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Non differentiable Solutions for Nonlinear Lane–Emden Equation and Emden–Fowler Equation within Local Fractional Derivative

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Abstract

The basic idea of the present study is to apply the local fractional Sumudu decomposition method (LFSDM) presented in [31] to solve Lane–Emden of index m ***and Emden–Fowler equation of index*** m ***with local fractional derivative, in order to obtain non differentiable analytical solutions. The results of the solved local differential fractional equations show the effectiveness of this method.***

Keywords: *Local fractional operators, Local fractional Sumudu decomposition method, Lane–Emden equation, Emden–Fowler equation.*

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1 Introduction

The search for solutions to nonlinear differential equations, whether ordinary differential equations, partial differential equations or integral differential equations with integer order or fractional order, is one of the most difficult steps that researchers face in the field of mathematics or physics.

For example, the Lane–Emden equation of index m and Emden–Fowler equation of index m are a nonlinear equations either integer order or fractional order, which has taken a large amount of scientific research, for example, among the articles that discussed the solutions of the Lane–Emden equation, we find ([12], [13], [16], [17], [18], [20], [29], [30]), as well as among the articles that discussed the solutions of the Emden–Fowler, there are ([5], [6], [8], [9], [14], [15], [19], [22]). Among the methods that have taken a wide space from the researchers' work and have been used a lot in solving differential equations of all kinds and of different orders, there is the Adomian decomposition method and in the abbreviation (ADM), which is among the most famous methods developed recently, where it was developed between 1970s and 1990s by George Adomian ([2]-[4]).

With the new concepts of fractional derivative and fractional integral, as well as local fractional derivative and local fractional integral, researchers were able to use the ADM method to solve these new type of equations or systems which include, local fractional differential equations, local fractional partial differential equations and local fractional integro-differential equations ([7], [10], [26], [27], [28]). One of the additions made to this method, is to combine it with some transformations such as Laplace transform and Sumudu transform, so the results of using this new method were effective. Among these works we find the local fractional Laplace decomposition method ([11], [38]) and the local fractional Sumudu decomposition method ([31]-[37]).

The objective of this study is to apply the local fractional Sumudu decomposition method suggested by Ziane, D., et al..[31], to solve nonlinear Lane–Emden equation of index m with local fractional derivative and nonlinear Emden–Fowler equation of index m with local fractional derivative, then we compared the obtained results with the results of other works in the case of $\sigma = 1$.

2 Preliminaries

In this section, we will present the basics of local fractional calculus, these concepts include: Local fractional derivative, local fractional integral, some important results and local fractional Sumudu transform.

2.1 Local fractional derivative

Definition 2.1. *The local fractional derivative of* $\Phi(r)$ *of order* σ *at* $r = r_0$ *is defined as ([24],[25])*

$$\Phi^{(\sigma)}(\varkappa) = \left.\frac{d^\sigma \Phi}{d\varkappa^\sigma}\right|_{\varkappa=\varkappa_0} = \frac{\Delta^\sigma(\Phi(\varkappa) - \Phi(\varkappa_0))}{(\varkappa - \varkappa_0)^\sigma}, \tag{1}$$

where

$$\Delta^\sigma(\Phi(\varkappa) - \Phi(\varkappa_0)) \cong \Gamma(1+\sigma)\left[(\Phi(\varkappa) - \Phi(\varkappa_0))\right]. \tag{2}$$

For any $\varkappa \in (\alpha, \beta)$, *there exists*

$$\Phi^{(\sigma)}(\varkappa) = D^\sigma_\varkappa \Phi(\varkappa),$$

denoted by

$$\Phi(\varkappa) \in D^\sigma_\varkappa(\alpha, \beta).$$

Local fractional derivative of high order is written in the form

$$\Phi^{(m\sigma)}(\varkappa) = \overbrace{D^{(\sigma)}_\varkappa \cdots D^{(\sigma)}_\varkappa \Phi(\varkappa)}^{m\ times}, \tag{3}$$

and local fractional partial derivative of high order

$$\frac{\partial^{m\sigma}\Phi(\varkappa)}{\partial\varkappa^{m\sigma}} = \overbrace{\frac{\partial^\sigma}{\partial\varkappa^\sigma} \cdots \frac{\partial^\sigma}{\partial\varkappa^\sigma}\Phi(\varkappa)}^{m\ times}. \tag{4}$$

