Int. J. Open Problems Compt. Math., Vol. 17, No. 1, March 2024
Print ISSN: 1998-6262, Online ISSN: 2079-0376

The Laplace Decomposition Method for Solving Nonlinear Conformable Fractional Evolution Equations

Ahmed ANBER and Zoubir DAHMANI

Department of Mathematics University of USTO, Oran, 31000, Algeria

Department of Mathematics, Unversity of Blida 1, Blida 09000, Algeria
e-mail: ah.anber@gmail.com, zzdahmani@yahoo.fr

Received 20 January 2024; Accepted 19 March 2024

Abstract

In the present paper, we discuss the application of the Laplace decomposition method (LDM) to find approximate solutions for some classes of time and space-conformable fractional evolution equations. Some illustrative examples are presented showing that the LDM is an efficient method for finding solutions of nonlinear conformable fractional problems.

Keywords: *Conformable derivative, Laplace Decomposition Method, Fractional differential equation.*

2010 Mathematics Subject Classification: 26A33, 44A10, 35R11.

1 Introduction

Some applications in sciences are modeled by nonlinear fractional differential equations and it is very difficult to find solutions to these equations. For this reason, there are several methods that can be applied to find approximate solutions [5, 6, 10, 11, 28]. In particular, one of these methods, is the Laplace decomposition method (LDM), for more information and also for some applications, we cite the research papers [7, 8, 13, 14, 29, 31, 34]. The principle of the LDM method is in combining the Laplace transform method [12, 21, 23] and the Adomian decomposition method [2, 4, 22, 26].
In this paper, we apply the LDM method to solve some nonlinear conformable

fractional differential equations of the form:

$$T_t^{\beta}u(x,t) + aT_x^{\alpha}u(x,t) + bu(x,t) + g(u(x,t)) = h(x,t), \tag{1.1}$$

with the initial conditions

$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \tag{1.2}$$

where $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivatives in the sense of Khalil [18], with $1 < \alpha, \beta \leq 2$, a, b are constants, $h(x,t)$ is the inhomogeneous part, $g(u(x,t))$ is a nonlinear function of $u(x,t)$ and $f_1(x), f_2(x)$ are a given functions.
The paper is organized as follows: In the second section, some preliminaries related to conformable fractional derivative approach are recalled. In the third section, the Laplace decomposition method is discussed for solving some important conformable fractional evolution equations. In the fourth section, numerical examples are presented. Finally, some graphs of the obtained solutions are plotted, and a conclusion follows.

2 Conformable Fractional Concepts

In this section, we introduce some definitions and properties, see [1, 3, 9, 18, 20, 25, 32].

Definition 2.1 *Let $f:(0,\infty)\to\mathbb{R}$. The conformable fractional derivative of order α is defined by*

$$(T^{\alpha}f)(t) = \frac{\partial^{\alpha}f(t,x)}{\partial t^{\alpha}} = \lim_{\varepsilon\to 0}\left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}\right), \quad t>0, \quad 0<\alpha\leq 1. \tag{2.1}$$

Definition 2.2 *The conformable fractional integral of a function $f:(0,\infty)\to\mathbb{R}$ of order α is defined as*

$$(I^{\alpha}f)(t) = \int_0^t \tau^{\alpha-1}f(\tau)\,d\tau, \qquad 0<\alpha\leq 1. \tag{2.2}$$

In this paper, we need to recall the properties:

$$I^{\alpha}T^{\alpha}f(t) = f(t) - f(0) \tag{2.3}$$

and

$$(T^{\alpha}f)(t) = t^{1-\alpha}\frac{df(t)}{dt}. \tag{2.4}$$

Definition 2.3 *Let* $f : (0, \infty) \to \mathbb{R}$ *be a real valued function and* $0 < \alpha \leq 1$. *Then the conformable fractional Laplace transform of f is defined as:*

$$L_\alpha [f(x)](s) =_\alpha (s) = \int_0^\infty e^{-s\left(\frac{x^\alpha}{\alpha}\right)} f(x)\, d_\alpha x = \int_0^\infty e^{-s\left(\frac{x^\alpha}{\alpha}\right)} f(x)\, x^{\alpha-1} dx \tag{2.5}$$

The Laplace transform for the conformable fractional-order derivative is described as follows:

$$L_\alpha (T^\alpha f(x)) = s L_\alpha [f(x)] - f(0) \tag{2.6}$$

The relation between the usual and the fractional Laplace transforms is given below.

