Int. J. Open Problems Compt. Math., Vol. 17, No. 1, March 2024
Print ISSN: 1998-6262, Online ISSN: 2079-0376
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Analysing Edge-Equitable Antimagic connection number of certain Graph families

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Received 04 December 2023; Accepted 17 February 2024

Abstract

The edge-equitable antimagic coloring has undergone comprehensive investigation across multiple graph structures, revealing that its outcomes align optimally with its equitable chromatic index. A graph, labeled using antimagic labeling, and satisfying the conditions: (i) ***no two adjacent edges incident on a vertex share the same edge weight, and*** (ii) ***for any distinct values*** i ***and*** j***, the absolute difference between the cardinalities of the color classes associated with the edge weights is limited to no more than 1, is said to possess an edge-equitable antimagic coloring. This paper analyses the above coloring in specific graph families, that includes the triangular ladder graph*** (TL_p)***, open triangular ladder graph*** (OTL_p)***, comb graph*** (P_p^+)***, and double comb graph*** (P_p^{++})***.***

Keywords: *Antimagic labeling, Comb graph families, Equitable chromatic index, Ladder graph families.*

2010 Mathematics Subject Classification: 05C15.

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1 Introduction

This study examines a finite, connected, and undirected graph denoted by G. The set of nodes and lines in G are usually called vertices $V(G)$ of order p and edges $E(G)$ of order q. *Hartsfield* and *Ringel* [3], introduced antimagic graphs, by which, antimagic labeling is an assignment of unique positive integers to each edge of the graph, such that the sum of the labels of the vertices adjacent to each other is always different. On extending this concept to antimagic labeling of vertices, which is an assignment of unique positive integers to the vertices of a graph, such that the sum of the labels of the edges incident to any two vertices is always different. On the other hand, an equitable edge coloring refers to a graph coloring problem where the goal is to assign k colors to the edges of a graph such that each color appears on an equal number of edges. In other words, for an undirected graph $G = (V(G), E(G))$, an equitable edge coloring is a coloring of the edges with k colors such that for any two colors i and j (where $1 \leq i \leq k$ and $1 \leq j \leq k$) and $i \neq j$, the difference in the number of edges colored between i and j is atmost 1. The concept of equitable edge coloring was initially defined by *Hilton* and *de Werra* [4]. It is worth noting that finding an antimagic labeling and equitable chromatic index for a given graph is considered to be an NP-hard problem, and various algorithms have been proposed to address this issue. Significant progress in the realm of antimagic labeling can be observed in diverse works including, [1, 8, 9, 10].

The concept of local antimagic vertex coloring, which merges antimagic labeling and vertex coloring, was introduced by *Arumugam et al.*[1]. Building upon this idea, *Dafik et al.*[2] proposed a new notion of Rainbow Antimagic coloring, for which *Septory et al.*[12] provided a general lower bound. Motivated by these developments, we have put forth a novel coloring scheme termed, Edge-Equitable Antimagic coloring, which combines the principles of antimagic vertex labeling and equitable edge coloring. Noteworthy advancements in the field of equitable chromatic index can be found in various works, including [4, 5, 6, 7, 11, 13, 14, 15, 16, 17].

The initial works on fundamental graphs revealed that its outcomes align optimally with its equitable chromatic index. Further results serve as a foundation for understanding the applicability and effectiveness of our proposed coloring technique across different graph types. The paper includes the exploration of general lower bounds and provides essential proofs in the subsequent sections.

2 Preliminaries

Definition 2.1. *For a bijective function $\varsigma : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ the corresponding edge weights are defined by, $W_i(xy) = \varsigma(x) + \varsigma(y)$, such that,* *(i) If $e, e' \in E(G)$, incident on a common vertex $x \in V(G)$, then $W_i(e) \neq W_j(e')$ and*

(ii) $\left| |\mathcal{C}_{W_i}| - |\mathcal{C}_{W_j}| \right| \leq 1$, where $|\mathcal{C}_{W_i}|$ is the cardinality of the color class having i^{th} edge weight and $i \neq j$.