2.2 Local fractional integral

Definition 2.2. *The local fractional integral of* $\Phi(\varkappa)$ *of order* σ *in the interval* $[\alpha, \beta]$ *is defined as ([24],[25])*

$$\begin{aligned} {}_\alpha I^{(\sigma)}_\beta \Phi(\varkappa) &= \frac{1}{\Gamma(1+\sigma)} \int_\alpha^\beta \Phi(\tau)(d\tau)^\sigma \\ &= \frac{1}{\Gamma(1+\sigma)} \lim_{\Delta\tau \longrightarrow 0} \sum_{j=0}^{N-1} f(\tau_j)(\Delta\tau_j)^\sigma, \end{aligned} \tag{5}$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j$, $\Delta\tau = \max\{\Delta\tau_0, \Delta\tau_1, \Delta\tau_2, \cdots\}$ *and* $[\tau_j, \tau_{j+1}]$, $\tau_0 = \alpha$, $\tau_N = \beta$, *is a partition of*
the interval $[\alpha, \beta]$. *For any* $\varkappa \in (\alpha, \beta)$, *there exists*

$${}_\alpha I^{(\sigma)}_\varkappa \Phi(\varkappa),$$

denoted by

$$\Phi(\varkappa) \in I^{(\sigma)}_\varkappa(\alpha, \beta).$$

2.3 Some important results

Definition 2.3. *In fractal space, the Mittage Leffler function, Hyperbolic sine and hyperbolic cosine are defined as ([24],[25])*

$$E_\sigma(\varkappa^\sigma) = \sum_{m=0}^{+\infty} \frac{\varkappa^{m\sigma}}{\Gamma(1+m\sigma)}, \quad 0 < \sigma \leqslant 1, \tag{6}$$

$$\sin_\sigma(\varkappa^\sigma) = \sum_{m=0}^{+\infty} (-1)^m \frac{\varkappa^{(2m+1)\sigma}}{\Gamma(1+(2m+1)\sigma)}, \quad 0 < \sigma \leqslant 1, \tag{7}$$

$$\cos_\sigma(\varkappa^\sigma) = \sum_{m=0}^{+\infty} (-1)^m \frac{\varkappa^{2m\sigma}}{\Gamma(1+2m\sigma)}, \quad 0 < \sigma \leqslant 1, \tag{8}$$

The properties of local fractional derivatives and integral of non-differentiable functions are given by ([24],[25])

$$\frac{d^\sigma}{d\varkappa^\sigma} \frac{\varkappa^{m\sigma}}{\Gamma(1+m\sigma)} = \frac{\varkappa^{(m-1)\sigma}}{\Gamma(1+(m-1)\sigma)}. \tag{9}$$

$$\frac{d^\sigma}{d\varkappa^\sigma} E_\sigma(\varkappa^\sigma) = E_\sigma(\varkappa^\sigma). \tag{10}$$

$$\frac{d^\sigma}{d\varkappa^\sigma} \sin_\sigma(\varkappa^\sigma) = \cos_\sigma(\varkappa^\sigma). \tag{11}$$

$$\frac{d^\sigma}{d\varkappa^\sigma} \cos_\sigma(\varkappa^\sigma) = -\sin_\sigma(\varkappa^\sigma). \tag{12}$$

$${}_0I_\varkappa^{(\sigma)} \frac{\varkappa^{m\sigma}}{\Gamma(1+m\sigma)} = \frac{\varkappa^{(m+1)\sigma}}{\Gamma(1+(m+1)\sigma)}. \tag{13a}$$

2.4 Local fractional Sumudu transform

We present here the definition of local fractional Sumudu transform (denoted in this paper by ${}^{LF}S_\sigma$) and some properties concerning this transformation [21].

