follows:

$$(ax + by)_{p,q}^{uvw}$$
$$= \sum_{m=0}^{u} \sum_{n=0}^{v} \sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}$$
$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)}b^{m+n+k}x^{(u-m)+(v-n)+(w-k)}y^{m+n+k},$$

$$(x+y)^{uvw}_{p,q} = (x+y)(px+qy)\left(p^2x+q^2y\right)$$
$$\cdots \left(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)^{uvw}_{p,q} = (1-x)\left(p-qx\right)\left(p^2-q^2x\right)$$
$$\cdots \left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)^{mnk}_{p,q} = x\,(px)\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order rst is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x)$$
$$= \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}$$
$$\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right) \qquad (1)$$
$$f\left(\frac{[m]_{p,q}\,[n]_{p,q}\,[k]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}\,[r]_{p,q}\,[s]_{p,q}\,[t]_{p,q}+\mu}\right),\; x \in [0,1]$$

Also, we have

$$(1-x)^{rst}_{p,q}$$
$$= \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}\,p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}}$$
$$q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}.$$

(p, q)-Bernstein-Stancu operators are defined as follows:

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \binom{r}{m} \binom{s}{n} \binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k}$$

$$\prod_{u_1=0}^{(r-m-1)} (p^{u_1} - q^{u_1}x) \prod_{u_2=0}^{(s-n-1)} (p^{u_2} - q^{u_2}x) \prod_{u_3=0}^{(t-k-1)} (p^{u_3} - q^{u_3}x) \quad (2)$$

$$f\left(\frac{p^{(r-m)+(s-n)+(t-k)} [m]_{p,q} [n]_{p,q} [k]_{p,q} + \eta}{[r]_{p,q} [s]_{p,q} [t]_{p,q} + \mu}\right), x \in [0,1]$$

Note that for $\eta = \mu = 0$, (p, q)-Bernstein-Stancu operators given by (2) reduces into (p, q)-Bernstein-Stancu operators. Also for $p = 1$, (p, q)-Bernstein-Stancu operators given by (1) turn out to be q-Bernstein-Stancu operators.

In this paper we construct Chlodowsky type (λ, q)-Bernstein-Stancu operators of triple sequence is defined as

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x) = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \widehat{b}_{rst,mnk}(x;q) f\left(\frac{[mnk]_q + \alpha}{[rst]_q + \beta} b_{rst}\right), \quad (3)$$

where $r, s, t \in \mathbb{N}$, $0 < q \le 1$, $0 \le x \le b_{rst}$ and b_{rst} is a sequence of positive numbers such that $\lim_{r,s,t\to\infty} b_{rst} = \infty$, $\lim_{rst\to\infty} \frac{b_{rst}}{[rst]_q} = 0$,

$$\widehat{b}_{rst,mnk}(x;q) = \binom{r}{m} \binom{s}{n} \binom{t}{k} \left(\frac{x}{b_{rst}}\right)^{m+n+k} \left(1 - \frac{x}{b_{rst}}\right)^{(r-m)+(s-n)+(t-k)}$$

and $\alpha, \beta \in \mathbb{R}$ and $0 \le \alpha \le \beta$. For $\alpha = \beta = 0$ we obtain the Chlodowsky type (λ, q)-Bernstein-Stancu polynomials.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein-Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ is said to be statistically convergent to *zero*, written as $st - \lim x = 0$, provided that the set

$$K_\epsilon := \left\{(m, n, k) \in \mathbb{N}^3 : \left|B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x) - (f,x)\right| \ge \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, *zero* is called the statistical limit of the triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$, i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{r,s,t\to\infty} \frac{1}{pqj} \left|\left\{m \le p, n \le q, k \le j : \left|B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x) - (f,x)\right| \ge \epsilon\right\}\right| = 0.$$

In this case, we write $\delta - \lim B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x) = (f,x)$ or $B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x) \to^{S_S} (f,x)$.

