Int. J. Open Problems Compt. Math., Vol. 17, No. 1, March 2024
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Blow-up analysis for a generalized Degasperis-Procesi equation with weak dissipation term

Nurhan Dündar

The Ministry of National Education, Diyarbakir, Turkey
e-mail:nurhandundar@hotmail.com

Received 25 January 2024; Accepted 1 March 2024

Abstract

Our main purpose in this study is to examine the blow up formation of a generalized version of the Degasperis-Procesi equation, which is a shallow water wave equation, with weak dissipation term.

Keywords: *The generalized Degasperis-Procesi equation, the weak dissipation term, Blow-up.*
2010 Mathematics Subject Classification: 35A01, 35G25, 35B44.

1 Introduction

In the current work, we examine the Cauchy problem of the following generalized Degasperis-Procesi (DP) equation with the weak dissipation:

$$\begin{cases} w_t - w_{xxt} + [h(w)]_x - 3w_x w_{xx} - w w_{xxx} + \lambda(w - w_{xx}) = 0, & t > 0,\ x \in \mathbb{R}, \\ w(0, x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

where $h : \mathbb{R} \to \mathbb{R}$ is given $\mathcal{C}^b$-function, $b \geq 2$ and $\lambda(w - w_{xx})$ is the dissipative term, $\lambda > 0$ is a constant.

If we take $h(w) = 2w^2$ and $\lambda = 0$, we obtain the classical DP equation [4]

$$w_t - w_{xxt} + 4ww_x - 3w_x w_{xx} - w w_{xxx} = 0. \tag{2}$$

Eq. (2) is one of the important equations modeling shallow water waves. DP equation has Hamiltonian structure and an infinite sequence of conserved quantities and is integrable [5]. The local well-posedness, global existence, weak solutions and explosion of the solutions of a Cauchy problem for the Eq. (2) have been studied in Sobolev and Besov spaces ([9], [16], [17]). For Eq. (2), the existence and uniqueness of global weak solutions, wave breaking phenomenon and global strong solutions were discussed in ([8], [13]). Wave breaking is defined as follows: for some partial differential equations, while the solution (w) of the equation remains finite, the derivative of the solution with respect to the space variable (w_x) tends to be infinite at the moment of explosion (see [1], [2]). This type of blow-up, which takes the derivative of the solution with respect to the space variable into account, is best known as wave breaking.

Generally, energy dissipation mechanics are hard to avoid in a real world. Therefore, it makes sense to investigate the model with energy dissipation in the propagation of nonlinear waves. In recent years, it has become very important to study nonlinear wave equations with dissipative terms.

In ([15], [10]) the mathematical behaviors of the following weakly dissipative DP equation, such as local well-posedness, global existence, blow up, persistence properties and propagation speed are studied:

$$w_t - w_{xxt} + 4ww_x - 3w_x w_{xx} - ww_{xxx} + \lambda(w - w_{xx}) = 0. \tag{3}$$

Wu and Yin studied the following problem involving the generalized DP equation in [14]:

$$\begin{cases} w_t - w_{xxt} + [h(w)]_x = \gamma(3w_x w_{xx} + ww_{xxx}), & t > 0,\ x \in \mathbb{R}, \\ w\left(0, x\right) = w_0\left(x\right), & x \in \mathbb{R}. \end{cases} \tag{4}$$

They demonstrated local well-posedness of (4) and derived a precise blow-up scenario and showed a few blow-up results.

As can be seen, if $h(w) = 2w^2$ and $\lambda \neq 0$ in Eq. (1), Eq. (3) is obtained. Also, if we take $\lambda = 0$ in Eq. (1), Eq. (4) is obtained. Therefore, we can say that Eq. (1) is a more general equation.

Our main purpose in this study is to examine the mathematical properties of problem (1), which has not been examined yet.

The basic framework of the article is as follows: In the second part, we will give some preliminary information. In the third part, we will examine the blow-up phenomena for problem (1).

2 Preliminaries

First of all, we introduce the notations we will use throughout the article. The convolution is denoted by $*$. $\|.\|_{\mathfrak{B}}$ indicates the norm of Banach space $\mathfrak{B}$. Since the entire function space is on $\mathbb{R}$, we drop $\mathbb{R}$ in our function space representations for convenience if there is no ambiguity. H^ℓ is the classical Sobolev space by norm $\|.\|_{H^\ell} = \|.\|_\ell$, $\ell \in \mathbb{R}$. The norm in Lebesgue space L^p is shown $\|.\|_{L^p}$, $1 \le p \le \infty$. $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$.

