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On Prime Ideals and Semiderivations

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Abstract

Let R ***be a ring,*** P ***a prime ideal of*** R ***and*** $d : R \to R$ ***a semiderivation associated with an automorphism*** g ***of*** R***. If any one of the following holds then*** $d(R) \subseteq P$ ***or*** R/P ***is commutative integral domain: i)*** $d([x, y]) \in P$, ***ii)*** $d(xoy) \in P$, ***iii)*** $d([x, y]) \pm [x, y] \in P$, ***iv)*** $d(xoy) \pm (xoy) \in P$, ***v)*** $d([x, y]) \pm (xoy) \in P$, ***vi)*** $d(xoy) \pm [x, y] \in P$, ***vii)*** $d([x, y]) \pm x^m[x, y]x^n \in P$, ***viii)*** $d(xoy) \pm x^m(xoy)x^n \in P$, ***ix)*** $d([x, y]) \pm x^m(xoy)x^n \in P$, ***x)*** $d(xoy) \pm x^m[x, y]x^n \in P$, ***xi)*** $d(xy) \pm xy \in P$, ***xii)*** $d(xy) \pm yx \in P$, ***xiii)*** $d(x)d(y) \pm xy \in P$, ***xiv)*** $d(x)d(y) \pm yx \in P$, ***for all*** $x, y \in R, m, n \in \mathbb{Z}$.

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1 Introduction

Let R will be an associative ring with center Z. Recall that a proper ideal P of R is said to be prime if for any $x, y \in R, xRy \subseteq P$ implies that $x \in P$ or $y \in P$. The ring R is prime if and only if (0) is a prime ideal of R, or equiently a ring R is prime if for $x, y \in R$, $xRy = (0)$ implies either $x = 0$ or $y = 0$. For any $x, y \in R$ the symbol $[x, y]$ represents the Lie commutator $xy - yx$ and the Jordan product $xoy = xy + yx$.

The derivations and their generalizations play major role in mathematics, economics, quantum physics and biology such as chemotheraphy. Because of that the correlation between derivations and the algebraic structures has become an exciting subject in the the last years. Over the last few decades, a number of authors have investigated the commutativity of the ring R or some appropriate subsets of R satisfying certain differantial identities with derivation. With the development of the theory, different definitions of derivations have been made (generalized derivation, $(\alpha, \beta)-$derivation, homoderivation etc.). The concept of derivation in rings was introduced by Posner in [4]. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. This is the very first example of derivation. One of the new derivation definitions is the semiderivation definition. The notion of semiderivations first time introduced by in Bergen in [5]. An additive mapping $d : R \rightarrow R$ is said to be a semiderivation if there exists a function $g : R \rightarrow R$ such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$ hold for all $x, y \in R$. In case g is an identity map of R, then all semiderivations associated with g are merely derivations. Hence semiderivation covers concept of the derivation. On the other hand, if g other main motivating examples are of the form $d = g - 1$ where g is any ring endomorphism of R such that $g \neq 1$. Then d is a semiderivation with associated map g which is not a derivation.

Recently, some authors adopted a new approach by considering algebraic identities with derivations involving prime ideal without primenees assumption on the considered ring (see, e.g., [1], [2], [3], [8] and references therein). They characterized the commutativity of a quotient ring R/P.

In [7], Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R contains a nonzero central ideal. On the other hand, in [6], Ashraf and Rehman showed that R is prime ring with a nonzero ideal I of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in I$, then R is commutative.

In the present paper is motivated by the previous results. We aim to investigate the commutativity of quotient ring R/P where R any ring and P is prime ideal of R which admits a semiderivations associated with an automorphism g of R are satisfying some identities acting on prime ideal P. The material in this work is a part of first author's Master Thesis which is supervised by Prof. Dr. Öznur Gölbaşı.