Lemma 2.2. *For any $p \in N$, the edge-equitable 2-coloring of a triangular ladder graph is $\chi'_e(TL_p) = 2p - 1$.*

Lemma 2.3. *For any $p \in N$, the edge-equitable 2-coloring of an open triangular ladder graph is $\chi'_e(OTL_p) = 2p - 2$.*

3 Bounds on Edge-Equitable Antimagic Coloring

Proposition 3.1. *For any connected graph G, $\chi'_{eac}(G) \geq \chi'_e(G)$, where $\chi'_e(G)$ is the equitable chromatic index of G.*

Theorem 3.2. *For any connected graph G, $\chi'_{eac}(G) \geq max\{\chi'_e(G), \Delta(G)\}$.*

Proof: Let G be a connected graph and $u, v, w \in V(G)$. Let $\varsigma : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ be a bijective function, such that $\varsigma(u) \neq \varsigma(v) \neq \varsigma(w)$. For uv, $vw \in E(G)$, incident on a common vertex $v \in V(G)$, their corresponding edge weights $W_i(uv) \neq W_j(vw)$. Hence, $\chi'_{eac}(G) \geq \Delta(G)$. Using this result and Proposition 3.1, we obtain the general lower bound, $\chi'_{eac}(G) \geq max\{\chi'_e(G), \Delta(G)\}$.

4 Main Results

Theorem 4.1. *For any positive integer $p \geq 3$, the edge-equitable antimagic connection number of a triangular ladder graph is $2p - 1 \leq \chi'_{eac}(TL_p) \leq 2p$.*

Proof: A triangular ladder graph (TL_p), is similar to that of a ladder graph with diagonal edges connecting the parallel paths with $2p$ vertices and $4p - 3$ edges. The vertex set and edge set is given by, $V(TL_p) = \{x_t, y_t : 1 \leq t \leq p\}$ and $E(TL_p) = \{x_t x_{t+1},\, y_t y_{t+1},\, x_t y_{t+1} : 1 \leq t \leq p - 1\} \cup \{x_t y_t : 1 \leq t \leq p\}$ respectively. The minimum and maximum degrees are: $\delta(TL_p) = 2$ and $\Delta(TL_p) = 4$ respectively. By Theorem 3.2 and Lemma 2.2, $\chi'_{eac}(TL_p) \geq max\{4, 2p - 1\} \geq 2p - 1$. To prove the upper bound, we consider four cases.

Case 1. *For $p \equiv 1 (mod\, 4)$*

Define an antimagic labeling $\varsigma : V(TL_p) \to \{1, 2, 3, \dots, 2p\}$ by,

$$\varsigma(x_t) = \begin{cases} 4t-3 & \text{for } t = 1, 3, 5, \dots, \frac{p+1}{2} \\ 4p-4t+3 & \text{for } t = \frac{p+5}{2}, \frac{p+9}{2}, \frac{p+13}{2}, \dots, p \\ 4t-1 & \text{for } t = 2, 4, 6, \dots, \frac{p-1}{2} \\ 4p-4t+1 & \text{for } t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \dots, p-1 \end{cases}$$

$$\varsigma(y_t) = \begin{cases} 4t & \text{for } t = 1, 3, 5, \dots, \frac{p-3}{2} \\ 4p-4t+2 & \text{for } t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \dots, p \\ 4t-2 & \text{for } t = 2, 4, 6, \dots, \frac{p-1}{2} \\ 4p-4t+4 & \text{for } t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \dots, p-1 \end{cases}$$

The total number of edge weights are obtained using n-term formula: $n = \frac{l-a}{d} + 1$. Define a positive integer N, such that, $N = |W_i| = \frac{l-a}{d} + 1$ corresponds to the cardinality of edge weights. Based on the above labelling, the edge weights and N values are:

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_1(x_t x_{t+1}) = 8t$	$1 \le t \le \frac{p-1}{2}$	$\|W_1\| = \frac{p-1}{2}$
$W_2(x_t x_{t+1}) = 8(p-t)$	$t = \frac{p+3}{2}, \frac{p+5}{2}, \frac{p+7}{2}, \dots, p-1$	in $\|W_1\|$
$W_3(x_t x_{t+1}) = 8t - 10$	$t = \frac{p+1}{2}$	1
$W_4(y_t y_{t+1}) = 8t + 2$	$1 \le t \le \frac{p-3}{2}$	in $\|W_5\|$
$W_5(y_t y_{t+1}) = 8p - 8t + 2$	$t = \frac{p+1}{2}, \frac{p+3}{2}, \frac{p+5}{2}, \dots, p-1$	$\|W_5\| = \frac{p-1}{2}$
$W_6(y_t y_{t+1}) = 8t$	$t = \frac{p-1}{2}$	in $\|W_1\|$
$W_7(x_t y_t) = 8t - 3$	$1 \le t \le \frac{p-1}{2}$	$\|W_7\| = \frac{p-1}{2}$
$W_8(x_t y_t) = 8p - 8t + 5$	$t = \frac{p+3}{2}, \frac{p+5}{2}, \frac{p+7}{2}, \dots, p$	same as $\|W_7\|$
$W_9(x_t y_t) = 8t - 5$	$t = \frac{p+1}{2}$	1
$W_{10}(x_t y_{t+1}) = 8t - 1$	$t = 1, 3, 5, \dots, \frac{p-3}{2}$	$\|W_{10}\| = \frac{p-1}{4}$
$W_{11}(x_t y_{t+1}) = 8p - 8t - 1$	$t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \dots, p-1$	same as $\|W_{10}\|$
$W_{12}(x_t y_{t+1}) = 8t + 3$	$t = 2, 4, 6, \dots, \frac{p-5}{2}; p \ge 9$	$\|W_{12}\| = \frac{p-5}{4}$
$W_{13}(x_t y_{t+1}) = 8p - 8t + 3$	$t = \frac{p+5}{2}, \frac{p+9}{2}, \frac{p+13}{2}, \dots, p-2; p \ge 9$	same as $\|W_{12}\|$
$W_{14}(x_t y_{t+1}) = 4p - 3$	$t = \frac{p-1}{2}, \frac{p+1}{2}$	1

