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On $\alpha-$Centralizer Mappings in Semiprime Rings

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Abstract

Let** R **be a** $2-$**torsion free semiprime ring,** α **an automorphism and** U **a noncentral square-closed Lie ideal of** R**. An additive mapping** $T : R \to R$ **is called a left (resp. right)** $\alpha-$**centralizer of R if** $T(xy) = T(x)\alpha(y)$ **(resp.** $T(xy) = \alpha(x)T(y)$**) holds for all** $x, y \in R$**. In this paper, we proved the following result: i) If** $T(uvu) = \alpha(u)T(v)\alpha(u)$ **for all** $u, v \in U$**, then** T **is a left** $\alpha-$**centralizer, ii) If** $T(uvu) = T(u)\alpha(vu)$ **(resp.** $T(uvu) = \alpha(uv)T(u)$**) for all** $u, v \in U$**, then** T **is a left (resp. right)** α**-centralizer on** U**.

Keywords: *Semiprime Ring, Lie Ideal, Centralizer, Jordan Centralizer,* $\alpha-$*Centralizer,* $\alpha-$*Jordan Centralizer.*
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1 Introduction

Throughout, R will represent an associative ring with center Z. Recall that a ring R is prime if $xRy = (0)$ implies $x = 0$ or $y = 0$, and semiprime if $xRx = (0)$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$, $r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$. According Zalar [13], an additive mapping $T : R \to R$ is called a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both left as well right centralizer, then it is called a centralizer. This concept appears naturally C^*-algebras. In ring theory, it is more common to work with module homorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself instead of a left centralizer. In case $T : R \to R$ is a centralizer, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ and $\lambda \in C$, where C is the extended centroid of R. A left (resp. right) Jordan centralizer $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Zalar proved that any left (right)Jordan centralizer on a $2-$torsion free semiprime ring is a left (right) centralizer.

Recently, in [1], Albaş introduced the definition of $\alpha-$centralizer of R, i. e. an additive mapping $T : R \to R$ is called a left (resp. right) $\alpha-$centralizer of R if $T(xy) = T(x)\alpha(y)$ (resp. $T(xy) = \alpha(x)T(y)$) holds for all $x, y \in R$, where is α an endomorphism of R. If T is left and right $\alpha-$centralizer then it is natural to call $\alpha-$centralizer. Clearly every centralizer is a special case of a $\alpha-$centralizer with $\alpha = id_R$. Also, an additive mapping $T : R \to R$ associated with a homomorphism $\alpha : R \to R$ if $L_a(x) = a\alpha(x)$and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left $\alpha-$centralizer and R_a is a right $\alpha-$centralizer. A left (resp. right) Jordan $\alpha-$centralizer $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)\alpha(x)$ (resp. $T(x^2) = \alpha(x)T(x)$) holds for all $x \in R$, where is α an endomorphism of R. Albaş generalized the result of Zalar as follows: If $\alpha(Z) = Z$ then each left Jordan $\alpha-$centralizer of R is a left $\alpha-$centralizer. In [4], Cortes and Haetinger proved this question changing the semiprimality condition on R by the existence of a commutator right (resp. left) nonzero divisor. By removing these conditions, Koç and Gölbaşı have proved this theorem for the Lie ideal of the semiprime ring in [8].

On other hand, if $T : R \to R$ is a centralizer, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = xT(y)x, \text{ for all } x, y \in R.$$

It seems natural to ask whether the converse is true. More precisely, asking for whether an additive mapping T on a ring R satisfying the above relation is a centralizer. In [16], Vukman proved that the answer is affirmative in case R is a $2-$torsion free semiprime ring. The first aim of the present article is a generalization of above result to the case $\alpha-$centralizer of Lie ideal on semiprime rings.

In [9], Molnar proved that if R is a 2-torsion free prime ring and $T : R \to R$

is an additive function such that

$$T(xyx) = T(x)yx \text{ for all } x, y \in R$$

then T is a left (right) centralizer. Ali and Haetinger generalized the above mentioned result for semiprime rings in [3]. The second aim of this paper is a generalization of above result to the case $\alpha-$centralizer of Lie ideal on semiprime rings.