2 Results

Throughout the paper, we will make some extensive use of the basic commutator identities for all $x, y, z \in R :$

$[x, yz] = y[x, z] + [x, y]z$

$[xy,z] = [x,z]y + x[y,z]$
$xo(yz) = (xoy)z - y[x,z] = y(xoz) + [x,y]z$
$(xy)oz = x(yoz) - [x,z]y = (xoz)y + x[y,z].$

Lemma 2.1 *[8, Lemma 1.3]Let R be a ring, P a prime ideal of R. If any of the following conditions is satisfied for all $x,y \in R$, then R/P is commutative integral domain.*
i) $[x,y] \in P$

ii) $xoy \in P$

Theorem 2.2 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with a map g of R. If $d([x,y]) \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$d([x,y]) \in P, \quad \text{for all } x,y \in R. \tag{1}$$

Writing yx for y in (1) and using this, we obtain that

$$d([x,y]x) = d([x,y])g(x) + [x,y]d(x) \in P$$

and so

$$[x,y]d(x) \in P, \quad \text{for all } x,y \in R. \tag{2}$$

Taking $zy, z \in R$ for y in (2) and using (2), we get

$$[x,z]Rd(x) \subseteq P, \quad \text{for all } x,z \in R. \tag{3}$$

Since P is prime, we get

$$[x,z] \in P \text{ or } d(x) \in P, \quad \text{for all } x,z \in R.$$

Let $L = \{x \in R \mid [x,z] \in P, \text{ for all } z \in R\}$ and $K = \{x \in R \mid d(x) \in P\}$. Clearly each of L and K is additive subgroup of R such that $R = L \cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L = R$ or $K = R$. In the first case, we have $[x,z] \in P$, for all $z \in R$, and so R/P is an integral domain by Lemma 2.1. In the second case, we get $d(R) \subseteq P$. This completes the proof.

Theorem 2.3 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with a map g of R. If $d(xoy) \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$d(xoy) \in P, \quad \text{for all } x, y \in R. \tag{4}$$

Writing yx for y in (4) and using this, we find that

$$d((xoy)x) = d(xoy)g(x) + (xoy)d(x) \in P$$

and so

$$(xoy)d(x) \in P, \ \text{for all } x, y \in R. \tag{5}$$

Substituting $zy, z \in R$ for y in (5) and using this expression, we arrive at

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Arguing the same methods after (3) in the proof of Theorem 2.2, we obtain the required result.

Theorem 2.4 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d([x, y]) \pm [x, y] \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we get

$$[x, y] \in P, \quad \text{for all } x, y \in R.$$

By Lemma 2.1, we get R/P is commutative integral domain.
Now, we assume that $d \neq 0$. By our hypothesis, we have

$$d([x, y]) \pm [x, y] \in P, \quad \text{for all } x, y \in R. \tag{6}$$

Replacing y by yx in (6) and using this equation, we arrive that

$$d([x, y]x) \pm [x, y]x = d([x, y])x + g([x, y])d(x) \pm [x, y]x \in P$$

and so

$$g([x, y])d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $zy, z \in R$ for y in this expression and using this, we get

$$g([x, z])g(y)d(x) \in P, \quad \text{for all } x, y \in R.$$

Since g is an automorphism of R, we have

$$g([x, z])Rd(x) \subseteq P, \quad \text{for all } x, z \in R. \tag{7}$$

Let $L = \{x \in R \mid g([x, z]) \in P, \text{ for all } z \in R\}$ and $K = \{x \in R \mid d(x) \in P\}$. Clearly each of L and K is additive subgroup of R such that $R = L \cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L = R$ or $K = R$. In the first case, we obtain that $[x, z] \in P$, for all $z \in R$ using g is an automorphism or R, and so R/P is an integral domain by Lemma 2.1. In the second case, we get $d(R) \subseteq P$. This completes the proof.