The edge weights W_2, W_6 and W_4 are already included in W_1 and W_5 respectively. Also the edge weights W_8, W_{11} and W_{13} shares the same colors as

W_7, W_{10} and W_{12} respectively, so they are excluded as superfluous. We consider only, W_i, for $i = 1, 3, 5, 7, 9, 10, 12, 14$ and all the color classes exhibits a cardinality of atmost 2, i.e, $|\mathcal{C}_{W_i}| \leq 2$. Hence, $\sum_{i=1,3,5,7,9,10,12,14} |W_i| = 2p.$

Case 2. *For* $p \equiv 2(mod\, 4)$

Define an antimagic labeling $\varsigma : V(TL_p) \rightarrow \{1, 2, 3, \ldots, 2p\}$ by,

$$\varsigma(x_t) = \begin{cases} 4t-3 & \text{for } t = 1,3,5,\ldots,\frac{p}{2} \\ 4p-4t+1 & \text{for } t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p-1 \\ 4t-1 & \text{for } t = 2,4,6,\ldots,\frac{p-2}{2} \\ 4p-4t+3 & \text{for } t = \frac{p+6}{2}, \frac{p+10}{2}, \frac{p+14}{2}, \ldots, p \\ 2p & \text{for } t = \frac{p+2}{2} \end{cases}$$

$$\varsigma(y_t) = \begin{cases} 4t & \text{for } t = 1,3,5,\ldots,\frac{p-4}{2} \\ 4p-4t+4 & \text{for } t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p-1 \\ 4t-2 & \text{for } t = 2,4,6,\ldots,\frac{p-2}{2} \\ 4p-4t+2 & \text{for } t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \ldots, p \\ 2p-1 & \text{for } t = \frac{p}{2} \end{cases}$$

Based on the above labelling, the edge weights and N values are:

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_1(x_t x_{t+1}) = 8t$	$1 \leq t \leq \frac{p-2}{2}$	$\|W_1\| = \frac{p-2}{2}$
$W_2(x_t x_{t+1}) = 8(p-t)$	$t = \frac{p+4}{2}, \frac{p+6}{2}, \frac{p+8}{2}, \ldots, p-1$	in $\|W_1\|$
$W_3(x_t x_{t+1}) = 6p-4t-3$	$t = \frac{p}{2}, \frac{p+2}{2}$	2
$W_4(y_t y_{t+1}) = 8t+2$	$1 \leq t \leq \frac{p-4}{2}$	in $\|W_5\|$
$W_5(y_t y_{t+1}) = 8p-8t+2$	$t = \frac{p+2}{2}, \frac{p+4}{2}, \frac{p+6}{2}, \ldots, p-1$	$\|W_5\| = \frac{p-2}{2}$
$W_6(y_t y_{t+1}) = 2p+4t-3$	$t = \frac{p-2}{2}, \frac{p}{2}$	in $\|W_3\|$
$W_7(x_t y_t) = 8t-3$	$1 \leq t \leq \frac{p-2}{2}$	$\|W_7\| = \frac{p-2}{2}$
$W_8(x_t y_t) = 8p-8t+5$	$t = \frac{p+4}{2}, \frac{p+6}{2}, \frac{p+8}{2}, \ldots, p$	same as $\|W_7\|$
$W_9(x_t y_t) = 3p+2t-4$	$t = \frac{p}{2}, \frac{p+2}{2}$	2
$W_{10}(x_t y_{t+1}) = 8t-1$	$t = 1,3,5,\ldots,\frac{p-4}{2}$	$\|W_{10}\| = \frac{p-2}{4}$
$W_{11}(x_t y_{t+1}) = 8p-8t-1$	$t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p-1$	same as $\|W_{10}\|$
$W_{12}(x_t y_{t+1}) = 8t+3$	$t = 2,4,6,\ldots,\frac{p-6}{2}; p \geq 10$	$\|W_{12}\| = \frac{p-6}{4}$
$W_{13}(x_t y_{t+1}) = 8p-8t+3$	$t = \frac{p+6}{2}, \frac{p+10}{2}, \frac{p+14}{2}, \ldots, p-2; p \geq 10$	same as $\|W_{12}\|$
$W_{14}(x_t y_{t+1}) = \frac{7p}{2} + t - 5$	$t = \frac{p-2}{2}, \frac{p}{2}, \frac{p+2}{2}$	1