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [18, 19], Esi et al. [7, 3, 6, 4, 5, 31], Dutta et al. [9], Subramanian et al. [23, 8, 30, 28, 20, 26, 22, 25, 29, 27], Debnath et al. [10]* and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

The Borel summability of fuzzy real numbers is denoted by $(\zeta, X)(\mathbb{R})$, and d denotes the supremum metric on $(\zeta, X)(\mathbb{R}^3)$. Now let r be nonnegative real number. A Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of fuzzy numbers is r-convergent to a fuzzy number (ζ, X) and we write

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) \to^r (\zeta, X)$$

provided that for every $\epsilon > 0$ there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

The set

$$\mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) :=$$
$$\left\{ (\zeta, X) \in (\zeta, X)(\mathbb{R}^3) : B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) \to^r (\zeta, X), \right\}$$

is called the r-limit set of the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right)$.

A Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) = \begin{cases} \eta(X), & \text{if } m, n, k \text{ are odd integers,} \\ \mu(X), & \text{otherwise} \end{cases},$$

where

$$\eta(X) = \begin{cases} X, & \text{if } X \in [0, 1], \\ -X + 2, & \text{if } X \in [1, 2], \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu(X) = \begin{cases} X - 3, & \text{if } X \in [3, 4], \\ -X + 5, & \text{if } X \in [4, 5], \\ 0, & \text{otherwise} \end{cases} \quad .$$

Then we have

$$\text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) = \begin{cases} \phi, & \text{if } r < \frac{3}{2}, \\ [\mu - r_1, \eta + r_1], & \text{otherwise} \end{cases},$$

where r_1 is nonnegative real number with

$$[\mu - r_1, \eta + r_1]$$
$$:= \left\{ B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \in (\zeta, X)\left(\mathbb{R}^3\right) : \mu - r_1 \leq B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \leq \eta + r_1 \right\}.$$

The ideal of rough convergence of a Borel summability of triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) \right)$ can be interpreted as follows:

Let $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$ be a convergent triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) \right)$ of fuzzy numbers. Assume that $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$ cannot be determined exactly for every $(m, n, k) \in \mathbb{N}^3$. That is, $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$ cannot be calculated so we can use approximate value of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$ for simplicity of calculation. We only know that $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right) \in [\mu_{mnk}, \lambda_{mnk}]$, where $d(\mu_{mnk}, \lambda_{mnk}) \leq r$ for every $(m, n, k) \in \mathbb{N}^3$. The Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) \right)$ of $(S_{rst,p,q}(\zeta, X))$ satisfying $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \right) \in [\mu_{mnk}, \lambda_{mnk}]$, for all m, n, k. Then the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) \right)$ of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \right)$ may not be convergent, but the inequality

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X) \right) \leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$$
$$+ d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y), (\zeta, Y) \right)$$
$$\leq r + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y), (\zeta, Y) \right)$$

implies that the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X)\right)$ is r-convergent.

In this paper, we first define the concept of rough convergence of a Borel summability of triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of fuzzy numbers. Also obtain the relation between the set of rough limit and the extreme limit points of a Borel summability of triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of fuzzy numbers. We show that the rough limit set of a Borel summability of triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ is closed, bounded and convex.

2   Definitions and Preliminaries

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where $X^0; = \operatorname{cl}\{x \in \mathbb{R}^3 : X(x) > 0\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0, 1]$, the α-level set X^α defined by

$$X^\alpha = \left\{x \in \mathbb{R}^3 : X(x) \geq \alpha\right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$$

is a non empty compact convex subset of $\mathbb{R}^3$.

The supremum metric d on the set $L\left(\mathbb{R}^3\right)$ is defined by

$$d(X, Y) = \sup_{\alpha \in [0,1]} \max\left(\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right).$$

Now, given $X, Y \in L\left(\mathbb{R}^3\right)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0, 1]$. We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^3\right)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu'$ for all upper bounds μ'.