If there is a new transform operator $LFS_\sigma : \Phi(\varkappa) \longrightarrow \digamma(u)$, namely,

$$LFS_\sigma \left\{ \sum_{m=0}^{\infty} a_m \varkappa^{m\sigma} \right\} = \sum_{m=0}^{\infty} \Gamma(1+m\sigma) a_m u^{m\sigma}. \tag{14}$$

As typical examples, we have

$$LFS_\sigma \left\{ E_\sigma(i^\sigma \varkappa^\sigma) \right\} = \sum_{m=0}^{\infty} i^{\sigma m} u^{\sigma m}. \tag{15}$$

$$LFS_\sigma\left\{\frac{\varkappa^\sigma}{\Gamma(1+\sigma)}\right\}=u^\sigma. \tag{16}$$

Definition 2.4. *[21] The local fractional Sumudu transform of $\Phi(r)$ of order $\sigma$is defined as*

$$LFS_\sigma\left\{\Phi(\varkappa)\right\}= F_\sigma(u)=\frac{1}{\Gamma(1+\sigma)}\int_0^\infty E_\sigma(-u^{-\sigma}\varkappa^\sigma)\frac{\Phi(\varkappa)}{u^\sigma}(d\varkappa)^\sigma,\ 0<\sigma\leqslant 1 \tag{17}$$

Following (17), its inverse formula is defined as

$$LFS_\sigma^{-1}\left\{ F_\sigma(u)\right\}=\Phi(\varkappa)\ ,\quad 0<\sigma\leqslant 1\ . \tag{18}$$

Theorem 2.5. (*linearity*). *If $LFS_\sigma\left\{\Phi(\varkappa)\right\} = F_\sigma(u)$ and $LFS_\sigma\left\{\varphi(\varkappa)\right\} = \Psi_\sigma(u)$, then one has*

$$LFS_\sigma\left\{\Phi(\varkappa)+\varphi(\varkappa)\right\}=F_\sigma(u)+\Psi_\sigma(u). \tag{19}$$

Proof. Relying on the definition, we can easily prove the linearity. □

Theorem 2.6. (1) (*local fractional Sumudu transform of local fractional derivative*). *If $LFS_\sigma\left\{\Phi(\varkappa)\right\}=F_\sigma(u)$, then one has*

$$LFS_\sigma\left\{\frac{d^\sigma\Phi(\varkappa)}{d\varkappa^\sigma}\right\}=\frac{F_\sigma(u)-F(0)}{u^\sigma}. \tag{20}$$

As the direct result of (20), we have the following results. If $LFS_\sigma\left\{\Phi(\varkappa)\right\}= F_\sigma(u)$, we obtain

$$LFS_\sigma\left\{\frac{d^{n\sigma}\Phi(\varkappa)}{d\varkappa^{n\sigma}}\right\}=\frac{1}{u^{n\sigma}}\left[F_\sigma(u)-\sum_{k=0}^{n-1}u^{k\sigma}\Phi^{(k\sigma)}(0)\right]. \tag{21}$$

When $n=2$, from (21), we get

$$LFS_\sigma\left\{\frac{d^{2\sigma}\Phi(\varkappa)}{d\varkappa^{2\sigma}}\right\}=\frac{1}{u^{2\sigma}}\left[F_\sigma(u)-\Phi(0)-u^\sigma\Phi^{(\sigma)}(0)\right]. \tag{22}$$

(2) (*local fractional Sumudu transform of local fractional integral*). *If $LFS_\sigma\left\{\Phi(\varkappa)\right\} = F_\sigma(u)$, then we have*

$$LFS_\sigma\left\{{}_0I_\varkappa^{(\sigma)}\Phi(\varkappa)\right\}=u^\sigma F_\sigma(u). \tag{23}$$

Proof 2.7. *see [21]*

Theorem 2.8. (*local fractional convolution*). *If* $LFS_{\sigma}\left\{\Phi(\varkappa)\right\} = F_{\sigma}(u)$ *and* $LFS_{\sigma}\left\{\varphi(\varkappa)\right\} = \Psi_{\sigma}(u)$*, then one has*

$$LFS_{\sigma}\left\{\Phi(\varkappa) * \varphi(\varkappa)\right\} = u^{\sigma} F_{\sigma}(u)\Psi_{\sigma}(u), \tag{24}$$

where

$$\Phi(\varkappa) * \varphi(\varkappa) = \frac{1}{\Gamma(1+\sigma)} \int_{0}^{\infty} \Phi(\tau)\varphi(\varkappa-\tau)(d\varkappa)^{\sigma}.$$

Proof 2.9. *see [21]*

3 Local Fractional Sumudu Decomposition Method

Let us consider the following nonlinear operator with local fractional derivative [31]:

$$L_{\sigma}\left(V(\varkappa)\right) + R_{\sigma}\left(V(\varkappa)\right) + N_{\alpha}\left(V(\varkappa)\right) = g(r), \tag{25}$$

where $L_{\sigma} = \frac{\partial^{m\sigma}}{\partial \varkappa^{m\sigma}}$, $(m \in \mathbb{N}^{*})$ denotes linear local fractional derivative operator of order $m\sigma$, R_{σ} is the remaining linear operator, N_{σ} denotes nonlinear operator and $g(\varkappa)$ is a source term.