The edge weights W_2, W_4 and W_6 are already included in W_1, W_5 and W_3 respectively. Also the edge weights W_8, W_{11} and W_{13} shares the same colors as W_7, W_{10} and W_{12} respectively, so they are excluded as superfluous. Further, $W_{14} = \frac{7p}{2} + t - 5$, for $t = \frac{p-2}{2}, \frac{p+2}{2}$ are already included in W_5 and W_9 respectively. We consider only, W_i, for $i = 1, 3, 5, 7, 9, 10, 12, 14$ and all the color classes exhibits a cardinality of atmost 2, i.e, $|\mathcal{C}_{W_i}| \leq 2$. Hence, $\sum\limits_{i=1,3,5,7,9,10,12,14} |W_i| = 2p$.

Case 3. *For* $p \equiv 3(mod\, 4)$

Let us define an antimagic labeling $\varsigma : V(TL_p) \to \{1, 2, 3, \ldots, 2p\}$ by,

$$\varsigma(x_t) = \begin{cases} 4t - 3 & \text{for } t = 1, 3, 5, \ldots, \frac{p-1}{2} \\ 4p - 4t + 3 & \text{for } t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \ldots, p \\ 4t - 1 & \text{for } t = 2, 4, 6, \ldots, \frac{p-3}{2}; p \geq 7 \\ 4p - 4t + 1 & \text{for } t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \ldots, p - 1 \end{cases}$$

$$\varsigma(y_t) = \begin{cases} 4t & \text{for } t = 1, 3, 5, \ldots, \frac{p-1}{2} \\ 4p - 4t + 2 & \text{for } t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \ldots, p \\ 4t - 2 & \text{for } t = 2, 4, 6, \ldots, \frac{p+1}{2} \\ 4p - 4t + 4 & \text{for } t = \frac{p+5}{2}, \frac{p+9}{2}, \frac{p+13}{2}, \ldots, p - 1; p \geq 7 \end{cases}$$

Based on the above labelling, the edge weights and N values are:

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_1(x_t x_{t+1}) = 8t$	$1 \leq t \leq \frac{p-3}{2}; p \geq 7$	in $\|W_2\|$
$W_2(x_t x_{t+1}) = 8(p - t)$	$t = \frac{p+1}{2}, \frac{p+3}{2}, \frac{p+5}{2}, \ldots, p - 1$	$\|W_2\| = \frac{p-1}{2}$
$W_3(x_t x_{t+1}) = 8t - 2$	$t = \frac{p-1}{2}$	1
$W_4(y_t y_{t+1}) = 8t + 2$	$1 \leq t \leq \frac{p-1}{2}$	$\|W_4\| = \frac{p-1}{2}$
$W_5(y_t y_{t+1}) = 8p - 8t + 2$	$t = \frac{p+3}{2}, \frac{p+5}{2}, \frac{p+7}{2}, \ldots, p - 1; p \geq 7$	in $\|W_4\|$
$W_6(y_t y_{t+1}) = 8(t - 1)$	$t = \frac{p+1}{2}$	in $\|W_2\|$
$W_7(x_t y_t) = 8t - 3$	$1 \leq t \leq \frac{p-1}{2}$	$\|W_7\| = \frac{p-1}{2}$