Theorem 2.5 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(xoy)\pm(xoy) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we get

$$xoy \in P, \quad \text{for all } x, y \in R.$$

By Lemma 2.1, we get R/P is commutative integral domain.
Now, we get $d \neq 0$. By our hypothesis, we have

$$d(xoy) \pm (xoy) \in P, \quad \text{for all } x, y \in R. \tag{8}$$

Replacing y by yx in equation (8) and using this, we arrive that

$$d((xoy)x) \pm (xoy)x = d(xoy)x + g(xoy)d(x) \pm (xoy)x \in P$$

and so

$$g(xoy)d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $zy, z \in R$ for y and using this equation, we find that

$$g([x, z])Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

We obtain the required result using the same arguments after (7) in the proof of Theorem 2.4.

Theorem 2.6 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d([x, y]) \pm (xoy) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we get

$$xoy \in P, \quad \text{for all } x, y \in R.$$

Hence R/P is commutative integral domain by Lemma 2.1.
Now, we assume $d \neq 0$. By our hypothesis, we have

$$d([x, y]) \pm (xoy) \in P, \quad \text{for all } x, y \in R. \tag{9}$$

Replacing yx by y in (9) and using this, we get

$$g([x, y])d(x) \in P, \quad \text{for all } x, y \in R.$$

Taking $zy, z \in R$ for y in this equation and using this, we have

$$g([x, z])Rd(x) \subseteq P \quad \text{for all } x, z \in R.$$

This equation is the same as (7). Arguing the same lines in the proof of Theorem 2.4, we get the required result.

Theorem 2.7 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(xoy) \pm [x, y] \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then $[x, y] \in P$, for all $x, y \in R$. By Lemma 2.1, we find that R/P is commutative integral domain.
Now, we have $d \neq 0$ and

$$d(xoy) \pm [x, y] \in P, \quad \text{for all } x, y \in R. \tag{10}$$

Replacing yx by y in (10) and using this, we get

$$g(xoy)d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $zy, z \in R$ for y in this equation and using this, we find that

$$g([x, z])Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Using the same arguments after (7) in the proof of Theorem 2.4, we obtain the required result.

Theorem 2.8 *Let R be a ring, P a prime ideal of R and d be a nonzero semiderivation associated with an automorphism g of R. If $d\left([x, y]\right) \pm x^m \left[x, y\right] x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By the hypothesis, we have

$$d\left([x, y]\right) \pm x^m \left[x, y\right] x^n \in P, \quad \text{for all } x, y \in R. \tag{11}$$

Replacing y by yx in equation (11), we get

$$d\left([x, y]\, x\right) \pm x^m \left[x, y\right] x^{n+1} \in P$$

and so

$$d\left([x, y]\right)x + g([x, y])d(x) \pm x^m \left[x, y\right] x^{n+1} \in P, \quad \text{for all } x, y \in R.$$

Using the hypothesis, we obtain that

$$g([x, y])d(x) \in P, \quad \text{for all } x, y \in R.$$

Taking $zy, z \in R$ for y in this equation and using this, we have

$$g([x, z])Rg(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Using the same arguments after (7) in the proof of Theorem 2.4, we get the required result.

Theorem 2.9 *Let R be a ring, P a prime ideal of R and d be a nonzero semiderivation associated with an automorphism g of R. If $d\,(x \circ y) \pm x^m(x \circ y)x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. We assume that

$$d\,(x \circ y) \pm x^m(x \circ y)x^n \in P, \quad \text{for all } x, y \in R. \tag{12}$$

Replacing y by yx in equation (12), we obtain

$$d\,(x \circ yx) \pm x^m(x \circ yx)x^n \in P$$

and so

$$d\,((x \circ y)x) \pm x^m((x \circ y)x)x^n \in P, \quad \text{for all } x, y \in R.$$

That is

$$d\,(x \circ y)x + g(x \circ y)d(x) \pm x^m(x \circ y)x^{n+1} \in P.$$

Using the hypothesis, we get

$$g(x \circ y)g(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $zy, z \in R$ for y in this equation and using this, we find that

$$g([x, z])Rg(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Arguing the same methods after (7) in the proof of Theorem 2.4, we obtain the required result.