Taking the local fractional Sumudu transform (denoted in this paper by LFS_{σ}) on both sides of (25), we get:

$$LFS_{\sigma}\left[L_{\sigma}\left(V(\varkappa)\right)\right] + LFS_{\sigma}\left[R_{\sigma}\left(V(\varkappa)\right) + N_{\alpha}\left(V(\varkappa)\right)\right] = LFS_{\sigma}\left[g(\varkappa)\right]. \tag{26}$$

Using the property of the local fractional Sumudu transform, we have:

$$\begin{aligned} LFS_{\sigma}\left[V(\varkappa)\right] &= \sum_{k=0}^{m-1} v^{k\sigma} \frac{\partial^{k\sigma} V(0)}{\partial \varkappa^{k\sigma}} + v^{m\sigma}\left(LFS_{\sigma}\left[g(\varkappa)\right]\right) \\ & -v^{m\sigma}\left(LFS_{\sigma}\left[R_{\sigma}\left(V(\varkappa)\right) + N_{\alpha}\left(V(\varkappa)\right)\right]\right), \end{aligned} \tag{27}$$

Taking the inverse local fractional Sumudu transform on both sides of (27), gives:

$$\begin{aligned} V(\varkappa) &= \sum_{k=0}^{m-1} v^{k\sigma} \frac{\partial^{k\sigma} V(0)}{\partial \varkappa^{k\sigma}} \frac{\varkappa^{k\sigma}}{\Gamma(1+k\sigma)} + \ LFS_{\sigma}^{-1}\left(v^{m\sigma}\left(LFS_{\sigma}\left[g(\varkappa)\right]\right)\right) \\ & - LFS_{\sigma}^{-1}\left(v^{m\sigma}\left(LFS_{\sigma}\left[R_{\sigma}\left(V(\varkappa)\right) + N_{\alpha}\left(V(\varkappa)\right)\right]\right)\right). \end{aligned} \tag{28}$$

According to the Adomian decomposition method [2], we decompose the unknown function U as an infinite series given by:

$$V(\varkappa) = \sum_{n=0}^{\infty} V_n(\varkappa). \tag{29}$$

and the nonlinear term can be decomposed as:

$$N_\sigma V(\varkappa) = \sum_{n=0}^{\infty} A_n(V), \tag{30}$$

where A_n are Adomian polynomials [39] of $V_0, V_1, V_2, ..., V_n$ and it can be calculated by the formula given below:

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N_\sigma\left(\sum_{i=0}^{\infty}\lambda^i V_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots . \tag{31}$$

Substituting (29) and (30) in (28), we get:

$$\begin{aligned}\sum_{n=0}^{\infty} V_n(\varkappa) \;&=\; \sum_{k=0}^{m-1}\left[\frac{\partial^{k\sigma}V(0)}{\partial \varkappa^{k\sigma}}\frac{\varkappa^{k\sigma}}{\Gamma(1+k\sigma)}\right] + \; LFS_\sigma^{-1}\left(\upsilon^{m\sigma}\left(LFS_\sigma\left[g(\varkappa)\right]\right)\right)\\ &\quad - LFS_\sigma^{-1}\left(\upsilon^{m\sigma}\left(LFS_\sigma\left[R_\sigma\sum_{n=0}^{\infty}V_n(\varkappa) + \sum_{n=0}^{\infty}A_n(V)\right]\right)\right)\end{aligned} \tag{32}$$