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_8(x_t y_t) = 8p - 8t + 5$	$t = \frac{p+3}{2}, \frac{p+5}{2}, \frac{p+7}{2}, \ldots, p$	same as $\|W_7\|$
$W_9(x_t y_t) = 8t - 5$	$t = \frac{p+1}{2}$	1
$W_{10}(x_t y_{t+1}) = 8t - 1$	$t = 1, 3, 5, \ldots, \frac{p-1}{2}$	$\|W_{10}\| = \frac{p+1}{4}$
$W_{11}(x_t y_{t+1}) = 8p - 8t - 1$	$t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \ldots, p - 1$	same as $\|W_{10}\|$
$W_{12}(x_t y_{t+1}) = 8t + 3$	$t = 2, 4, 6, \ldots, \frac{p-3}{2}; p \geq 7$	$\|W_{12}\| = \frac{p-3}{4}$
$W_{13}(x_t y_{t+1}) = 8p - 8t + 3$	$t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \ldots, p - 1; p \geq 7$	same as $\|W_{12}\|$

The edge weights W_1, W_6 and W_5 are already included in W_2 and W_4 respectively. Also the edge weights W_8, W_{11} and W_{13} shares the same colors as W_7, W_{10} and W_{12} respectively, so they are excluded as superfluous. We consider only, W_i, for $i = 2, 3, 4, 7, 9, 10, 12$ and all the color classes exhibits a cardinality of atmost 2, i.e, $|\mathcal{C}_{W_i}| \leq 2$. Hence, $\sum\limits_{i=2,3,4,7,9,10,12} |W_i| = 2p$.

Case 4. *For* $p \equiv 0 (mod\, 4)$

Let us define an antimagic labeling $\varsigma : V(TL_p) \to \{1, 2, 3, \ldots, 2p\}$ by,

$$\varsigma(x_t) = \begin{cases} 4t - 3 & \text{for } t = 1, 3, 5, \ldots, \frac{p-2}{2} \\ 4p - 4t + 1 & \text{for } t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \ldots, p - 1 \\ 4t - 1 & \text{for } t = 2, 4, 6, \ldots, \frac{p}{2} \\ 4p - 4t + 3 & \text{for } t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p \end{cases}$$

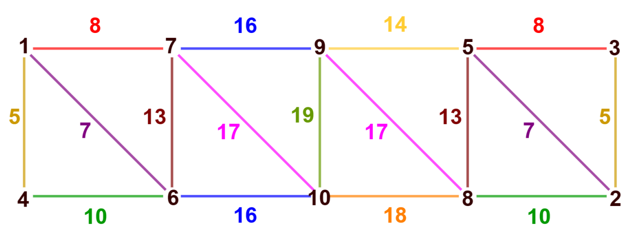
$$\varsigma(y_t) = \begin{cases} 4t & \text{for } t = 1, 3, 5, \ldots, \frac{p-2}{2} \\ 4p - 4t + 4 & \text{for } t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \ldots, p - 1 \\ 4t - 2 & \text{for } t = 2, 4, 6, \ldots, \frac{p}{2} \\ 4p - 4t + 2 & \text{for } t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p \end{cases}$$

Based on the above labelling, the edge weights and N values are:

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_1(x_t x_{t+1}) = 8t$	$1 \leq t \leq \frac{p-2}{2}$	$\|W_1\| = \frac{p-2}{2}$
$W_2(x_t x_{t+1}) = 8(p - t)$	$t = \frac{p+2}{2}, \frac{p+4}{2}, \frac{p+6}{2}, \ldots, p - 1$	same as $\|W_1\|$
$W_3(x_t x_{t+1}) = 8t - 4$	$t = \frac{p}{2}$	1

Edge Weights	Range	$N = \|W_i\| = \frac{l-a}{d} + 1$
$W_4(y_ty_{t+1}) = 8t + 2$	$1 \leq t \leq \frac{p-2}{2}$	$\|W_4\| = \frac{p-2}{2}$
$W_5(y_ty_{t+1}) = 8p - 8t + 2$	$t = \frac{p+2}{2}, \frac{p+4}{2}, \frac{p+6}{2}, \ldots, p-1$	same as $\|W_4\|$
$W_6(y_ty_{t+1}) = 8t - 2$	$t = \frac{p}{2}$	1
$W_7(x_ty_t) = 8t - 3$	$1 \leq t \leq \frac{p}{2}$	$\|W_7\| = \frac{p}{2}$
$W_8(x_ty_t) = 8p - 8t + 5$	$t = \frac{p+2}{2}, \frac{p+4}{2}, \frac{p+6}{2}, \ldots, p$	same as $\|W_7\|$
$W_9(x_ty_{t+1}) = 8t - 1$	$t = 1, 3, 5, \ldots, \frac{p-2}{2}$	$\|W_9\| = \frac{p}{4}$
$W_{10}(x_ty_{t+1}) = 8p - 8t - 1$	$t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \ldots, p-1$	same as $\|W_9\|$
$W_{11}(x_ty_{t+1}) = 8t + 3$	$t = 2, 4, 6, \ldots, \frac{p-4}{2}; p \geq 8$	$\|W_{11}\| = \frac{p-4}{4}$
$W_{12}(x_ty_{t+1}) = 8p - 8t + 3$	$t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \ldots, p-2; p \geq 8$	same as $\|W_{11}\|$
$W_{13}(x_ty_{t+1}) = 8t - 1$	$t = \frac{p}{2}$	1