Theorem 2.10 *Let R be a ring, P a prime ideal of R and d be a nonzero semiderivation associated with an automorphism g of R. If $d\,([x, y]) \pm x^m(xoy)x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. Let assume that

$$d\,([x, y]) \pm x^m(xoy)x^n \in P, \quad \text{for all } x, y \in R. \tag{13}$$

Writing y by yx in (13) and using this equation, we arrive that

$$d\,([x, yx]) \pm x^m(x \circ yx)x^n \in P$$

and so

$$d\,([x, y]x) \pm x^m((x \circ y)x)x^n \in P, \quad \text{for all } x, y \in R.$$

Hence we get

$$d([x,y])x + g([x,y])d(x) \pm x^m(x \circ y)x^{n+1} \in P.$$

Using the hypothesis, we have

$$g([x,y])d(x) \in P, \quad \text{for all } x,y \in R. \tag{14}$$

Taking $zy, z \in R$ for y in equation (14) and using this, we have

$$g([x,z])Rg(x) \subseteq P, \quad \text{for all } x,z \in R.$$

This equation is same as (7) in the proof of Theorem 2.4. Arguing the same techniques therein, we get the required result.

Theorem 2.11 *Let R be a ring, P a prime ideal of R and d be a nonzero semiderivation associated with an automorphism g of R. If $d(xoy) \pm x^m [x,y] x^n \in P$, for all $x,y \in R, m,n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. Let assume that

$$d(xoy) \pm x^m [x,y] x^n \in P, \quad \text{for all } x,y \in R. \tag{15}$$

Writing y by yx in (15) and using this equation, we arrive that

$$d(x \circ yx) \pm x^m[x,yx]x^n \in P$$

and so

$$d((x \circ y)x) \pm x^m([x,y]x)x^n \in P, \quad \text{for all } x,y \in R.$$

That is

$$d(x \circ y)x + g(x \circ y)d(x) \pm x^m[x,y]x^{n+1} \in P.$$

Using the hypothesis, we get

$$g(x \circ y)d(x) \in P, \quad \text{for all } x,y \in R.$$

Substituting $zy, z \in R$ for y in this equation and using this, we find that

$$g([x,z])Rd(x) \subseteq P, \quad \text{for all } x,z \in R.$$

By the same arguments after the equation (7) in the proof of Theorem 2.4, we obtain the required result.

Theorem 2.12 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(xy) \pm xy \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we get $xy \in P$, for all $x, y \in R$ from the hypothesis, and so $[x, y] \in P$, for all $x, y \in R$. By Lemma 2.1, we find that R/P is commutative integral domain.
Now, we assume that $d \neq 0$.
By our hypothesis, we get

$$d(xy) \pm xy \in P, \text{ for all } x, y \in R. \tag{16}$$

Replacing y by yz in (16), we get

$$(d(xy) \pm xy)z + g(xy)d(z) \in P \tag{17}$$

and so

$$g(xy)d(z) \in P, \text{ for all } x, y \in R. \tag{18}$$

Since g is an automorphism of R, we have

$$xRd(z) \in P, \text{ for all } x, y \in R.$$

Since P is prime, we get

$$x \in P \text{ or } d(z) \in P, \text{ for all } x, z \in R.$$

If $x \in P$, for all $x \in R$, then we obtain that $P = R$, and it contradicts that P is prime ideal of R. So, we get $d(R) \subseteq P$. This completes the proof.