Problem (1) can be rewritten as:

$$\begin{cases} w_t + ww_x = -\partial_x(1-\partial_x^2)^{-1}(h(w) - \frac{1}{2}w^2) - \lambda w, & t > 0,\ x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

It is note that $(1-\partial_x^2)^{-1} q = r * q$, $\forall q \in L^2$, where $r(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. Utilizing this identity, we rewrite problem (5) the following:

$$\begin{cases} w_t + ww_x = -\partial_x r * (h(w) - \frac{1}{2}w^2) - \lambda w, & t > 0,\ x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{6}$$

Theorem 2.1 *Suppose that $h \in \mathcal{C}^b$, $b \ge 2$. Given $w_0 \in H^\ell$, $\frac{3}{2} < \ell \le b$, there is a maximum $T = T(h, w_0, \lambda) > 0$ which is independent on ℓ, and a unique solution w to (5) (or (1)) such that*

$$w = (., w_0) \in \mathcal{C}([0,T); H^\ell) \cap \mathcal{C}^1([0,T); H^{\ell-1}).$$

Also, the solution is constantly dependent on the initial data.

The theorem above is the local well-posedness theorem. We applied Kato theory [11] to obtain this theorem. The proof is done similarly to ([6], [7], [14]) with minor changes. For this reason we skip this proof.

Now, we will list the lemmas we will use to prove our main results.

Lemma 2.2 *([12]) Let $\ell > 0$. Then we have*

$$\left\| \left[\Lambda^\ell, u\right] v \right\|_{L^2} \le k \left(\|\partial_x u\|_{L^\infty} \left\|\Lambda^{\ell-1} v\right\|_{L^2} + \left\|\Lambda^\ell u\right\|_{L^2} \|v\|_{L^\infty} \right).$$

Here k is constant depending only on ℓ.

Lemma 2.3 *([12]) Let $\ell > 0$. Then $H^\ell \cap L^\infty$ is an algebra. Moreover*

$$\|vu\|_\ell \le k \left(\|v\|_{L^\infty} \|u\|_\ell + \|v\|_\ell \|u\|_{L^\infty} \right).$$

Here k is a constant depending only on ℓ.

Lemma 2.4 *([3]) Suppose that $Q \in \mathcal{C}^{b+2}$ with $Q(0) = 0$. Then for every $\frac{1}{2} < \ell \le b$, we have*

$$\|Q(w)\|_\ell \le \tilde{Q}(\|w\|_{L^\infty}) \|w\|_\ell, \qquad w \in H^\ell$$

where $\tilde{Q}$ is a monotone increasing function depending only on Q and ℓ.

Lemma 2.5 *([1]) Let $T > 0$ and $w \in C^1([0,T); H^2)$. Then for every $t \in [0,T)$, there exist at least one pair points $\gamma(t)$, $\Gamma(t) \in \mathbb{R}$, such that*

$$z(t) = \inf_{x\in\mathbb{R}} [w_x(t,x)] = w_x(t,\gamma(t)),$$

$$Z(t) = \sup_{x\in\mathbb{R}} [w_x(t,x)] = w_x(t,\Gamma(t)),$$

and $z(t)$, $Z(t)$ are absolutely continuous in $(0,T)$. Additionally,

$$\frac{dz(t)}{dt} = w_{tx}(t,\gamma(t)), \qquad \frac{dZ(t)}{dt} = w_{tx}(t,\Gamma(t)), \qquad a.e.on \ (0,T).$$

3 Blow-up analysis

Blow up analysis for (1) is made in this section. For this purpose, our first goal will be to create a blow up scenario. Then we will present a blow-up result.