The edge weights W_2, W_5, W_8, W_{10} and W_{12} shares the same colors as W_1, W_4, W_7, W_9 and W_{11} respectively, so they are excluded as superfluous. We consider only, W_i, for $i = 1, 3, 4, 6, 7, 9, 11, 13$ and all the color classes exhibits a cardinality of atmost 2, i.e, $|\mathcal{C}_{W_i}| \leq 2$. Hence, $\sum\limits_{i=1,3,4,6,7,9,11,13} |W_i| = 2p$.



$\chi'_{eac}(TL_5) = 10$

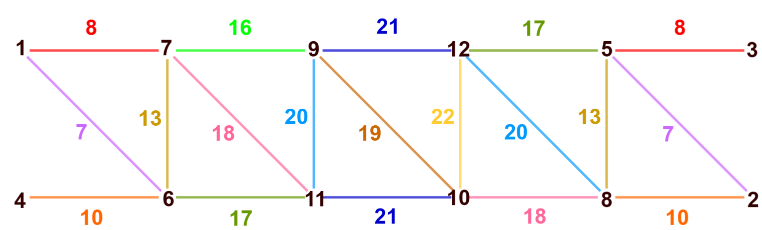
Figure 1: Edge-equitable antimagic coloring of triangular ladder graph

From Case 1, Case 2, Case 3 and Case 4, it is evident that the total number of edge weights are $2p$. Thus, $\chi'_{eac}(TL_p) \leq 2p$. On combining both the upper and lower bound, we get $2p - 1 \leq \chi'_{eac}(TL_p) \leq 2p$.

Theorem 4.2. *For any positive integer $p \geq 3$, the edge-equitable antimagic connection number of an open triangular ladder graph is $2p-2 \leq \chi'_{eac}(OTL_p) \leq 2p - 1$.*

Proof: The vertex set and edge set of an open triangular ladder is given by, $V(OTL_p) = \{x_t, y_t : 1 \leq t \leq p\}$ and $E(OTL_p) = \{x_t x_{t+1},\ y_t y_{t+1},\ x_t y_{t+1} : 1 \leq t \leq p-1\} \cup \{x_t y_t : 2 \leq t \leq p-1\}$ respectively. The minimum and maximum degrees are: $\delta(OTL_p) = 1$ and $\Delta(OTL_p) = 4$ respectively. An open triangular ladder graph involves neglecting the first and last edges from a triangular ladder graph, thus resulting in the exclusion of one edge weight 5 as in Theorem 4.1. So the total number of edge weights in an open triangular ladder is $2p-1$. The labelling of vertices for $p \equiv 0, 1, 2, 3\,(mod\,4)$ are same as in Theorem 4.1. The distinction arises in the case of $W_7(x_t y_t)$, where edge weight 5 is the only color absent from the edge weight sets of the triangular ladder graph. As a result, Lemma 2.3 proves that the edge-equitable antimagic coloring number of an open triangular ladder graph is bounded between $2p-2$ and $2p-1$.

Case	$p \equiv 0(mod4)$	$p \equiv 1(mod4)$	$p \equiv 2(mod4)$	$p \equiv 3(mod4)$
$W_7(x_t y_t) =$	$8t-3$ $(2 \leq t \leq \frac{p}{2})$	$8t-3$ $(2 \leq t \leq \frac{p-1}{2})$	$8t-3$ $(2 \leq t \leq \frac{p-2}{2})$	$8t-3$ $(2 \leq t \leq \frac{p-1}{2})$
$\lvert W_7 \rvert = \frac{l-a}{d} + 1$	$\frac{p-2}{2}$	$\frac{p-3}{2}$	$\frac{p-4}{2}$	$\frac{p-3}{2}$



$\chi'_{eac}(OTL_6) = 11$

Figure 2: Edge-equitable antimagic coloring of open triangular ladder graph

Theorem 4.3. *For any positive integer $p \geq 3$, the edge-equitable antimagic connection number of a comb graph is $\chi'_{eac}(P_p^+) = 4$.*