2 Results

Lemma 2.1 *[6, Corollary 2.1] Let R be a $2-$torsion free semiprime ring, U a Lie ideal of R such that $U \not\subseteq Z(R)$ and $a, b \in U$.*
i) If $aUa = (0)$, then $a = 0$.
ii) If $aU = (0)$ (or $Ua = (0)$), then $a = 0$.
iii) If U is square closed and $aUb = (0)$, then $ab = 0$ and $ba = 0$.

Lemma 2.2 *[8, Theorem 1] Let R be a $2-$torsion free semiprime ring, U a square closed Lie ideal of R, α an automorphism of R and $T : R \to R$ a left (resp. right) Jordan $\alpha-$centralizer mapping of U into R. Then T is a left (resp. right) $\alpha-$centralizer mapping of U into R.*

Throughout the study, since R is $2-$torsion free ring, uv will be written instead of $2uv$ for each $u, v \in U$ in order to facilitate the equations.

Lemma 2.3 *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $a, b, c \in U$. If $avb + bvc = 0$ for all $v \in U$, then $(a + c)vb = 0$ for all $v \in U$.*

Proof. By the hypothesis, we get

$$avb + bvc = 0, \text{ for all } v \in U.$$

Replacing v by vbu in this equation, we have

$$avbub + bvbuc = 0.$$

Multiplying the hypothesis on the right by ub, we obtain that

$$avbub + bvcub = 0.$$

If the last two equations are subtracted from each other, we find that

$$bv(buc - cub) = 0. \tag{1}$$

Replacing v by ucv in the last equation, we get

$$bucv(buc - cub) = 0.$$

Multiplying (1) on the left by cu, we have

$$cubv(buc - cub) = 0.$$

If the last two equations are subtracted from each other, we find that

$$(buc - cub)v(buc - cub) = 0.$$

By Lemma 2.1, we get $buc = cub$. Using the hypothesis, we obtain that $avb + cvb = 0$. That is, $(a + c)vb = 0$ for all $v \in U$.

Theorem 2.4 *Let R be a $2-$torsion free semiprime ring, U a noncentral square-closed Lie ideal of R, α an automorphism of R and $\alpha(U) = U$, $T(U) \subset U$. If $T : R \to R$ is an additive mapping such that $T(uvu) = \alpha(u)T(v)\alpha(u)$ for all $u, v \in U$, then T is a left $\alpha-$centralizer.*

Proof. By the hypothesis, we get

$$T(uvu) = \alpha(u)T(v)\alpha(u), \text{ for all } u, v \in U. \tag{2}$$

Replacing u by $u + w, w \in U$ in the last equation, we have

$$T((u + w)v(u + w)) = \alpha(u + w)T(v)\alpha(u + w)$$

and so,

$$\begin{aligned} T(uvu + uvw + wvu + wvw) = {} & \alpha(u)T(v)\alpha(u) + \alpha(u)T(v)\alpha(w) \\ & + \alpha(w)T(v)\alpha(u) + \alpha(w)T(v)\alpha(w). \end{aligned}$$

Using the hypothesis, we obtain that

$$T(uvw + wvu) = \alpha(u)T(v)\alpha(w) + \alpha(w)T(v)\alpha(u). \tag{3}$$

Taking v by u and w by v respectively in equation (3), we have

$$T\left(u^2v + vu^2\right) = \alpha(u)T(u)\alpha(v) + \alpha(v)T(u)\alpha(u). \tag{4}$$

Thus, replacing w by u^3 in equation (3), we see that

$$T\left(uvu^3 + u^3vu\right) = \alpha(u)T(v)\alpha\left(u^3\right) + \alpha\left(u^3\right)T(v)\alpha(u). \tag{5}$$

Writting v by uvu in equation (4), we have

$$T\left(u^3vu + uvu^3\right) = \alpha(u)T(u)\alpha(uvu) + \alpha(uvu)T(u)\alpha(u). \tag{6}$$

Replacing v by u^2v+vu^2 in the hypothesis, we have

$$T\left(u\left(u^2v+vu^2\right)u\right)=\alpha\left(u\right)T\left(u^2v+vu^2\right)\alpha\left(u\right)$$

and so,

$$T\left(u^3vu+uvu^3\right)=\alpha\left(u\right)T\left(u^2v+vu^2\right)\alpha\left(u\right). \tag{7}$$