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L\left(\mathbb{R}^3\right)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence spaces (X_{mnk}) is defined by

$$\lim_{m,n,k\to\infty} \inf X_{mnk} := \inf A_X$$

$$\lim_{m,n,k\to\infty} \sup X_{mnk} := \inf B_X$$

where

$$A_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{mnk} < \mu\right\} \text{ is infinite}\right\}$$
$$B_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{mnk} > \mu\right\} \text{ is infinite}\right\}.$$

Now, given two fuzzy numbers $X, Y \in L\left(\mathbb{R}^3\right)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0,1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L\left(\mathbb{R}^3\right)$, which is defied by

$$a_1(x) = \left\{ \begin{array}{ll} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{array} \right. .$$

An order interval in $L\left(\mathbb{R}^3\right)$ is defined by $[X,Y] := \{Z \in L\left(\mathbb{R}^3\right) : X \leq Z \leq Y\}$, where $X, Y \in L\left(\mathbb{R}^3\right)$.

A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in E$.

3   Main Results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of real numbers. If $(\zeta, X) \in \text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$, then $\text{diam}\left(\lim\sup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta,X), (\zeta,X)\right) \leq r$ and $\text{diam}\left(\lim\inf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta,X), (\zeta,X)\right) \leq r$.*

Proof. We assume that $\text{diam}\left(\lim\sup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X)\right) > r$. Define $\tilde{\epsilon} := \frac{\left(\lim\sup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta,X),(\zeta,X)\right) - r}{2}$. By definition of limit supremum, we have that given $m'_{\tilde{\epsilon}}, n'_{\tilde{\epsilon}}, k'_{\tilde{\epsilon}} \in \mathbb{N}^3$ there exists an $(m,n,k) \in \mathbb{N}^3$ with $m \geq m'_{\tilde{\epsilon}}$, $n \geq n'_{\tilde{\epsilon}}$, $k \geq k'_{\tilde{\epsilon}}$ such that $\text{diam}\left(\lim\sup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X)\right) \leq \tilde{\epsilon}$. Also, since $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \to^r (\zeta, X)$ as $m,n,k \to \infty$, there is an integer $m''_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}}$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X)\right) < r + \tilde{\epsilon}$$

whenever $m \geq m_{\tilde{\epsilon}}''$, $n \geq n_{\tilde{\epsilon}}''$, $k \geq k_{\tilde{\epsilon}}''$. Let

$$(m_{\tilde{\epsilon}}, n_{\tilde{\epsilon}}, k_{\tilde{\epsilon}}) := \max\left\{\left(m_{\tilde{\epsilon}}', n_{\tilde{\epsilon}}', k_{\tilde{\epsilon}}'\right), \left(m_{\tilde{\epsilon}}'', n_{\tilde{\epsilon}}'', k_{\tilde{\epsilon}}''\right)\right\}.$$

There exists $(m, n, k) \in \mathbb{N}^3$ such that $m \geq m_{\tilde{\epsilon}}$, $n \geq n_{\tilde{\epsilon}}$, $k \geq k_{\tilde{\epsilon}}$ and

$$\text{diam}\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) \leq (\zeta, X)$$

$$\text{diam}\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) + \text{diam}\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right)$$

$$< \tilde{\epsilon} + r + \tilde{\epsilon}$$

$$< r + 2\tilde{\epsilon}$$

$$= r + \text{diam}\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) - r$$

$$= \text{diam}\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right).$$

The contradiction proves the theorem.

Similarly, $\text{diam}\left(\liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) \leq r$ can be proved using definition of limit infimum. $\square$

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f; x)\right)$ of real numbers. If $\text{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) \neq \phi$, then we have*

$$\text{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)$$

$$\subseteq \left[\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) - r_1, \left(\liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) + r_1\right].$$

Proof. To prove that

$$(\zeta, X) \in \left[\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) - r_1, \left(\liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) + r_1\right]$$

for an arbitrary $(\zeta, X) \in \text{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)$, i.e.,

$$\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) - r_1 \leq (\zeta, X) \leq \left(\liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) + r_1.$$