Theorem 2.4 *[1]Let* $f : (0, \infty) \to \mathbb{R}$ *be a real valued function such that* $L_\alpha [f(x)](s) =_\alpha (s)$ *exists. Then:*

$$_\alpha (s) = L\left[f\left((\alpha x)^{\frac{1}{\alpha}}\right)\right](s),$$

where $L[g(x)](s) = \int_0^\infty e^{-st} f(x)\, dx$

It is easy to show that:

Theorem 2.5 *If* $L_\alpha [f(x)](s) =_\alpha (s)$ *exists. Then:*

$$L_\alpha [c](s) = \frac{c}{s}$$

$$L_\alpha [x^p](s) = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{s^{1+\frac{p}{\alpha}}}$$

, c and p are arbitrary constants.

3 Main Steps of LDM Method

In this section, we present the Laplace Decomposition methods in the case of Khalil fractional theory. For more details on the two methods, one can consult [15, 16, 19, 27, 30].
Firstly, Laplace transform is applied on both sides of the equation (1.1). Then, we get

$$L_\beta \left[T_t^\beta u(x,t)\right] + a L_\beta [T_x^\alpha u(x,t)] + b L_\beta [u(x,t)] + L_\beta [g(u(x,t))] = L_\beta [h(x,t)], \tag{3.1}$$

hence,

$$s^\beta u(x,s) - s^{\beta-1} u(x,0) - s^{\beta-2} u_t(x,0) = L_\beta [h(x,t)] - a L_\beta [T_x^\alpha u(x,t)]$$
$$-b L_\beta [u(x,t)] - L_\beta [g(u(x,t))], \tag{3.2}$$

equ. (3.2) can be written as

$$\begin{aligned} u(x,s) &= \tfrac{1}{s}u(x,0) + \tfrac{1}{s^2}u_t(x,0) + \tfrac{1}{s^\beta}L_\beta[h(x,t)] - \tfrac{a}{s^\beta}L_\beta[T_x^\alpha u(x,t)] \\ &\quad - \tfrac{b}{s^\beta}L_\beta[u(x,t)] - \tfrac{1}{s^\beta}L_\beta[g(u(x,t))], \end{aligned} \tag{3.3}$$

Now, we use the fact that the Adomian decomposition method assumes that the function $u(x,t)$ can be decomposed into an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.4}$$

It also assumes that

$$g(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{3.5}$$

where A_n are Adomian polynomials given by

$$A_n = \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\left[g\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{3.6}$$

where λ is a parameter.

So, we have

$$A_0 = \tfrac{1}{0!}\tfrac{d^0}{d\lambda^0}\left[g\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} = g(u_0)$$

$$A_1 = \tfrac{1}{1!}\tfrac{d^1}{d\lambda^1}\left[g\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} = u_1 g'(u_0)$$

$$A_2 = \tfrac{1}{2!}\tfrac{d^2}{d\lambda^2}\left[g\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} = \tfrac{1}{2!}(u_1)^2 g''(u_0) + u_2 g'(u_0)$$

.
.
.

Substitution Equation (3.4), Equation (3.5) and Equation (3.6) in Equation (3.3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,s) &= \tfrac{1}{s}f_1(x) + \tfrac{1}{s^2}f_2(x) + \tfrac{1}{s^\beta}L_\beta[h(x,t)] - \tfrac{a}{s^\beta}L_\beta\left[T_x^\alpha\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\right] \\ &\quad - \tfrac{b}{s^\beta}L_\beta\left[\sum_{n=0}^{\infty}u_n(x,t)\right] - \tfrac{1}{s^\beta}L_\beta\left[\sum_{n=0}^{\infty}A_n\right], \end{aligned} \tag{3.7}$$

Hence,

$$\begin{aligned}
&u_0(x,s) = \tfrac{1}{s}f_1(x) + \tfrac{1}{s^2}f_2(x) + \tfrac{1}{s^\beta}L_\beta[h(x,t)] = f_3(x,s)\\
&u_1(x,s) = -\tfrac{a}{s^\beta}L_\beta[T_x^\alpha(u_0(x,t))] - \tfrac{b}{s^\beta}L_\beta[u_0(x,t)] - \tfrac{1}{s^\beta}L_\beta[A_0]\\
&.\\
&.\\
&.\\
&u_{n+1}(x,s) = -\tfrac{a}{s^\beta}L_\beta[T_x^\alpha(u_n(x,t))] - \tfrac{b}{s^\beta}L_\beta[u_n(x,t)] - \tfrac{1}{s^\beta}L_\beta[A_n]
\end{aligned} \quad (3.8)$$

When the inverse Laplace transform Equation (3.8) is applied, we get

$$\begin{aligned}
&u_0(x,t) = f_3(x,t)\\
&u_{n+1}(x,t) = -L^{-1}\left[\tfrac{a}{s^\beta}L_\beta\left(aT_x^{2\alpha}(u_n(x,t)) + bu_n(x,t) + A_n\right)\right]
\end{aligned} \quad (3.9)$$

4 Nonlinear Applications

As application, we begin by the following nonlinear example:

4.1 Example 1

We consider the nonlinear fractional equation

$$T_t^\beta u(x,t) - T_x^\alpha u(x,t) = \lambda u^\gamma(x,t), \quad 1 < \alpha, \beta \leq 2. \quad (4.1)$$

with the initial conditions:

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x). \quad (4.2)$$

We have

$$u(x,s) = \tfrac{1}{s}u(x,0) + \tfrac{1}{s^2}u_t(x,0) + \tfrac{1}{s^\beta}L_\beta[T_x^\alpha u(x,t)] + \tfrac{\lambda}{s^\beta}L_\beta[u^\gamma(x,t)] \quad (4.3)$$

By using the initial condition, the following recurrence relational is obtained

$$u(x,s) = \tfrac{g_1(x)}{s} + \tfrac{g_2(x)}{s^2} + \tfrac{1}{s^\beta}L_\beta(T_x^\alpha u(x,t) + \lambda u^\gamma(x,t)) \quad (4.4)$$

Applying the inverse Laplace transform, to both sides of Equation (4.4) yields:

$$u(x,t) = g_1(x) + tg_2(x) + L_\beta^{-1}[T_x^\alpha u(x,t) + \lambda u^\gamma(x,t)] \quad (4.5)$$

Case 1: For $\gamma = 1$
We have

$$\sum_{n=0}^{\infty} u_n(x,t) = g_1(x) + tg_2(x) + +L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[\sum_{n=0}^{\infty}T_x^\alpha u_n(x,t) + \lambda\sum_{n=0}^{\infty}u_n(x,t)\right]\right]$$

Hence,

$$\begin{aligned} &u_0(x,t) = g_1(x) + tg_2(x) \\ &u_{n+1}(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left[\sum_{n=0}^{\infty} T_x^\alpha u_n(x,t) + \lambda\sum_{n=0}^{\infty} u_n(x,t)\right]\right] \end{aligned} \tag{4.6}$$

Consequently

$$u_1(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_0(x,t) + \lambda u_0(x,t)\right)\right] = \left(\lambda g_1(x) + T^\alpha(g_1(x))\right)\frac{t^\beta}{\Gamma(\beta+1)} + \left(\lambda g_2(x) + T^\alpha(g_2(x))\right)\frac{t^{\beta+1}}{\Gamma(\beta+2)}$$

$$u_2(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1(x,t) + \lambda u_1(x,t)\right)\right]$$

$$= L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(\begin{array}{c} T_x^\alpha\left(\begin{array}{c}\left(\left(\lambda g_1(x) + T_x^\alpha(g_1(x))\right)\frac{t^\beta}{\Gamma(\beta+1)}\right) \\ + \left(\lambda g_2(x) + T_x^\alpha(g_2(x))\right)\frac{t^{\beta+1}}{\Gamma(\beta+2)}\end{array}\right) \\ +\lambda\left(\begin{array}{c}\left(\lambda g_1(x) + T_x^\alpha(g_1(x))\right)\frac{t^\beta}{\Gamma(\beta+1)} \\ + \left(\lambda g_2(x) + T^\alpha(g_2(x))\right)\frac{t^{\beta+1}}{\Gamma(\beta+2)}\end{array}\right)\end{array}\right)\right]$$

$$= \left[2\lambda T_x^\alpha(g_1(x)) + T_x^{2\alpha}(g_1(x)) + \lambda^2 g_1(x)\right]\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \left[2\lambda T_x^\alpha(g_2(x)) + T_x^{2\alpha}(g_2(x)) + \lambda^2 g_2(x)\right]\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}$$

$$u_3(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(\frac{\partial^\alpha u_2(x,t)}{\partial x^\alpha} + \lambda u_2(x,t)\right)\right]$$

$$= L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(\begin{array}{c} T_x^\alpha\left(\begin{array}{c}\left[2\lambda T_x^\alpha(g_1(x)) + T_x^{2\alpha}(g_1(x)) + \lambda^2 g_1(x)\right]\frac{t^{2\beta}}{\Gamma(2\beta+1)} \\ \left[2\lambda T_x^\alpha(g_2(x)) + T_x^{2\alpha}(g_2(x)) + \lambda^2 g_2(x)\right]\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\end{array}\right) \\ +\lambda\left(\begin{array}{c}\left[2\lambda T_x^\alpha(g_1(x)) + T_x^{2\alpha}(g_1(x)) + \lambda^2 g_1(x)\right]\frac{t^{2\beta}}{\Gamma(2\beta+1)} \\ \left[2\lambda T_x^\alpha(g_2(x)) + T_x^{2\alpha}(g_2(x)) + \lambda^2 g_2(x)\right]\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\end{array}\right)\end{array}\right)\right]$$

$$= \left[3\lambda T_x^{2\alpha}(g_1(x)) + 3\lambda^2 T_x^\alpha(g_1(x)) + T_x^{3\alpha}(g_1(x)) + \lambda^3 g_1(x)\right]\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \left[3\lambda T_x^{2\alpha}(g_2(x)) + 3\lambda^2 T_x^\alpha(g_2(x)) + T_x^{3\alpha}(g_2(x)) + \lambda^3 g_2(x)\right]\frac{t^{3\beta+1}}{\Gamma(3\beta+2)}$$

.
.
.