If (4) equation is used in (7) equation, we get

$$T\left(u^3vu+uvu^3\right)=\alpha\left(u^2\right)T\left(u\right)\alpha\left(vu\right)+\alpha\left(uv\right)T\left(u\right)\alpha\left(u^2\right). \tag{8}$$

If (6) and (8) equations are used, we obtain that

$$\alpha\left(u\right)T\left(u\right)\alpha\left(uvu\right)+\alpha\left(uvu\right)T\left(u\right)\alpha\left(u\right)=\alpha\left(u^2\right)T\left(u\right)\alpha\left(vu\right)+\alpha\left(uv\right)T\left(u\right)\alpha\left(u^2\right).$$

That is,

$$\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u\right)+\alpha\left(u\right)\alpha\left(v\right)\left[\alpha\left(u\right),T\left(u\right)\right]\alpha\left(u\right)=0.$$

That is,

$$\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u\right)-\alpha\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u\right)=0. \tag{9}$$

Since $\alpha(U)=U$, we get

$$\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]v\alpha\left(u\right)-\alpha\left(u\right)v\left[T\left(u\right),\alpha\left(u\right)\right]\alpha(u)=0.$$

By Lemma 2.3, we get

$$(\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\left[T\left(u\right),\alpha\left(u\right)\right]\alpha(u))v\alpha\left(u\right)=0.$$

That is,

$$\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]v\alpha\left(u\right)=0. \tag{10}$$

Replacing v by $v\left[T\left(u\right),\alpha\left(u\right)\right]$ in equation (10), we have

$$\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]v\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u\right)=0. \tag{11}$$

Multiplying (10) on the right by $\left[T\left(u\right),\alpha\left(u\right)\right]$, we obtain that

$$\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]v\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]=0. \tag{12}$$

Subtracting (11) from (12), we arrive at

$$\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]v\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]=0.$$

By Lemma 2.1, we have

$$\left[\alpha\left(u\right),\left[T\left(u\right),\alpha\left(u\right)\right]\right]=0. \tag{13}$$

Replacing u by $u+v$ in this equation and using this equation, we get

$$\begin{aligned} 0 &= [\alpha(u), [T(u), \alpha(v)]] + [\alpha(u), [T(v), \alpha(u)]] + [\alpha(u), [T(v), \alpha(v)]] \\ &\quad + [\alpha(v), [T(u), \alpha(u)]] + [\alpha(v), [T(u), \alpha(v)]] + [\alpha(v), [T(v), \alpha(u)]]. \end{aligned}$$

Putting $-u$ for u in last equation, we have

$$\begin{aligned} 0 &= [\alpha(u), [T(u), \alpha(v)]] + [\alpha(u), [T(v), \alpha(u)]] - [\alpha(u), [T(v), \alpha(v)]] \\ &\quad + [\alpha(v), [T(u), \alpha(u)]] - [\alpha(v), [T(u), \alpha(v)]] - [\alpha(v), [T(v), \alpha(u)]]. \end{aligned}$$

By comparing in two last equations and since R is $2-$torsion free, we obtain that

$$[\alpha(u), [T(u), \alpha(v)]] + [\alpha(u), [T(v), \alpha(u)]] + [\alpha(v), [T(u), \alpha(u)]] = 0. \quad (14)$$

