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Univalence Criteria for Analytic Functions related to Hurwitz Lerch Zeta function

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Abstract
In this paper we obtain sufficient condition for univalence of analytic functions defined by Hurwitz-Lerach Zeta function.

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1 Introduction

Let A denote the class of analytic functions f defined on the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U. For $f \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in U). \tag{3}$$

Note that $f * g \in A$.

In geometric function theory, the univalence of complex functions is an important property, but it is difficult, and in many cases impossible, to show directly that a certain complex function is univalent. For this reason, many authors found different types of sufficient conditions of univalence. One of the most important of these conditions of univalence in the domains E and the exterior of a closed unit disc is the well-known criterion of Becker [5]. Becker's work depends upon a clever use of the theory of Loewner chains and the generalized Loewner differential equation. Extensions of this criterion were given by Deniz and Orhan [9], Ali et al. [2] and Nehari [14].
In [13] Mustafa and Darus hzve recently introduced a new generalized integral operator $\mathfrak{J}^{\alpha}_{\mu,b} f(z)$ as we show in the following:

Definition 1.1 *A general Hurwitz- Lerch Zeta function $\Phi(z, \mu, b)$ defined by*

$$\Phi(z, \mu, b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^{\mu}},$$

where $(\mu \in C, b \in C - Z_0^-)$ when $|z| < 1$, and $R(b) > 1$ when $(|z| = 1)$.

We define the function

$$\Phi^*(z, \mu, b) = (b^{\mu} z \Phi(z, \mu, b)) * f(z),$$

then

$$\Phi^*(z, \mu, b) = z + \sum_{n=2}^{\infty} \frac{a_n}{(n+b-1)^{\mu}} z^n$$

Definition 1.2 *Let the function f be analytic in a simply connected domain of the z-plane containg the orgin. The fractional derivative of f of order α is defined by*

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \quad (0 \leq \alpha < 1),$$

where the multiplycity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Using Definition 1.2 and and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [9] introduced the operator $\Omega^{\alpha} : A \to A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned}\Omega^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \quad (\alpha \neq 2,3,4,...)\\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, \quad (z \in U)\end{aligned}$$

For $\alpha \in C, b \in C - Z_0^-$, and $0 \leq \alpha < 1$, the generalized integral operator $\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} f : A \to A$, is defined by

$$\begin{aligned}\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z,\alpha,b), \quad (\alpha \neq 2,3,4,...)\\ &= z + \sum_{n=2}^{\infty} \Phi_n(\mu,b,\alpha)\ a_n z^n, \quad (z \in U).\end{aligned}$$

where $\Phi_n(\mu,b,\alpha) = \frac{\Gamma(n+1) z^{\alpha} D_z^{\alpha} \Phi^*(z,\alpha,b)}{\Gamma(n+1-\alpha)} \left(\frac{b}{n-1+b}\right)^{\mu}$
Note that : $\mathfrak{J}_{\mathfrak{o},\mathfrak{b}}^{\mathfrak{o}} f(z) = f(z)$.
Special cases of this operator include :

(i). $\mathfrak{J}_{\mathfrak{o},\mathfrak{b}}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z)$ is Owa and Srivastava operator [16].

(ii). $\mathfrak{J}_{\mu,\mathfrak{b}+\mathfrak{1}}^{\mathfrak{o}} f(z) \equiv J_{\mu,b} f(z)$ is the Srivastava and Attiya integral operator[18].

(iii). $\mathfrak{J}_{\mathfrak{1},\mathfrak{1}}^{\mathfrak{o}} f(z) \equiv A(f)(z)$ is the Alexander integral operator [1].

(iv). $\mathfrak{J}_{\mu+\mathfrak{1},\mathfrak{1}}^{\mathfrak{o}} f(z) \equiv L(f)(z)$ is the Libera integral operator [12].

(v). $\mathfrak{J}_{\mathfrak{1},\delta}^{\mathfrak{o}} f(z) \equiv L_{\delta}(f)(z)$ is the Bernardi integral operator [6].

(vi). $\mathfrak{J}_{\sigma,\mathfrak{2}}^{\mathfrak{o}} f(z) \equiv I^{\sigma} f(z)$ is the Jung-Kim-Kim-Srivastava integral operator [11].

Now, by making use of the Hurwitz - Lerch zeta operator $\mathfrak{J}_{\mu,b}^{\alpha} f$, we define a new subclass of functions belonging to the class A. In this paper we derive sufficient conditions of univalence for the generalized operator $\mathfrak{J}_{\mu,b}^{\alpha} f(z)$. Also, a number of known univalent conditions would follow upon specializing the parameters involved. In order to prove our results wee need the following Lemmas.

Lemma 1.3 *[5] Let $f \in A$. If for all $z \in E$*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{4}$$

then the function f is univalent in E.

Lemma 1.4 *[18] Let $f \in A$. If for all $z \in E$*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1 \tag{5}$$

then the function f is univalent in E.

Lemma 1.5 *[19] Let μ be a real number $\mu > \frac{1}{2}$ and $f \in A$. If for all $z \in E$*

$$(1 - |z|^{2\mu})\left|\frac{zf''(z)}{f'(z)} + 1 - \mu\right| \leq \mu \tag{6}$$

then the function f is univalent in E.