On comparing both sides of (32), we have:

$$\begin{aligned} V_0(\varkappa) \;&=\; \sum_{k=0}^{m-1}\left[\frac{\partial^{k\sigma}V(0)}{\partial \varkappa^{k\sigma}}\frac{\varkappa^{k\sigma}}{\Gamma(1+k\sigma)}\right] + LFS_\sigma^{-1}\left(\upsilon^{m\sigma}\left(LFS_\sigma\left[g(\varkappa)\right]\right)\right),\\ V_1(\varkappa) \;&=\; - LFS_\sigma^{-1}\left(\upsilon^{m\sigma}\left(LFS_\sigma\left[R_\sigma(V_0(\varkappa)) + A_0(V)\right]\right)\right),\\ V_2(\varkappa) \;&=\; - LFS_\sigma^{-1}\left(\upsilon^{m\sigma}(LFS_\sigma\left[R_\sigma(V_1(\varkappa)) + A_1(V)\right]\right)),\\ V_3(\varkappa) \;&=\; - LFS_\sigma^{-1}\left(\upsilon^{m\sigma}\left(LFS_\sigma\left[R_\sigma(V_2(\varkappa)) + A_2(V)\right]\right)\right),\\ &\;\;\vdots \end{aligned} \tag{33}$$

The local fractional recursive relation in its general form is:

$$\begin{aligned} V_0(\varkappa) &= \sum_{k=0}^{m-1}\tfrac{\partial^{k\sigma}U(0)}{\partial \varkappa^{k\sigma}}\tfrac{\varkappa^{k\sigma}}{\Gamma(1+k\sigma)} + \; LFS_\sigma^{-1}(\upsilon^{m\sigma}(LFS_\sigma\left[g(\varkappa)\right])),\\ V_n(\varkappa) &= - \; LFS_\sigma^{-1}(\upsilon^{m\sigma}\left(LFS_\sigma\left[R_\sigma(V_{n-1}(\varkappa)) + A_{n-1}(V)\right]\right)), \end{aligned} \tag{34}$$

where $0 < \sigma \leqslant 1$ and $n, m \in \mathbb{N}^*$.

4 Applications

In this section, we will apply the method presented in the previous paragraph [31], which is "local fractional sumudu decomposition method (LFSDM)", to the Lane–Emden equation of index m with local fractional derivative and Emden–Fowler equation of index m with local fractional derivative.

Lane–Emden equation of index m with local fractional derivative

Consider the following Lane–Emden equation of index m with local fractional derivative:

$$\frac{\partial^{2\sigma}V(\varkappa)}{\partial\varkappa^{2\sigma}}+\frac{2\Gamma(1+\sigma)}{\varkappa^{\sigma}}\frac{\partial^{\sigma}V(\varkappa)}{\partial\varkappa^{\sigma}}+V^{m}(\varkappa)=0,\ 0<\sigma\leqslant 1, \tag{35}$$

with the initial conditions:

$$V(0)=1,\quad \frac{\partial^{\sigma}V(0)}{\partial\varkappa^{\sigma}}=0. \tag{36}$$

It should be noted here, that the exact solutions in the case $\sigma=1$, exist only for $m=0;1$, and 5.

To facilitate the solution of Eq. (35), we use the following transformation:

$$W(\varkappa)=\frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}V(\varkappa), \tag{37}$$

so that

$$\begin{aligned}\frac{\partial^{\sigma}W(\varkappa)}{\partial\varkappa^{\sigma}}&=\frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}\frac{\partial^{\sigma}V(\varkappa)}{\partial\varkappa^{\sigma}}+V(\varkappa),\\ \frac{\partial^{2\sigma}W(\varkappa)}{\partial\varkappa^{2\sigma}}&=\frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}\frac{\partial^{2\sigma}V(\varkappa)}{\partial\varkappa^{2\sigma}}+2\frac{\partial^{\sigma}V(\varkappa)}{\partial\varkappa^{\sigma}}.\end{aligned} \tag{38}$$

Substituting (37) and (38) into Eq.(35), gives the following new local fractional differential equation:

$$\frac{\partial^{2\sigma}W(\varkappa)}{\partial\varkappa^{2\sigma}}+\frac{\varkappa^{(1-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}W^{m}=0,\qquad m=0,1,2,..., \tag{39}$$

with the initial conditions

$$W(0)=0,\quad \frac{\partial^{\sigma}W(0)}{\partial\varkappa^{\sigma}}=1. \tag{40}$$

From (34) and (39), the successive approximations take the form:

$$\begin{aligned}W_0(\varkappa)&=\frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)},\\ W_n(\varkappa)&=-LFS_{\sigma}^{-1}\left(w^{2\sigma}\left(LFS_{\sigma}\left[\frac{\varkappa^{(1-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_{n-1}(W)\right]\right)\right),\quad n\geqslant 1,\end{aligned} \tag{41}$$

where $A_{n-1}(W)$ denotes the polynomials of Adonmian [39], and therefore, the first three terms are given as follow:

$$\begin{aligned}A_0(W)&=W_0^{m},\\ A_1(W)&=mW_1W_0^{m-1},\\ A_2(W)&=mW_2W_0^{m-1}+\frac{(m-1)m}{2}W_1^2W_0^{m-2},\\ &\vdots\end{aligned} \tag{42}$$