Replacing v by uvu in (14) and using (14), (1) and (14), we get

$$\begin{aligned} 0 &= [\alpha(u), [T(u), \alpha(uvu)]] + [\alpha(u), [T(uvu), \alpha(u)]] \\ &\quad + [\alpha(uvu), [T(u), \alpha(u)]] \\ &= [\alpha(u), [T(u), \alpha(uvu)]] + [\alpha(u), [\alpha(u)T(v)\alpha(u), \alpha(u)]] \\ &\quad + [\alpha(uvu), [T(u), \alpha(u)]] \\ &= [\alpha(u), [T(u), \alpha(u)]\alpha(vu)] + [\alpha(u), \alpha(u)[T(u), \alpha(vu)]] \\ &\quad + [\alpha(u), \alpha(u)[T(v), \alpha(u)]\alpha(u)] + [\alpha(u), [T(u), \alpha(u)]]\alpha(vu) \\ &\quad + \alpha(u)[\alpha(vu), [T(u), \alpha(u)]] \\ &= [T(u), \alpha(u)][\alpha(u), \alpha(v)\alpha(u)] + \alpha(u)[\alpha(u), [T(u), \alpha(v)\alpha(u)]] \\ &\quad + \alpha(u)[\alpha(u), [T(v)\alpha(u), \alpha(u)]] + \alpha(u)[\alpha(v), [T(u), \alpha(u)]]\alpha(u) \\ &= [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - [T(u), \alpha(u)]\alpha(v)\alpha(u)\alpha(u) \\ &\quad + \alpha(u)[\alpha(u), [T(u), \alpha(v)]\alpha(u) + \alpha(v)[T(u), \alpha(u)]] \\ &\quad + \alpha(u)[\alpha(u), [T(v), \alpha(u)]]\alpha(u) + \alpha(u)[\alpha(v), [T(u), \alpha(u)]]\alpha(u) \\ &= [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - [T(u), \alpha(u)]\alpha(v)\alpha(u)\alpha(u) \\ &\quad + \alpha(u)[\alpha(u), [T(u), \alpha(v)]]\alpha(u) + \alpha(u)[\alpha(u), \alpha(v)][T(u), \alpha(u)] \\ &\quad + \alpha(u)[\alpha(u), [T(v), \alpha(u)]]\alpha(u) + \alpha(u)[\alpha(v), [T(u), \alpha(u)]]\alpha(u) \\ &= [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - [T(u), \alpha(u)]\alpha(v)\alpha(u)\alpha(u) \\ &\quad + \alpha(u)[\alpha(u), \alpha(v)][T(u), \alpha(u)] + \alpha(u)[\alpha(u), [T(u), \alpha(v)]]\alpha(u) \\ &\quad + \alpha(u)[\alpha(u), [T(v), \alpha(u)]]\alpha(u) + \alpha(u)[\alpha(v), [T(u), \alpha(u)]]\alpha(u) \\ &= [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - [T(u), \alpha(u)]\alpha(v)\alpha(u)\alpha(u) \\ &\quad + \alpha(u)[\alpha(u), \alpha(v)][T(u), \alpha(u)] \\ &= [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - [T(u), \alpha(u)]\alpha(v)\alpha(u^2) \\ &\quad + \alpha(u^2)\alpha(v)[T(u), \alpha(u)] - \alpha(u)\alpha(v)\alpha(u)[T(u), \alpha(u)] \\ &= \alpha(u^2)\alpha(v)[T(u), \alpha(u)] - [T(u), \alpha(u)]\alpha(v)\alpha(u^2) \\ &\quad + [T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u) - \alpha(u)\alpha(v)\alpha(u)[T(u), \alpha(u)]. \end{aligned}$$

Using equation (13), we get

$$\begin{aligned} 0 &= \alpha(u^2)\alpha(v)[T(u), \alpha(u)] - [T(u), \alpha(u)]\alpha(v)\alpha(u^2) \\ &\quad + \alpha(u)[T(u), \alpha(u)]\alpha(v)\alpha(u) - \alpha(u)\alpha(v)[T(u), \alpha(u)]\alpha(u). \end{aligned}$$

Using equation (9), we obtain that

$$\alpha\left(u^2\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u^2\right)=0.$$

Multiplying last equation on the left by $\alpha(u)$, we get

$$\alpha\left(u^3\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u^2\right)=0 \qquad (15)$$

and using equation (9), we get

$$\alpha\left(u^3\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\alpha\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right)=0. \qquad (16)$$

Multiplying equation (15) on the left by $T\left(u\right)$, we see that

$$T\left(u\right)\alpha\left(u^3\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-T\left(u\right)\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u^2\right)=0.$$

Replacing v by $\alpha^{-1}\left(T\left(u\right)\right)v$ in (16), we get

$$\alpha\left(u^3\right)T\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\alpha\left(u\right)T\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right)=0.$$