Let us assume that $\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) - r_1 \leq (\zeta, X)$ does not hold. Thus, there exists an $\alpha_0 \in [0, 1]$ such that

$$\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)^{\alpha_0}\right) - r_1 > (\zeta, X)^{\alpha_0}$$

or

$$\left(\overline{\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^{\alpha_0}\right) - r_1 > \le \overline{\left(\zeta, X\right)}^{\alpha_0}$$

holds i.e.,

$$\left(\underline{\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^{\alpha_0}\right) - \underline{\left(\zeta, X\right)}^{\alpha_0} > r_1$$

or

$$\left(\overline{\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^{\alpha_0}\right) - \le \overline{\left(\zeta, X\right)}^{\alpha_0} > r_1.$$

On the other hand, by theorem 3.1 we have

$$\left| \left(\underline{\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^{\alpha_0}\right) - \underline{\left(\zeta, X\right)}^{\alpha_0} \right| \le r_1$$

and

$$\left| \left(\overline{\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^{\alpha_0}\right) - \le \overline{\left(\zeta, X\right)}^{\alpha_0} \right| \le r_1.$$

We obtain a contradiction. Hence we get $\left(\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)\right) - r_1 \le (\zeta, X)$. By using the similar arguments and get it for second part. $\square$

Note 3.3. *The converse inclusion in this theorem holds for f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of real numbers, but it may not hold for Borel summability of rough triple sequences of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of fuzzy numbers as in the following example:*

Example 3.4. *Define*

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right) = \begin{cases} \frac{-1}{2(mnk)}X + 1, & \text{if } X \in [0, 1], \\ 0, & \text{otherwise} \end{cases}$$

and

$$\left(\zeta, X\right) = \begin{cases} 1, & \text{if } X \in [0, 1], \\ 0, & \text{otherwise} \end{cases}.$$

Then we have

$$\left| \overline{\left(\zeta, X\right)}^1 - \overline{B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta, X\right)}^1 \right| = |1 - 0| = 1,$$

i.e., $d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) \ge 1$ for all $(m, n, k) \in \mathbb{N}^3$. Although the Borel summability of rough triple sequence spaces of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ is not convergent to (ζ, X), $\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)$ and $\liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)$ of this Borel summability of

rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ are equal to (ζ, X). Hence we get

$$L \in \left[\limsup B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) - \left(\frac{1}{2}\right)_1, \liminf B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) + \left(\frac{1}{2}\right)_1\right],$$

but $(\zeta, X) \notin \mathrm{LIM}^{\frac{1}{2}} B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)$.

Theorem 3.5. *Let f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of real numbers converges to the fuzzy number (f, X), then*

$$\mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) = \bar{S}_r\left((\zeta, X)\right) := \left\{\mu \in (\zeta, X)\left(\mathbb{R}^3\right) : d\left(\mu, (\zeta, X)\right) \le r\right\}.$$

Proof. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu operators of $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right)$ is convergent to (ζ, X), there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) < \epsilon \text{ whenever } m \ge m_\epsilon, n \ge n_\epsilon, k \ge k_\epsilon.$$

Let $Y \in \bar{S}_r\left((\zeta, X)\right),$ we have

$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), Y\right) \le d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right) + d((\zeta, X), Y) < \epsilon + r$$

for every $m \ge m_\epsilon, n \ge n_\epsilon, k \ge k_\epsilon.$

Hence we have $Y \in \mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X).$

Now let $Y \in \mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X).$ Hence there is an integer $m'_\epsilon, n'_\epsilon, k'_\epsilon$ so that

$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), Y\right) < r + \epsilon$$

whenever $m \ge m'_\epsilon, n \ge n'_\epsilon, k \ge k'_\epsilon.$ Let

$$\left(m''_\epsilon, n''_\epsilon, k''_\epsilon\right) := \max\left\{(m_\epsilon, n_\epsilon, k_\epsilon), \left(m'_\epsilon, n'_\epsilon, k'_\epsilon\right)\right\}$$

for all $m \ge m''_\epsilon, n \ge n''_\epsilon, k \ge k''_\epsilon,$ we obtain

$$d\left(Y, \zeta(X)\right) \le d\left(Y, S_{rst,p,q}(\zeta, X)\right) + d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X), (\zeta, X)\right)$$

$$< r + \epsilon + \epsilon < r + 2\epsilon.$$

Since ϵ is arbitrary, we have $d\left(Y, (\zeta, X)\right) \le r.$ Hence we get $Y \in \bar{B}_r\left((\zeta, X)\right).$ Thus, if the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ of $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X)\right) \to^r (\zeta, X),$ then $\mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta, X) = \bar{B}_r\left((\zeta, X)\right).$ $\quad\square$