Proof: A comb graph (P_p^+) is a particular case of a caterpillar graph on $2p$ vertices obtained by joining the p vertices to form a path graph and the other p vertices serves as leaves of a tree. The vertex set and edge set is given by, $V(P_p^+) = \{x_t, y_t : 1 \leq t \leq p\}$ and $E(P_p^+) = \{x_t x_{t+1} : 1 \leq t \leq p-1\} \cup \{x_t y_t : 1 \leq t \leq p\}$ respectively. By Theorem 3.2, for $p \geq 3$,

$\chi'_{eac}(P_p^+) \geq max\{3,4\} \geq 4$, which forms the lower bound. To ascertain the upper bound, define an antimagic labeling $\varsigma(P_p^+) : V(P_p^+) \to \{1,2,3,\ldots,2p\}$ by:

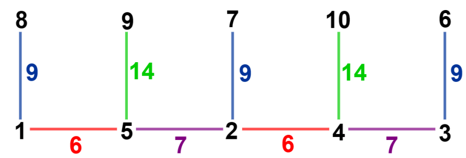
$$\varsigma(x_t) = \begin{cases} \frac{t+1}{2} & \text{for t = odd} \\ p - \frac{t}{2} + 1 & \text{for t = even} \end{cases}$$

$$\varsigma(y_t) = \begin{cases} \frac{3p-t}{2} + 1 & \text{for t = odd; p = odd} \\ \frac{3p+t+1}{2} & \text{for t = even; p = odd} \\ \frac{3p-t+1}{2} & \text{for t = odd; p = even} \\ \frac{3p+t}{2} & \text{for t = even; p = even} \end{cases}$$

Define a positive integer M, such that $M = |\mathcal{C}_{W_i}| = \frac{l-a}{d} + 1$, that corresponds to the cardinality of each color class. Based on the above labelling, the edge weights and M values are:

Edges	Edge Weights		$M = \lvert\mathcal{C}_{W_i}\rvert = \frac{l-a}{d} + 1$	
	$p = odd$	$p = even$	$p = odd$	$p = even$
$W_1(x_tx_{t+1})$	$p+1$ $(t = odd)$	$p+1$ $(t = odd)$	$\frac{p-1}{2}$	$\frac{p}{2}$
$W_2(x_tx_{t+1})$	$p+2$ $(t = even)$	$p+2$ $(t = even)$	$\frac{p-1}{2}$	$\frac{p-2}{2}$
$W_3(x_ty_t)$	$\frac{3p+3}{2}$ $(t = odd)$	$\frac{3p}{2} + 1$ $(t = odd)$	$\frac{p+1}{2}$	$\frac{p}{2}$
$W_4(x_ty_t)$	$\frac{5p+3}{2}$ $(t = even)$	$\frac{5p}{2} + 1$ $(t = even)$	$\frac{p-1}{2}$	$\frac{p}{2}$

On verifying, $\left||\mathcal{C}_{W_i}| - |\mathcal{C}_{W_j}|\right|$, for $1 \leq i \leq 4$; $1 \leq j \leq 4$ and $i \neq j$ results in 0 and 1, thus satisfying the equitable edge coloring condition. Further, the total number of edge weights are: $|W_i| = 1$, for $i = 1,2,3,4$. Hence, $\sum_{i=1}^{4} |W_i| = 4$, i.e., $\chi'_{eac}(P_p^+) \leq 4$. On combining the upper and lower bounds, $\chi'_{eac}(P_p^+) = 4$.



$\chi'_{eac}(P_5^+) = 4$

Figure 3: Edge-equitable antimagic coloring of comb graph

Theorem 4.4. *For any positive integer $p \geq 3$, the edge-equitable antimagic connection number of a double comb graph is $\chi'_{eac}(P_p^{++}) = 6$*

Proof: A double comb graph (P_p^{++}) is also a particular case of caterpillar graph on $3p$ vertices obtained through unification of two pendant edges to each vertex of a path graph. The vertex set and edge set is given by, $V(P_p^{++}) = \{x_t, y_t, z_t : 1 \leq t \leq p\}$ and $E(P_p^{++}) = \{x_t x_{t+1} : 1 \leq t \leq p-1\} \cup \{x_t y_t, x_t z_t : 1 \leq t \leq p\}$ respectively. By Theorem 3.2, for $p \geq 3$, $\chi'_{eac}(P_p^{++}) \geq max\{4, 6\} \geq 6$ forms the lower bound. To ascertain the upper bound, define an antimagic labeling $\varsigma(P_p^{++}) : V(P_p^{++}) \to \{1, 2, 3, \ldots, 3p\}$ by:

$$\varsigma(x_t) = \begin{cases} \frac{t+1}{2} & \text{for t = odd ; } \forall p \\ p - \frac{t}{2} + 1 & \text{for t = odd ; } \forall p \end{cases}$$