If the last two equations are used, we see that

$$\begin{aligned} 0 &= \left[T\left(u\right),\alpha\left(u^3\right)\right]\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-T\left(u\right)\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\alpha\left(u^2\right) \\ &\quad +\alpha\left(u\right)T\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right) \end{aligned}$$

and using equation (9), we get

$$\begin{aligned} 0 &= \left[T\left(u\right),\alpha\left(u^3\right)\right]\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-T\left(u\right)\alpha\left(u\right)\alpha\left(v\right)\alpha\left(u^2\right) \\ &\quad +\alpha\left(u\right)T\left(u\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right) \end{aligned}$$

and so

$$\left[T\left(u\right),\alpha\left(u^3\right)\right]\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]-\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right)=0.$$

By Lemma 2.3, we have

$$(\left[T\left(u\right),\alpha\left(u^3\right)\right]-\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u^2\right))\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]=0,$$

and so

$$(\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u\right)+\alpha\left(u^2\right)\left[T\left(u\right),\alpha\left(u\right)\right])\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]=0.$$

Using equation (13), we get

$$\left(\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u\right)+\alpha\left(u\right)\left[T\left(u\right),\alpha\left(u\right)\right]\alpha\left(u\right)\right)\alpha\left(v\right)\left[T\left(u\right),\alpha\left(u\right)\right]=0.$$

That is,

$$2\alpha(u)[T(u), \alpha(u)]\alpha(u)\alpha(v)[T(u), \alpha(u)] = 0.$$

Since R is 2-torsion free , we get

$$\alpha(u)[T(u), \alpha(u)]\alpha(u)\alpha(v)[T(u), \alpha(u)] = 0. \tag{17}$$

Replacing v by vu in this equation, we get

$$\alpha(u)[T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u)[T(u), \alpha(u)] = 0.$$

Multiplying equation this equation on the right by $\alpha(u)$, we get

$$\alpha(u)[T(u), \alpha(u)]\alpha(u)\alpha(v)\alpha(u)[T(u), \alpha(u)]\alpha(u) = 0,$$

and so,

$$\alpha(u)[T(u), \alpha(u)]\alpha(u)U\alpha(u)[T(u), \alpha(u)]\alpha(u) = (0).$$

By Lemma 2.1, we get

$$\alpha(u)[T(u), \alpha(u)]\alpha(u) = 0. \tag{18}$$

Replacing v by vu in (9), we get

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)\alpha(u)\alpha(u) - \alpha(u)\alpha(v)\alpha(u)[T(u), \alpha(u)]\alpha(u) = 0.$$

Using equation (18), we have

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)\alpha\left(u^2\right) = 0. \tag{19}$$

Replacing v by $v\alpha^{-1}(T(u))$ in this equation, we have

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)T(u)\alpha\left(u^2\right) = 0.$$

Multiplying (19) on the right by $T(u)$, we have

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)\alpha\left(u^2\right)T(u) = 0.$$

If the last two equations are used, we see that

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)\left[T(u), \alpha\left(u^2\right)\right] = 0.$$

That is,

$$\alpha(u)[T(u), \alpha(u)]\alpha(v)[T(u), \alpha(u)]\alpha(u) + \alpha(u)[T(u), \alpha(u)]\alpha(v)\alpha(u)[T(u), \alpha(u)] = 0.$$

Using equation (13), we have

$$2\alpha(u)[T(u),\alpha(u)]\alpha(v)\alpha(u)[T(u),\alpha(u)]=0.$$

Since R is $2-$torsion free, we get

$$\alpha(u)[T(u),\alpha(u)]\alpha(v)\alpha(u)[T(u),\alpha(u)]=0$$

and so

$$\alpha(u)[T(u),\alpha(u)]U\alpha(u)[T(u),\alpha(u)]=(0).$$

By Lemma 2.1, we get

$$\alpha(u)[T(u),\alpha(u)]=0. \tag{20}$$

By equation (13), we have

$$[T(u),\alpha(u)]\alpha(u)=0.$$

Replacing u by $u+v$ in this equation, we get

$$\begin{aligned} 0 \;=\; & [T(u),\alpha(u)]\alpha(v)+[T(v),\alpha(u)]\alpha(u)+[T(v),\alpha(u)]\alpha(v) \\ & +[T(u),\alpha(v)]\alpha(u)+[T(u),\alpha(v)]\alpha(v)+[T(v),\alpha(v)]\alpha(u) \end{aligned}.$$