Theorem 3.6. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of real numbers of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x) \right)$ and $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right) \in (\zeta, X)(\mathbb{R}^3)$. If $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \to^r (\zeta, X)$ then*

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \to^r (\zeta, Y)$$

and

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right) \leq r$$

for every $(m, n, k) \in \mathbb{N}^3$.

Proof. Assume that $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \to^r (\zeta, Y)$, as $m, n, k \to \infty$ and

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)(\zeta, Y) \right) \leq r$$

for every $(m, n, k) \in \mathbb{N}^3$. We have $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \to^r (\zeta, Y)$, as $m, n, k \to \infty$ means that for every $\epsilon > 0$ there exists an $m_\epsilon, n_\epsilon, k_\epsilon$ such that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y), (\zeta, Y) \right) < \epsilon \text{ for all } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

If the in equality $d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right) \leq r$ yields then

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X) \right) \leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \right)$$
$$+ d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y), (\zeta, Y) \right)$$
$$< r + \epsilon$$

for all $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

Hence the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \right)$ is r-convergent to the fuzzy number (ζ, X). $\square$

Theorem 3.7. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \right)$ of real numbers and the diameter of an r-limit set is not greater than $3r$.*

Proof. We have to prove that

$$\sup \left\{ d(W, Z) : W, Y, Z \in LIM^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \right\} \leq 3r.$$

Assume on the contrary that

$$\sup\left\{d\left(W,Z\right):W,Y,Z\in LIM^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right)\right\}>3r.$$

By this assumption, there exists, $W,Y,Z\in\mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right)$ satisfying $\lambda:=d\left(W,Z\right)>3r$. For an arbitrary $\epsilon\in\left(0,\frac{\lambda}{3}-r\right)$, we have

$$\exists\left(m_\epsilon^{'},n_\epsilon^{'},k_\epsilon^{'}\right)\in\mathbb{N}^3:\forall\left(m,n,k\right)\geq\left(m_\epsilon^{'},n_\epsilon^{'},k_\epsilon^{'},\right)\Rightarrow$$
$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right),W\right)\leq r+\epsilon,$$

$$\exists\left(m_\epsilon^{''},n_\epsilon^{''},k_\epsilon^{''}\right)\in\mathbb{N}^3:\forall\left(m,n,k\right)\geq\left(m_\epsilon^{''},n_\epsilon^{''},k_\epsilon^{''},\right)\Rightarrow$$
$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right),Y\right)\leq r+\epsilon,$$

$$\exists\left(m_\epsilon^{'''},n_\epsilon^{'''},k_\epsilon^{'''}\right)\in\mathbb{N}^3:\forall\left(m,n,k\right)\geq\left(m_\epsilon^{'''},n_\epsilon^{'''},k_\epsilon^{'''},\right)\Rightarrow$$
$$d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right),Z\right)\leq r+\epsilon.$$

Define $(m_\epsilon,n_\epsilon,k_\epsilon):=\max\left\{\left(m_\epsilon^{'},n_\epsilon^{'},k_\epsilon^{'}\right),\left(m_\epsilon^{''},n_\epsilon^{''},k_\epsilon^{''}\right),\left(m_\epsilon^{'''},n_\epsilon^{'''},k_\epsilon^{'''}\right)\right\}$. Thus we get

$$d(W,Z)\leq d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X),W\right)+d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X),Y\right)$$
$$+d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X),Z\right)$$
$$<(r+\epsilon)+(r+\epsilon)+(r+\epsilon)<3(r+\epsilon)$$
$$<3r+3\left(\frac{\lambda}{3}-r\right)<3r+\lambda-3r$$
$$=\lambda\text{ for all }m\geq m_\epsilon,n\geq n_\epsilon,k\geq k_\epsilon$$

which contradicts to the fact that $\lambda=d\left(W,Z\right)$.   □

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of Chlodowsky type (λ,q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right)\right)$ of real numbers is analytic if and only if there exists an $r\geq0$ such that $\mathrm{LIM}^r B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right)\neq\phi$.*