Lemma 1.6 *[10] If $f \in S$ (the class of univalent functions) and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{7}$$

then $\sum\limits_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1$.

Lemma 1.7 *[17] Let $\nu \in C, Re\{\nu\} \geq 0$ and $f \in A$. If for all $z \in E$*

$$\frac{1 - |z|^{2Re(\nu)}}{Re(\nu)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{8}$$

then the function

$$F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u)du\right)^{\frac{1}{\nu}}$$

is univalent in E.

2 Main Results

In this section, we establish the sufficient conditions to obtain a univalence for analytic functions involving the differential operator.

Theorem 2.1 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} \Phi_n(\mu, b, \alpha)[n(2n-1)]|a_n| \leq 1 \tag{9}$$

then $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}} f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^2)\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}\right| \le (1+|z|^2)\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}\right|$$
$$\le \frac{2\sum\limits_{n=2}^{\infty} n(n-1)\Phi_n(\mu,b,\alpha)|a_n|}{1-\sum\limits_{n=2}^{\infty} n\Phi_n(\mu,b,\alpha)|a_n|}$$

the last inequality is less than 1 if the assertion (9) is hold. Thus is view of Lemma 1.3, $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

Theorem 2.2 *Let $f \in A$. If for all $z \in E$*

$$\Phi_n(\mu,b,\alpha)|a_n| \le \frac{1}{\sqrt{7}} \tag{10}$$

then $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

Let $f \in A$. It sufficient to show that

$$\left|\frac{z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}{2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))^2}\right| \le 1.$$

Now

$$\left|\frac{z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}{2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))^2}\right| \le \frac{1+\sum\limits_{n=2}^{\infty} n\Phi_n(\mu,b,\alpha)|a_n|}{2(1-2\sum\limits_{n=2}^{\infty}[\Phi_n(\mu,b,\alpha)])^m|a_n| - (\sum\limits_{n=2}^{\infty}\Phi_n(\mu,b,\alpha)|a_n|^2)}.$$

The last inequality is less than 1 if the assertion (10) is hold. Thus in view of Lemma 1.4, $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

Theorem 2.3 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1)+(2\mu-1)]\Phi_n(\mu,b,\alpha)|a_n| \le 2\mu-1,\ \ \mu > \frac{1}{2} \tag{11}$$

then $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$(1-|z|^{2\mu})\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}+1-\mu\right| \le (1+|z|^2)\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'}\right| + |1-\mu|$$
$$\le \frac{2\sum\limits_{n=2}^{\infty}\Phi_n(\mu,b,\alpha)[n(n-1)]|a_n|}{1-\sum\limits_{n=2}^{\infty} n\Phi_n(\mu,b,\alpha)|a_n|} + |1-\mu|$$

the last inequality is less than μ if the assertion (11) is hold. Thus is view of Lemma 1.5, $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

As applications of Theorems 2.1, 2.2 and 2.3, we have the following Theorem.

Theorem 2.4 *Let $f \in A$. If for all $z \in E$ one of the inequality (9-11) holds then*

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1, \qquad (12)$$

where $\frac{z}{\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$.

Proof. Let $f \in A$. Then in view of Theorems 2.1, 2.2 or 2.3, $\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)$ is univalent in E.

Hence by Lemma 1.6, we obtain the result.

Theorem 2.5 *Let $f \in A$. If for all $z \in E$*

$$\sum_{n=1}^{\infty} n[2(n-1) + Re(v)]\Phi_n(\mu, b, \alpha)|a_n| \leq Re(v), \;\; Re(v) > 0 \qquad (13)$$

then

$$G_v(z) = \left(v \int_0^z u^{v-1}[\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z)]' du \right)^{\frac{1}{v}}$$

is univalent in E.

Let $f \in A$. Then for all $z \in E$,

$$\frac{1-|z|^{2Re(v)}}{Re(v)} \left| \frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'} \right| \leq \frac{1+|z|^{2Re(v)}}{Re(v)} \left| \frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))''}{(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}f(z))'} \right|$$
$$\leq \frac{2\sum\limits_{n=2}^{\infty} n(n-1)\Phi_n(\mu, b, \alpha)|a_n|}{1 - \sum\limits_{n=2}^{\infty} n\Phi_n(\mu, b, \alpha)|a_n|}$$

the last inequality is less than 1 if the assertion (13) is hold. Thus is view of Lemma 1.7, $G_v(z)$ is univalent in E.

3 Conclusion

Special functions such as HurwitzLerch zeta functions have been continuously developed. Indeed, the theme of developments formula for Hurwitz- Lerach Zeta functions and correlated functions has a long history, which can be traced back to Goldbach and Euler. We obtained univalence conditions for analytic functions connected with Hurwitz- Lerach Zeta functions.The proposed operators can be employed to generalize other types of convolution, differential, and integral operators such as fractional operators or to establish several classes of normalized regular functions.

4 Open Problem

The authors suggest to find necessary and sufficient conditions for negative coefficients and study geometric and algebric properties, partial sums, subordination and neighborhood results.

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