Theorem 3.1 *Assume that $h \in \mathcal{C}^{b+2}$, $b \geq 2$, and $w_0 \in H^\ell$, $\frac{3}{2} < \ell \leq b$. Then the solution w of (5) (or (1)) with the initial datum w_0 blows up in finite $T < \infty$ if and only if*

$$\overline{\lim}_{t\uparrow T}(\|w(t,x)\|_{L^\infty} + \|w_x(t,x)\|_{L^\infty}) = \infty.$$

Proof. Let w is the solution of (1) with the initial datum $w_0 \in H^\ell$, $\frac{3}{2} < \ell \leq b$, which is guaranteed by Theorem 2.1. If $\overline{\lim}_{t\uparrow T}(\|w(t,x)\|_{L^\infty} + \|w_x(t,x)\|_{L^\infty}) = \infty$, by Sobolev imbedding theorem, we acquire that the solution w will blow up in finite time.

Now, applying the operator Λ^ℓ to (5), multiplying by $\Lambda^\ell w$, and integrating by parts on $\mathbb{R}$, we have

$$\frac{d}{dt}(w,w)_\ell = -2(ww_x, w) + 2(a(w), w)_\ell, \tag{7}$$

where $a(w) = -\partial_x(1-\partial_x^2)^{-1}(h(w) - h(0) - \frac{w^2}{2}) - \lambda w$. Suppose there exists a $K > 0$, such that $\overline{\lim}_{t\uparrow T}(\|w(t,x)\|_{L^\infty} + \|w_x(t,x)\|_{L^\infty}) \leq K$. At that case, we get

$$\begin{aligned}
|(ww_x, w)| &= |(\Lambda^\ell(ww_x), \Lambda^\ell w)_0| \\
&= |([\Lambda^\ell, w]w_x, \Lambda^\ell w)_0 + (w\Lambda^\ell w_x, \Lambda^\ell w)_0| \\
&\leq \|[\Lambda^\ell, w]w_x\|_{L^2}\|\Lambda^\ell w\|_{L^2} + \frac{1}{2}\|w_x\|_{L^\infty}\|\Lambda^\ell w\|_{L^2}^2 \\
&\leq k(\|w_x\|_{L^\infty}\|\Lambda^{\ell-1}w_x\|_{L^2} + \|\Lambda^\ell w\|_{L^2}\|w_x\|_{L^\infty})\|w\|_\ell \\
&+ \frac{1}{2}\|w_x\|_{L^\infty}\|w\|_\ell^2 \\
&\leq k\|w_x\|_{L^\infty}\|w\|_\ell^2 \leq kK\|w\|_\ell^2,
\end{aligned} \tag{8}$$

where we used Lemma 2.2.

Now, we estimate the $(a(w), w)_\ell$.

$$\begin{aligned}(a(w), w)_\ell &= (-\partial_x(1-\partial_x^2)^{-1}(h(w)-h(0)-\frac{w^2}{2})-\lambda w, w)_\ell \\ &\le k\|w\|_\ell(\|h(w)-h(0)\|_{\ell-1}+\frac{1}{2}\|w^2\|_{\ell-1}+\lambda\|w\|_\ell) \\ &\le k(\tilde{Q}(\|w\|_{L^\infty})\|w\|_{\ell-1}+\|w\|_{L^\infty}\|w\|_{\ell-1}+\lambda\|w\|_\ell)\|w\|_\ell \\ &\le k((\tilde{Q}(\|w\|_{L^\infty})+\|w\|_{L^\infty}+\lambda)\|w\|_\ell^2 \\ &\le k(\tilde{Q}(K)+K+\lambda)\|w\|_\ell^2, \end{aligned} \tag{9}$$

where we used Lemma 2.4 with $Q(w) = h(w) - h(0)$ and Lemma 2.3. From (7)-(9), we have

$$\frac{d}{dt}\|w\|_\ell^2 \le k(\tilde{Q}(K)+K+\lambda)\|w\|_\ell^2.$$

Applying Gronwall's inequality to this inequality, we obtain the following inequality, which ends Theorem 3.1

$$\|w(t)\|_\ell^2 \le \|u_0\|_\ell^2 \exp(k(\tilde{Q}(K)+K+\lambda))t.$$

Theorem 3.2 *Suppose that $h \in \mathcal{C}^{b+2}$, $b \ge 3$. Given $w_0 \in H^\ell$, $3 \le \ell \le b$. If there is a $K > 0$ such that $\|w\|_{L^\infty} \le K$, $\forall t \in [0, T)$, then the solution w of (1) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\uparrow T}\inf(\inf_{x\in\mathbb{R}} w_x(t,x)) = -\infty.$$

Proof. If the slope of the solution w becomes unbounded from below in finite time, then by Theorem 2.1 and Sobolev imbedding theorem, we get the solution w will blow up in finite time.