Writing u by $-u$ in tis equation, we have

$$\begin{aligned} 0 \;=\; & [T(u),\alpha(u)]\alpha(v)+[T(v),\alpha(u)]\alpha(u)-[T(v),\alpha(u)]\alpha(v) \\ & +[T(u),\alpha(v)]\alpha(u)-[T(u),\alpha(v)]\alpha(v)-[T(v),\alpha(v)]\alpha(u) \end{aligned}.$$

If the last two equations are used, we see that

$$2([T(u),\alpha(u)]\alpha(v)+[T(v),\alpha(u)]\alpha(u)+[T(u),\alpha(v)]\alpha(u))=0.$$

Since R is 2-torsion free, we have

$$[T(u),\alpha(u)]\alpha(v)+[T(v),\alpha(u)]\alpha(u)+[T(u),\alpha(v)]\alpha(u)=0.$$

Multiplying the last equation on the right by $[T(u),\alpha(u)]$ and using (20), we have

$$[T(u),\alpha(u)]\alpha(v)[T(u),\alpha(u)]=0.$$

By Lemma 2.1, we obtain that

$$[T(u),\alpha(u)]=0. \tag{21}$$

Replacing w by u^2 in (3), we obtain that

$$T\left(uvu^2+u^2vu\right)=\alpha(u)T(v)\alpha\left(u^2\right)+\alpha\left(u^2\right)T(v)\alpha(u). \tag{22}$$

Taking v by $uv+vu$ in the hypothesis, we get

$$T\left(u^2vu+uvu^2\right)=\alpha(u)T(uv+vu)\alpha(u) \tag{23}$$

Subtracting (22) from (23), we arrive at

$$\alpha(u)T(uv+vu)\alpha(u)-\alpha(u)T(v)\alpha\left(u^2\right)-\alpha\left(u^2\right)T(v)\alpha(u)=0.$$

That is, $\alpha(u)\beta(u,v)\alpha(u)=0$, where $\beta(u,v)=T(uv+vu)-T(v)\alpha(u)-\alpha(u)T(v)$. Replacing u by $u+w, w\in U$ in this equation and using this equation, we get

$$\begin{aligned}0 &= \alpha(u)\beta(u,v)\alpha(w)+\alpha(u)\beta(w,v)\alpha(u)\\ &+\alpha(u)\beta(w,v)\alpha(w)+\alpha(w)\beta(u,v)\alpha(u)\\ &+\alpha(w)\beta(u,v)\alpha(w)+\alpha(w)\beta(w,v)\alpha(u).\end{aligned} \tag{24}$$

Replacing u by $-u$ in the last equation, we see that

$$\begin{aligned}0 &= \alpha(u)\beta(u,v)\alpha(w)+\alpha(u)\beta(w,v)\alpha(u)\\ &-\alpha(u)\beta(w,v)\alpha(w)+\alpha(w)\beta(u,v)\alpha(u)\\ &-\alpha(w)\beta(u,v)\alpha(w)-\alpha(w)\beta(w,v)\alpha(u).\end{aligned} \tag{25}$$

Subtracting (24) from (25) and since R is 2-torsion free, we arrive at

$$\alpha(u)\beta(u,v)\alpha(w)+\alpha(u)\beta(w,v)\alpha(u)+\alpha(w)\beta(u,v)\alpha(u)=0$$

Multiplying the last equation on the right by $\beta(u,v)\alpha(u)$, we have

$$\begin{aligned}0 &= \alpha(u)\beta(u,v)\alpha(w)\beta(u,v)\alpha(u)+\alpha(u)\beta(w,v)\alpha(u)\beta(u,v)\alpha(u)\\ &+\alpha(w)\beta(u,v)\alpha(u)\beta(u,v)\alpha(u)\end{aligned}$$