According to the successive formula (41), we get:

$$
\begin{array}{c}
W_0(\varkappa) = \frac{\varkappa^\sigma}{\Gamma(1+\sigma)},\\
W_1(\varkappa) = -LFS_\sigma^{-1}\left(w^{2\sigma}\left(LFS_\sigma\left[\frac{\varkappa^{(1-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_0(W)\right]\right)\right),\\
W_2(\varkappa) = -LFS_\sigma^{-1}\left(w^{2\sigma}\left(LFS_\sigma\left[\frac{\varkappa^{(1-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_1(W)\right]\right)\right),\\
W_3(\varkappa) = -LFS_\sigma^{-1}\left(w^{2\sigma}\left(LFS_\sigma\left[\frac{\varkappa^{(1-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_2(W)\right]\right)\right),\\
\vdots
\end{array} \tag{43}
$$

and so on.

From the formulas (43), the first terms of local fractional sumudu decomposition method is given by:

$$
\begin{array}{c}
W_0(\varkappa) = \frac{\varkappa^\sigma}{\Gamma(1+\sigma)},\\
W_1(\varkappa) = -\frac{\varkappa^{3\sigma}}{\Gamma(1+3\sigma)},\\
W_2(\varkappa) = m\frac{\varkappa^{5\sigma}}{\Gamma(1+5\sigma)},\\
W_3(\varkappa) = -\left[m^2 + \frac{m(m-1)}{2}\frac{\Gamma(1+5\sigma)\Gamma(1+\sigma)}{(\Gamma(1+3\sigma))^2}\right]\frac{\varkappa^{7\sigma}}{\Gamma(1+7\sigma)},\\
\vdots
\end{array} \tag{44}
$$

and so on. According to the first terms, the solution in the form of a series for the equation (35), is given by:

$$
\begin{aligned}
V(\varkappa) &= 1 - \frac{\Gamma(1+\sigma)}{\Gamma(1+3\sigma)}\varkappa^{2\sigma} + m\frac{\Gamma(1+\sigma)}{\Gamma(1+5\sigma)}\varkappa^{4\sigma}\\
&\quad - \left[m^2 + \frac{m(m-1)}{2}\frac{\Gamma(1+5\sigma)\Gamma(1+\sigma)}{(\Gamma(1+3\sigma))^2}\right]\frac{\Gamma(1+\sigma)}{\Gamma(1+7\sigma)}\varkappa^{6\sigma} + \cdots
\end{aligned} \tag{45}
$$

Even for the equation with the local fractional derivative, we will discuss the solution in the three cases mentioned above.

Case 1: For $m = 0$, the exact solution of the equation (35) is given by:

$$
V(\varkappa) = 1 - \frac{\Gamma(1+\sigma)}{\Gamma(1+3\sigma)}\varkappa^{2\sigma}.
$$

Case 2: For $m = 1$ the solution $V(\varkappa)$ in series form is given by:

$$
V(\varkappa) = \Gamma(1+\sigma)\left(1 - \frac{\varkappa^{2\sigma}}{\Gamma(1+3\sigma)} + \frac{\varkappa^{4\sigma}}{\Gamma(1+5\sigma)} - \frac{\varkappa^{6\sigma}}{\Gamma(1+7\sigma)} + \cdots\right), \tag{46}
$$

therefore the exact solution is given in the form:

$$
V(\varkappa) = \frac{\sin_\sigma(\varkappa^\sigma)}{\frac{\varkappa^\sigma}{\Gamma(1+\sigma)}}. \tag{47}
$$

Case 3: For $m = 5$ the solution $V(\varkappa)$ in series form is given by:

$$V(\varkappa) = \Gamma(1+\sigma) \left(\begin{array}{c} 1 - \frac{\varkappa^{2\sigma}}{\Gamma(1+3\sigma)} + \frac{5\varkappa^{4\sigma}}{\Gamma(1+5\sigma)} - \left[25 + \frac{10\Gamma(1+5\sigma)\Gamma(1+\sigma)}{(\Gamma(1+3\sigma))^2}\right] \\ \times \frac{\varkappa^{6\sigma}}{\Gamma(1+7\sigma)} + \cdots \end{array} \right), \quad (48)$$

In cases where $\sigma = 1$, we obtain

$$V(\varkappa) = 1 - \frac{\varkappa^2}{6} + \frac{\varkappa^4}{24} - \frac{5\varkappa^6}{432} + \cdots, \quad (49)$$

which is the same results obtained in the work presented in articles [23].