Theorem 2.13 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(xy) \pm yx \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we find that R/P is commutative integral domain using the same arguments in begining of Theorem 2.12.
Now, we get $d \neq 0$.
Assume that

$$d(xy) \pm yx \in P, \text{ for all } x, y \in R.$$

Taking yz instead of y in this equation, we have

$$d(xy)z + g(xy)d(z) \pm yzx \in P, \text{ for all } x, y, z \in R.$$

For all $x, y, z \in R$, we can write this equation

$$d(xy)z + g(xy)d(z) \pm yzx + yxz - yxz \in P, \text{ for all } x, y, z \in R$$

and so

$$(d(xy) \pm yx)z + g(xy)d(z) \pm y[z, x] \in P, \text{ for all } x, y, z \in R.$$

Using the hypothesis, we arrive at

$$g(xy)d(z) + y[x,z] \in P, \text{ for all } x,y,z \in R. \tag{19}$$

Replacing z by x in (19) and using this, we get

$$g(xy)d(x) = 0, \text{ for all } x,y \in I.$$

This equation is same as (18) in the proof of Theorem 2.12. Arguing the same techniques therein, we get the required result.

Theorem 2.14 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(x)d(y) \pm xy \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $d = 0$, then we get $xy \in P$, for all $x,y \in R$ in the hypothesis. We had done in the proof of Theorem 2.12. So, we have $d \neq 0$.
By our hypothesis, we get

$$d(x)d(y) \pm xy \in P, \text{ for all } x,y \in R.$$

Replacing y by yz in this equation and using the hypothesis, we get

$$d(x)d(y)z + d(x)g(y)d(z) \pm xyz \in P,$$

and so

$$d(x)g(y)d(z) \in P, \text{ for all } x,y,z \in R.$$

Since g is an automorphism of R, we have

$$d(x)Rd(z) \in P, \text{ for all } x,y,z \in R$$

Using P is a prime ideal of R, we obtain that $d(R) \subseteq P$. This completes the proof.

Theorem 2.15 *Let R be a ring, P a prime ideal of R and d be a semiderivation associated with an automorphism g of R. If $d(x)d(y) \pm yx \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. Using the same arguments begining of the proof of Theorem 2.12, we arrive at R/P is commutative integral domain in the case of $d = 0$.
Now, we have $d \neq 0$. By our hypothesis, we get

$$d(x)d(y) \pm yx \in P, \text{ for all } x,y \in R.$$

Replacing y by yx in this equation and using the hypothesis, we have

$$d(x)d(y)x + d(x)g(y)d(x) \pm xyx \in P,$$

and so

$$(d(x)d(y) \pm xy)x + d(x)g(y)d(x) \in P.$$

By the hypothesis, we find that

$$d(x)g(y)d(z) \in P$$

Since g is an automorphism of R and P is a prime ideal of R, we obtain that $d(R) \subseteq P$. This completes the proof.

3 Open Problem

How to generalize these theorems for a semiprime ideals of R? Are the results remain valid if we suppose that g is only map ?

References

[1] F. A. Almahdi, A. Mamouni and A. Tamekkante, *A generalization of Posners theorem on derivations in rings,* Indian J. Pure Appl. Math., **51** (1), (2020), 187-194.

[2] A. Mamouni, L. Oukhtite and M. Zerra, *On derivations involving prime ideals and commutativity in rings,* São Paulo Journal of Mathematical Sciences, **14**, (2020), 675-688.

[3] H. E. Mir, *Commutativity with algebraic identities involving prime ideals,* Communications of the Korean Mathematical Society, **35** (3), (2020), 723–731.

[4] E. C. Posner, *Derivations in prime rings,* Proc Amer. Math. Soc., 8, (1957), 1093-1100.

[5] J. Bergen, *Derivations in prime rings,* Canad. Math. Bull., 26, (1983), 267-270.

[6] M. Ashraf and N. Rehman, *On derivations and commutativity in prime rings,* East-West J. Math., 3(1), (2001), 87-91.

[7] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings,* Int. J. Math. Math. Sci., 15(1), (1992), 205-206.

[8] N. Rehman and H. Al-Noghashi, *Action of prime ideals on generalized derivations-I,* arXiv preprint arXiv:2107.06769, (2021).