Next, if the slope of solution is bounded from below in finite time, then we understand that the solution will not blow up in finite time. Differentiating (6) with respect to x, in view of the identity $\partial_x^2(r*q) = r*q-q$, we get

$$w_{tx} + w_x^2 + ww_{xx} = -\partial_x^2[r*(h(w)-\frac{1}{2}w^2)] - \lambda w_x.$$

Then

$$w_{tx} = -w_x^2 - ww_{xx} + h(w) - \frac{w^2}{2} - r*(h(w)-\frac{1}{2}w^2) - \lambda w_x.$$

By Young's inequality, we obtain

$$\|r*h(w)\|_{L^\infty} \le \|r\|_{L^1}\|h(w)\|_{L^\infty} \le \|h(w)\|_{L^\infty} \le \sup_{|v|\le\|w\|_{L^\infty}} |h(v)| \tag{10}$$

and

$$\|r * w^2\|_{L\infty} \leq \|r\|_{L^1}\|w^2\|_{L^\infty} \leq \|w\|^2_{L^\infty}. \tag{11}$$

Define $Z(t) = w_x(t, \Gamma(t)) = \sup_{x\in\mathbb{R}} w_x(t,x)$. Since $w_{xx}(t,\Gamma(t)) = 0$, $\forall t \in [0,T)$, it follows that a.e. on $[0,T)$

$$Z'(t) = -Z^2(t) - \lambda Z(t) + h(w(t,\Gamma(t))) - \frac{1}{2}w^2(t,\Gamma(t)) - r * (h(w) - \frac{1}{2}w^2).$$

By (10) and (11), we obtain

$$Z'(t) \leq -Z^2(t) - \lambda Z(t) + \frac{1}{2}\|w\|^2_{L^\infty} + 2 \sup_{|v|\leq\|w\|_{L^\infty}} |h(v)|.$$

Set $\beta^2 = \frac{1}{2}\|w\|^2_{L^\infty} + 2\sup_{|v|\leq\|w\|_{L^\infty}} |h(v)|$. Then we get

$$Z'(t) \leq -Z^2(t) - \lambda Z(t) + \beta^2 = -\frac{1}{4}(2Z(t)+\lambda-\sqrt{\lambda^2+4\beta^2})(2Z(t)+\lambda+\sqrt{\lambda^2+4\beta^2})$$

If $Z(t) > -\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2+\beta^2}$, we get that $Z'(t) < 0$ which gives that $Z(t)$ is decreasing. Otherwise $Z(t) \leq -\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2+\beta^2}$. Thus, we obtain

$$z(t) \leq Z(t) \leq \max\{Z(0), -\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2+\beta^2}\}, \qquad t \in [0,T). \tag{12}$$

By Theorem 3.1 and from (12), we obtain that if w_x is bounded from below, then the solution of problem (1) will not blow up in finite time.

Next, we present a blow-up result.

Theorem 3.3 *Let $w_0 \in H^\ell$, $\ell > \frac{3}{2}$. Suppose that w_0 is odd and $w_0'(0) < -\lambda$. Given $h \in \mathcal{C}^b$, $b \geq 3$, h is even. If $r * (h(w) - h(0) - \frac{1}{2}w^2)(t,x) \geq 0$, then the solution w with initial data w_0 to (1) blows up in finite time.*

Proof. By Theorem 2.1 and a simple density argument, we only to prove that the theorem holds for $\ell = 3$. Let T be the maksimum time of the existence of the solution $w \in \mathcal{C}([0,T);H^\ell) \cap \mathcal{C}^1([0,T);H^{\ell-1})$ with the initial data w_0 of (1). Note that (1) has the symmetry $(w,x) \to (-w,-x)$. If $w_0(x)$ is odd, then $w(t,x)$ is odd for any $t \in [0,T)$. Since $\ell = 3$, the functions w and w_{xx} are continuous in x. So, we possess

$$w(t,0) = w_{xx}(t,0) = 0, \qquad \forall t \in [0,T).$$

Differentiating (6) with respect to x, we get

$$w_{tx}(t,x) + w_x^2(t,x) + ww_{xx}(t,x) = -\partial_x^2[r * (h(w) - h(0) - \frac{1}{2}w^2)](t,x) - \lambda w_x(t,x).$$