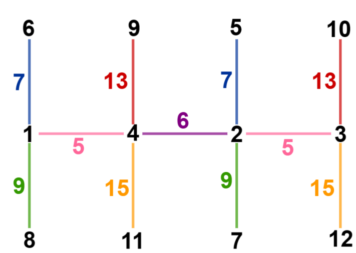
$$\varsigma(y_t) = \begin{cases} \frac{3p-t}{2} + 1 & \text{for t = odd ; p = odd} \\ 2p + \frac{t}{2} + 1 & \text{for t = even ; p = odd} \\ \frac{3p-t+1}{2} & \text{for t = odd ; p = even} \\ 2p + \frac{t}{2} & \text{for t = even ; p = even} \end{cases}$$

$$\varsigma(z_t) = \begin{cases} 2p - (\frac{t-3}{2}) & \text{for t = odd ; p = odd} \\ \frac{5p+t+1}{2} & \text{for t = even ; p = odd} \\ 2p - (\frac{t-1}{2}) & \text{for t = odd ; p = even} \\ \frac{5p+t}{2} & \text{for t = even ; p = even} \end{cases}$$

Define a positive integer M, such that $M = |\mathcal{C}_{W_i}| = \frac{l-a}{d} + 1$, that corresponds to the cardinality of each color class. Based on the above labelling, the edge weights and M values are:

Edges	Edge Weights		$M = \lvert\mathcal{C}_{W_i}\rvert = \frac{l-a}{d}+1$	
	$p = odd$	$p = even$	$p = odd$	$p = even$
$W_1(x_tx_{t+1})$	$p+1$ $(t = odd)$	$p+1$ $(t = odd)$	$\frac{p-1}{2}$	$\frac{p}{2}$
$W_2(x_tx_{t+1})$	$p+2$ $(t = even)$	$p+2$ $(t = even)$	$\frac{p-1}{2}$	$\frac{p-2}{2}$
$W_3(x_ty_t)$	$\frac{3p+3}{2}$ $(t = odd)$	$\frac{3p}{2}+1$ $(t = odd)$	$\frac{p+1}{2}$	$\frac{p}{2}$
$W_4(x_ty_t)$	$3p+2$ $(t = even)$	$3p+1$ $(t = even)$	$\frac{p-1}{2}$	$\frac{p}{2}$
$W_5(x_tz_t)$	$2p+2$ $(t = odd)$	$2p+1$ $(t = odd)$	$\frac{p+1}{2}$	$\frac{p}{2}$
$W_6(x_tz_t)$	$\frac{7p+3}{2}$ $(t = even)$	$\frac{7p}{2}+1$ $(t = even)$	$\frac{p-1}{2}$	$\frac{p}{2}$

On verifying, $\left\lvert\lvert\mathcal{C}_{W_i}\rvert - \lvert\mathcal{C}_{W_j}\rvert\right\rvert$, for $1 \leq i \leq 6$; $1 \leq j \leq 6$ and $i \neq j$ results in 0 and 1, thus satisfying the equitable edge coloring condition. Further, the total number of edge weights are: $\lvert W_i\rvert = 1$, for $i = 1, 2, 3, 4, 5, 6$. Hence, $\sum_{i=1}^{6} \lvert W_i\rvert = 6$, i.e., $\chi'_{eac}(P_p^{++}) \leq 6$. On combining the upper and lower bounds, $\chi'_{eac}(P_p^{++}) = 6$.



$\chi'_{eac}(P_4^{++}) = 6$

Figure 4: Edge-equitable antimagic coloring of double comb graph

5 Conclusion

The proposed coloring scheme introduces an innovative and unprecedented approach to the field of research, which remains wide open for exploration. The paper presents a comprehensive analysis with a specific focus on ladder and comb graph families, that includes the triangular and open triangular ladder graphs, comb and double comb graphs. Though initial investigations encompass general lower bounds further study can provide rigorous proofs of optimality for other graph structures.

6 Open Problem

The paper has explored only one particular type of caterpillar tree, though characterizing the existence of edge-equitable antimagic coloring across various trees (such as, regular trees, caterpillars, lobster trees) and multipartite graphs remains an open problem.

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