Emden–Fowler equation of index m with local fractional derivative

The Emden–Fowler equation of index m with local fractional derivative is given by:

$$\frac{\partial^{2\sigma} V(\varkappa)}{\partial \varkappa^{2\sigma}} + \frac{2\Gamma(1+\sigma)}{\varkappa^{\sigma}} \frac{\partial^{\sigma} V(\varkappa)}{\partial \varkappa^{\sigma}} + a\varkappa^{n\sigma} V^m(\varkappa) = 0, \; 0 < \sigma \leqslant 1, \quad (50)$$

and with the initial conditions:

$$V(0) = 1, \; V_{\varkappa}^{(\sigma)}(0) = 0. \quad (51)$$

It should be noted here, that the exact solutions in the case $\sigma = 1$, exist only for $m = 0; 1$, and 5.

The only value that cause us problems in solving this equation is $\varkappa = 0$, then we exclude this value and we solve the equation. To overcome this difficulty, we use the transformations (37) and (38) into (50), we have the following new local fractional differential equation:

$$\frac{\partial^{2\sigma} W(\varkappa)}{\partial \varkappa^{2\sigma}} + \frac{a\varkappa^{(1+n-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}} W^m = 0, \qquad m = 0, 1, 2, ..., \quad (52)$$

with the initial conditions

$$W(0) = 0, \;\; W_{\varkappa}^{(\sigma)}(0) = 1. \quad (53)$$

From (34) and (52), we obtain the following successive approximations:

$$\begin{aligned} W_0(\varkappa) &= \frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}, \\ W_n(\varkappa) &= -LFS_{\sigma}^{-1}\left(w^{2\sigma}\left(LFS_{\sigma}\left[\frac{a\varkappa^{(1+n-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}} A_{n-1}(W)\right]\right)\right), \;\; n \geqslant 1, \end{aligned} \quad (54)$$

where $A_{n-1}(W)$ denotes the polynomials of Adonmian [39], given by the formulas (42).

According to the successive formula (54), we get:

$$\begin{array}{c} W_0(\varkappa) = \frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}, \\ W_1(\varkappa) = -LFS_{\sigma}^{-1}\left(w^{2\sigma}\left(LFS_{\sigma}\left[\frac{a\varkappa^{(1+n-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_0(W)\right]\right)\right), \\ W_2(\varkappa) = -LFS_{\sigma}^{-1}\left(w^{2\sigma}\left(LFS_{\sigma}\left[\frac{a\varkappa^{(1+n-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_1(W)\right]\right)\right), \\ W_3(\varkappa) = -LFS_{\sigma}^{-1}\left(w^{2\sigma}\left(LFS_{\sigma}\left[\frac{a\varkappa^{(1+n-m)\sigma}}{(\Gamma(1+\sigma))^{1-m}}A_2(W)\right]\right)\right), \\ \vdots \end{array} \quad (55)$$

and so on.

From the formulas (55) and the polynomials of Adonmian (42), the first terms of local fractional sumudu decomposition method of Eq.(50), are given by:

$$\begin{array}{c} W_0(\varkappa) = \frac{\varkappa^{\sigma}}{\Gamma(1+\sigma)}, \\ W_1(\varkappa) = -a\frac{\Gamma[1+(n+1)\sigma]}{\Gamma(1+\sigma)\Gamma[1+(n+3)\sigma]}\varkappa^{(n+3)\sigma}, \\ W_2(\varkappa) = ma^2\frac{\Gamma[1+(n+1)\sigma]\times\Gamma[1+(2n+3)\sigma]}{\Gamma(1+\sigma)\Gamma[1+(n+3)\sigma]\times\Gamma[1+(2n+5)\sigma]}\varkappa^{(2n+5)\sigma}, \\ W_3(\varkappa) = -ma^3\left[m\frac{\Gamma[1+(n+1)\sigma]\times\Gamma[1+(2n+3)\sigma]}{\Gamma(1+\sigma)\Gamma[1+(n+3)\sigma]\times\Gamma[1+(2n+5)\sigma]} + \frac{m-1}{2}\frac{\Gamma[1+(n+1)\sigma]^2}{\Gamma(1+\sigma)\Gamma[1+(n+3)\sigma]^2}\right] \\ \times\frac{\Gamma[1+(3n+5)\sigma]}{\Gamma[1+(3n+7)\sigma]}\varkappa^{(3n+7)\sigma}, \\ \vdots \end{array} \quad (56)$$