Proof. (Necessity:) Let the Borel summability of rough triple sequence space of Chlodowsky type (λ,q)-Bernstein Stancu operator $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X)\right)$ of $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(\zeta,X)\right)$ be a analytic sequence and

$$s:=\sup\left\{d\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(\zeta,X\right)^{1/m+n+k},0\right):(m,n,k)\in\mathbb{N}^3\right\}<\infty.$$

Then we have $0 \in \text{LIM}^s B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$, i.e., $\text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \neq \phi$, where $r = s$.

(Sufficiency:) If $\text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \neq \phi$ for some $r \geq 0$, then there exists $(\zeta, X) \in \text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$. By definition, for every $\epsilon > 0$ there is an integer $(m_\epsilon, n_\epsilon, k_\epsilon)$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X)\right) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Define

$$t = t(\epsilon)$$
$$:= \max\left\{d((\zeta, X), 0), d(B_{111,\lambda,q}(\zeta, X), 0), \cdots, d(B_{r_\epsilon s_\epsilon t_\epsilon, \lambda, q}(\zeta, X), 0), r + \epsilon\right\}.$$

Then we have

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q} \in \left\{\mu \in (\zeta, X)\left(\mathbb{R}^3\right) : d(\mu, 0) \leq t + r + \epsilon\right\} \text{ for every } (m, n, k) \in \mathbb{N}^3,$$

which proves the boundedness of the Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)\right)$. $\square$

Theorem 3.9. *Let f be a continuous function defined on the closed interval* $[0, 1]$. *A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $(B_{u_m v_n w_k, \lambda, q}(\zeta, X))$ of real numbers is a sub sequence of a Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)\right)$, then*

$$\text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X) \subset \text{LIM}^r B_{u_m v_n w_k, \lambda, q}(\zeta, X).$$

Proof. Omitted. $\square$

Theorem 3.10. *Let f be a continuous function defined on the closed interval* $[0, 1]$. *A Borel summability of rough triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)\right)$ of real numbers, for all $r \geq 0$, the r-limit set $\text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$ of an arbitrary Borel summability of rough triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator of $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$ is closed.*

Proof. Let $(Y_{mnk}) \subset \text{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y)$ and $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y) \to (\zeta, Y)$ as $m, n, k \to \infty$. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y)\right) \to^r (\zeta, Y)$, there is an integer $i_\epsilon j_\epsilon \ell_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, Y), (\zeta, Y)\right) < \frac{\epsilon}{2} \text{ whenever } m \geq i_\epsilon, n \geq j_\epsilon, k \geq \ell_\epsilon.$$

Since $B_{i_\epsilon j_\epsilon \ell_\epsilon, \lambda, q}(\zeta, Y) \in \mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$, there is an integer $(m_\epsilon n_\epsilon k_\epsilon)$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B_{i_\epsilon j_\epsilon \ell_\epsilon, \lambda, q}(\zeta, Y)\right) < r + \frac{\epsilon}{2} \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Therefore, we have

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), (\zeta, X)\right) \leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X), B_{i_\epsilon j_\epsilon \ell_\epsilon, p, q}(\zeta, Y)\right)$$
$$< r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon$$

for every $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$.

Hence $L \in \mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$ implies that the set $\mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)$ is closed. $\square$

4   Open Problem

We introduced Borel summability of triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu operator $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(\zeta, X)\right)$ of rough convergence of fuzzy numbers. For the reference sections, consider the following introduction described the main results are motivating the research. Thus it will be possible to benefit from this article for new studies.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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