and so,

$$\alpha(u)\beta(u,v)\alpha(w)\beta(u,v)\alpha(u)=0. \tag{26}$$

Replacing u by $u+v$ in (21) and using (21), we get

$$[T(u),\alpha(v)]+[T(v),\alpha(u)]=0. \tag{27}$$

Writing v by $uv+vu$ in the last equation and using (21), we get

$$\alpha(u)[T(u),\alpha(v)]+[T(u),\alpha(v)]\alpha(u)+[T(uv+vu),\alpha(u)]=0.$$

Using (27) in the last equation, we have

$$-\alpha(u)[T(v),\alpha(u)]-[T(v),\alpha(u)]\alpha(u)+[T(uv+vu),\alpha(u)]=0.$$

That is,

$$\begin{aligned}0 &= T(uv+vu)\alpha(u)-\alpha(u)T(uv+vu)-\alpha(u)T(v)\alpha(u)\\ &+\alpha(u^2)T(v)-T(v)\alpha(u^2)+\alpha(u)T(v)\alpha(u)\end{aligned}$$

and so

$$\begin{aligned} 0 &= \{T(uv+vu)-T(v)\alpha(u)-\alpha(u)T(v)\}\alpha(u) \\ &\quad -\alpha(u)\{T(uv+vu)-T(v)\alpha(u)-\alpha(u)T(v)\}. \end{aligned}$$

We obtain that

$$[\beta(u.v),\alpha(u)]=0. \tag{28}$$

That is $\beta(u,v)\alpha(u)=\alpha(u)\beta(u,v)$. Using this equation in equation (26), we obtain that

$$\beta(u,v)\alpha(u)\alpha(w)\beta(u,v)\alpha(u)=0.$$

Since $\alpha(U)=U$, we have

$$\beta(u,v)\alpha(u)U\beta(u,v)\alpha(u)=(0).$$

By Lemma 2.1, we get

$$\beta(u,v)\alpha(u)=0. \tag{29}$$

Using equation (28), we get

$$\alpha(u)\beta(u,v)=0. \tag{30}$$

Replacing u by $u+w$ in (29) and using (29), we see that

$$\beta(u,v)\alpha(w)+\beta(w,v)\alpha(u)=0.$$

Multiplying the last equation on the right by $\beta(u,v)$, we get

$$0=\beta(u,v)\alpha(w)\beta(u,v)+\beta(w,v)\alpha(u)\beta(u,v).$$

Using equation (30), we have

$$\beta(u,v)\alpha(w)\beta(u,v)=0$$

and so $\beta(u,v)U\beta(u,v)=(0)$. By Lemma 2.1, we get $\beta(u,v)=0$, for all $u,v\in U$. That is, $T(uv+vu)=T(v)\alpha(u)+\alpha(u)T(v)$. Replacing v by u, we see that

$$T\left(u^2+u^2\right)=T(u)\alpha(u)+\alpha(u)T(u).$$

and so,

$$2T\left(u^2\right)=T(u)\alpha(u)+\alpha(u)T(u).$$

Using equation (21), we see that

$$2T\left(u^2\right)=2T(u)\alpha(u).$$

Since R is $2-$torsion free, we arrive at $T(u^2)=T(u)\alpha(u)$ for all $u\in U$. By Lemma 2.2, we conclude that T is a left $\alpha-$centralizer mapping of U into R. This proof the completed.

Theorem 2.5 *Let R be a $2-$torsion free semiprime ring, U a nocentral square-closed Lie ideal of R, α an automorphism of R and $\alpha(U) = U$, $T(U) \subset U$. If $T : R \to R$ is an additive mapping such that $T(uvu) = T(u)\alpha(vu)$ (resp. $T(uvu) = \alpha(uv)T(u)$) for all $u, v \in U$, then T is a left (resp. right) α-centralizer on U.*

Proof. By the hypothesis, we have

$$T(uvu) = T(u)\alpha(vu), \text{ for all } u, v \in U.$$

Replacing u by $u + w, w \in U$ in the hypothesis, we get

$$T((u + w)v(u + w)) = T(u)\alpha(vu) + T(u)\alpha(vw) + T(w)\alpha(vu) + T(w)\alpha(vw).$$

On the other hand, we have

$$\begin{array}{rcl} T((u + w)v(u + w)) & = & T(uvw + wvu + uvu + wvw) \\ & = & T(uvw + wvu) + T(u)\alpha(vu) + T(w)\alpha(vw) \end{array}.$$