and so on. Therefore, based on relationship (37), the non-differentiable solution series is given as:

$$\begin{aligned} V(\varkappa) &= 1 - a\frac{\Gamma[1+(n+1)\sigma]}{\Gamma[1+(n+3)\sigma]}\varkappa^{(n+2)\sigma} + ma^2\frac{\Gamma[1+(n+1)\sigma]\times\Gamma[1+(2n+3)\sigma]}{\Gamma[1+(n+3)\sigma]\times\Gamma[1+(2n+5)\sigma]}\varkappa^{(2n+4)\sigma} \\ &\quad -ma^3\left[m\frac{\Gamma[1+(n+1)\sigma]\times\Gamma[1+(2n+3)\sigma]}{\Gamma[1+(n+3)\sigma]\times\Gamma[1+(2n+5)\sigma]} + \frac{m-1}{2}\frac{\Gamma[1+(n+1)\sigma]^2}{\Gamma[1+(n+3)\sigma]^2}\right] \\ &\quad \times\frac{\Gamma[1+(3n+5)\sigma]}{\Gamma[1+(3n+7)\sigma]}\varkappa^{(3n+6)\sigma} + \cdots\cdots . \end{aligned} \quad (57)$$

Even for the equation with the local fractional derivative, we will discuss the solution in the three cases mentioned above.

Case 1: For $m = 0$ and $n = 0$, the exact solution of the equation (35), is given by:

$$V(\varkappa) = 1 - a\frac{\Gamma(1+\sigma)}{\Gamma(1+3\sigma)}\varkappa^{2\sigma}. \quad (58)$$

Case 2: For $n = 0$ and $m = 1$, the solution$V(r)$ in series form is given by:

$$V(\varkappa) = \Gamma(1+\sigma)\left(\frac{1}{\Gamma(1+\sigma)} - \frac{(\sqrt{a}\varkappa^{\sigma})^2}{\Gamma(1+3\sigma)} + \frac{(\sqrt{a}\varkappa^{\sigma})^4}{\Gamma(1+5\sigma)} - \frac{(\sqrt{a}\varkappa^{\sigma})^6}{\Gamma(1+7\sigma)} + \cdots\right), \quad (59)$$

therefore the exact solution of Eq.(35), it takes the following form:

$$V(\varkappa) = \frac{\sin_\sigma\left(\sqrt{a}\varkappa^\sigma\right)}{\frac{\sqrt{a}\varkappa^\sigma}{\Gamma(1+\sigma)}}. \tag{60}$$

Case 3: For $n = 0$ and $m = 5$, the solution $V(r)$ in series form, is given by:

$$V(\varkappa) = \Gamma(1+\sigma)\left(\begin{array}{c} \frac{1}{\Gamma(1+\sigma)} - a\frac{\varkappa^{2\sigma}}{\Gamma(1+3\sigma)} + a^2\frac{5\varkappa^{4\sigma}}{\Gamma(1+5\sigma)} \\ -a^3\left[25 + \frac{10\Gamma(1+5\sigma)\Gamma(1+\sigma)}{(\Gamma(1+3\sigma))^2}\right]\frac{\varkappa^{6\sigma}}{\Gamma(1+7\sigma)} + \cdots \end{array}\right), \tag{61}$$

When we substitute σ by 1 in the previous three cases, we get the same results as in articles [23].

5 Conclusion and open problem

In this work, we have applied the local fractional sumudu decomposition method (LFSDM) to solve nonlinear Lane-Emden equation of index m with local fractional derivative and nonlinear Emden-Fowler equation of index m with local fractional derivative. The method proved to be effective in solving these two equations with local fractional derivative, where we have seen that the solutions are precise and of the type of nondifferential functions. Based on these results, it can be said that this method is effective in solving nonlinear local fractional differential equations of this type.

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