Let $f(t) = w_x(t, 0)$, $t \in [0, T)$. Using $\partial_x^2(r * q) = r * q - q$, we get

$$\frac{df(t)}{dt} = -f^2(t) - \lambda f(t) - r * (h(w) - h(0) - \frac{1}{2}w^2)(t, 0).$$

Since $r * (h(w) - h(0) - \frac{1}{2}w^2)(t, 0) \geq 0$, we obtain

$$\frac{df(t)}{dt} \leq -f^2(t) - \lambda f(t) = -(f(t) + \lambda)f(t), \qquad t \in [0, T).$$

From the hypothesis, we have $f(0) < -\lambda$. Therefore, $f(t) < -\lambda$, $\forall t \in [0, T)$. Solving the above inequality, we obtain

$$1 - \frac{f(0)}{f(0) + \lambda}e^{-\lambda t} \leq \frac{\lambda}{f(t) + \lambda} \leq 0.$$

Since

$$\frac{f(0)}{f(0) + \lambda} > 1,$$

we deduce that there exists T and

$$T \leq \frac{1}{\lambda} \ln \frac{f(0)}{f(0) + \lambda}$$

such that $\lim_{t \uparrow T} f(t) = -\infty$. This ends the proof of the theorem.

4 Open Problem

We examined the blow up formation of a generalized version of the Degasperis-Procesi equation, which is a shallow water wave equation, with the weak dissipation term. The open problem here is that are there global strong solutions to (1)?

References

[1] A. Constantin, J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., 181 (1998), 229–243.

[2] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Rat. Mech. Anal., 192 (2009), 165–186.

[3] A. Constantin and L. Molinet, *The initial value problem for a generalized Boussinesq equation*, Differential and Integral Equations, 15(9) (2002), 1061–1072.

[4] A. Degasperis and M. Procesi, *Asymptotic integrability*, Symmetry and Perturbation Theory, (World Scientific, Singapore), (1999), 23–37.

[5] A. Degasperis, D.D. Holm and A.N.W. Hone, *A new integral equation with peakon solutions*, Theo. Math. Phys., 133 (2002), 1463–1474.

[6] N. Dündar and N. Polat, *Local well-posedness for a generalized integrable shallow water equation with strong dispersive term*, Appl. Math. Inf. Sci. 15 (4) (2021), 429–436.

[7] N. Dündar and N. Polat, *Local well-posedness for a generalized two-component Degasperis-Procesi system*, Int. J. Open Problems Compt. Math., 14 (2) (2021), 32–40.

[8] J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal., 241 (2006), 457–485.

[9] G. Gui and Y. Liu, *On the Cauchy problem for the Degasperis-Procesi equation*, Quart. Appl. Math., 69 (2011), 445–464.

[10] Z. Guo, *Some properties of solutions to the weakly dissipative Degasperis-Procesi equation*, J. Differ. Equ., 246 (2009), 4332–4344.

[11] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in: Spectral Theory and Differential Equations, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 448 (1975), 25-70.

[12] T. Kato and G. Ponce *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., 41(7) (1988), 891–907.

[13] Y. Liu and Z. Yin, *Global existence and blow-up phenomena for the Degasperis-Procesi equation*, Comm. Math. Phys., 267 (2006), 801–820.

[14] X. Wu and Z. Yin, *Well-posedness and blow-up phenomena for the generalized Degasperis–Procesi equation*, Nonlinear Analysis: Theory, Methods and Applications, 73 (2010), 136–146.

[15] S. Wu and Z. Yin, *Blow-up and decay of the solution of the weakly dissipative Degasperis–Procesi equation*, SIAM J. Math. Anal. 40 (2) (2008), 475–490

[16] Z. Yin, *Global existence for a new periodic integrable equation*, J. Math. Anal. Appl., 49 (2003), 129–139.

[17] Z. Yin, *On the Cauchy problem for an integrable equation with peakon solutions*, Illinois Journal of Mathematics, 47 (2003), 649–666.