If the last two equations are used, we see that

$$T(uvw + wvu) = T(u)\alpha(vw) + T(w)\alpha(vu).$$

Writing w by u^2 in the last equation, we find that

$$T(uvu^2 + u^2vu) = T(u)\alpha(vu^2) + T(u^2)\alpha(vu).$$

Replacing v by $uv + vu$ in the hypothesis and using the hypothesis, we get

$$T(uvu^2 + u^2vu) = T(u)\alpha(uvu) + T(u)\alpha(vu^2).$$

By comparing in two last equations, we obtain that

$$T(u^2)\alpha(vu) - T(u)\alpha(uvu) = 0.$$

That is,

$$A(u)\alpha(vu) = 0, \text{for all } u, v \in U,$$

where $A(u) = T(u^2) - T(u)\alpha(u)$. Using and $\alpha(U) = U$, we see that

$$A(u)w\alpha(u) = 0, \text{for all } u, w \in U. \tag{31}$$

Multiplying (31) on the left by $\alpha(u)$, we have

$$\alpha(u)A(u)w\alpha(u) = 0, \text{for all } u, w \in U.$$

Multiplying the last equation on the right by $A(u)$, we have

$$\alpha(u)A(u)w\alpha(u)A(u) = 0, \text{for all } u, w \in U.$$

By Lemma 2.1, we get

$$\alpha(u)A(u) = 0, \text{ for all } u \in U. \tag{32}$$

On the other hand, replacing w by $\alpha(u)wA(u)$ in equation (32), we get

$$A(u)\alpha(u)wA(u)\alpha(u) = 0.$$

Again, by Lemma 2.1, we have

$$A(u)\alpha(u) = 0, \text{for all } u \in U. \tag{33}$$

Replacing u by $u+v$ in this equation, we obtain that

$$A(u+v)\alpha(u) + A(u+v)\alpha(v) = 0. \tag{34}$$

That is,

$$\begin{aligned} A(u+v) &= T((u+v)^2) - T(u+v)\alpha(u+v) \\ &= (T(uv+vu) - T(u)\alpha(v) - T(v)\alpha(u)) \\ &\quad +T(u^2) - T(u)\alpha(u) + T(v^2) - T(v)\alpha(v) \\ &= B(u,v) + A(u) + A(v). \end{aligned}$$

where $B(u,v) = T(uv+vu) - T(u)\alpha(v) - T(v)\alpha(u)$. Using the last equation in equation (34), we get

$$A(u)\alpha(u) + A(u)\alpha(v) + B(u,v)\alpha(u) + A(v)\alpha(u) + B(u,v)\alpha(v) + A(v)\alpha(v) = 0.$$

Using equation (33), we see that

$$A(u)\alpha(v) + B(u,v)\alpha(u) + A(v)\alpha(u) + B(u,v)\alpha(v) = 0.$$

Replacing u by $-u$ in this equation, we have

$$A(u)\alpha(v) + B(u,v)\alpha(u) - A(v)\alpha(u) - B(u,v)\alpha(v) = 0.$$

If the last two equations are used, we see that

$$2(A(u)\alpha(v) + B(u,v)\alpha(u)) = 0.$$

Since R is a $2-$torsion free, we get

$$A(u)\alpha(v) + B(u,v)\alpha(u) = 0.$$

Multiplying the last equation on the right by $A(u)$, we have

$$A(u)\alpha(v)A(u) + B(u,v)\alpha(u)A(u) = 0.$$

Using equation (32), we get $A(u)\alpha(v)A(u) = 0$. Since $\alpha(U) = U$, we have $A(u)UA(u) = 0$. By Lemma 2.1, we conclude that $A(u) = 0$,for all $u \in U$. That is, $T(u^2) = T(u)\alpha(u)$,for all $u \in U$. We conclude that T is a left α-centralizer on U by Lemma 2.2. This completes the proof.

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3 Open Problem

Our hypotheses are dealt with on the semiprime ring. Considering all hypotheses on the semiprime ring gives more general results. The article is discussed for Lie ideal on semiprime ring. The conditions discussed here can be considered for (σ, τ)-Lie ideal.

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