For $\lambda = 1, g_1(x) = 1 + \sin x, g_2(x) = 0$ and $\alpha = 2$, we obtain

$$u_0(x,t) = 1 + \sin x$$

$$u_1(x,t) = \frac{t^\beta}{\Gamma(\beta+1)}$$

.
.
.

$$u_n(x,t) = \frac{t^{n\beta}}{\Gamma(n\beta+1)}$$

Thus,

$$u\left(x,t\right) = 1+\sin x+\tfrac{t^{\beta}}{\Gamma(\beta+1)}+\tfrac{t^{2\beta}}{\Gamma(2\beta+1)}+...+\tfrac{t^{n\beta}}{\Gamma(n\beta+1)}+... \qquad (4.7)$$

To find the validity of the approximated solution, when $\beta = 2$, the exact solution is:

$$u\left(x,t\right) = \sin x+1+\tfrac{t^{2}}{\Gamma(3)}+\tfrac{t^{4}}{\Gamma(5)}+...+\tfrac{t^{2n}}{\Gamma(2n+1)}+...=\sin x+\cosh t. \qquad (4.8)$$

Figure 1.1 is a graph of the solution (4.7), respectively for β equal to 1.25, 1.5, 1.75, and 2.

$itbpF2.2381in1.6639in0inFigure$ (a)	$itbpF2.2381in1.6639in0inFigure$ (b)
$itbpF2.207in1.6405in0inFigure$ (c)	$itbpF2.175in1.6172in0inFigure$ (d)

Fig1.1 Graph of Eq.(4.7) with different β values
(a) $\beta = 1.25$, (b) $\beta = 1.5$, (c) $\beta = 1.75$, (d) $\beta = 2$,

Case 2: For $\gamma \neq 1$

We have

$$\sum_{n=0}^{\infty} u_n\left(x,t\right) = g_1\left(x\right)+tg_2\left(x\right)+L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left[\sum_{n=0}^{\infty} T_x^{\alpha}u_n\left(x,t\right)+\lambda\sum_{n=0}^{\infty} A_n\right]\right] \qquad (4.9)$$

The few components of Adomian polynomials are given by

$$\begin{aligned} A_0 &= u_0^{\gamma} \\ A_1 &= \gamma u_1 u_0^{\gamma-1} \\ A_2 &= \gamma u_2 u_0^{\gamma-1}+\tfrac{\gamma(\gamma-1)}{2}u_1^2 u_0^{\gamma-2} \\ A_3 &= \gamma u_3 u_0^{\gamma-1}+\gamma\left(\gamma-1\right)u_1 u_2 u_0^{\gamma-2}+\tfrac{\gamma(\gamma-1)(\gamma-2)}{6}u_1^3 u_0^{\gamma-3} \end{aligned} \qquad (4.10)$$

Therefore,

$$\begin{aligned} u_0\left(x,t\right) &= g_1\left(x\right)+tg_2\left(x\right) \\ u_{n+1}\left(x,t\right) &= L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left[\sum_{n=0}^{\infty} T_x^{\alpha}u_n\left(x,t\right)+\lambda\sum_{n=0}^{\infty} A_n\right]\right] \end{aligned} \qquad (4.11)$$

Consequently

$$\begin{array}{l} u_1(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_0(x,t) + \lambda A_0\right)\right] \\ = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_0 + \lambda u_0^\gamma\right)\right] \\ \\ u_2(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1(x,t), + \lambda A_1\right)\right] \\ = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1 + \lambda\gamma u_1 u_0^{\gamma-1}\right)\right] \\ \\ u_3(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_2(x,t) + \lambda A_2\right)\right] \\ = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_2 + \lambda\gamma u_2 u_0^{\gamma-1} + \frac{\lambda\gamma(\gamma-1)}{2}u_1^2 u_0^{\gamma-2}\right)\right] \\ . \\ . \\ . \end{array}$$

For $\lambda = 1, \gamma = 2, g_1(x) = x^2, g_2(x) = 0$, we obtain

$$\begin{array}{l} u_0(x,t) = x^2 \\ u_1(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_0(x,t) + A_0\right)\right] \\ = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_0(x,t) + u_0^2\right)\right] = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(2x^{2-\alpha} + x^4\right)\right] \\ = \left(2x^{2-\alpha} + x^4\right)\frac{t^\beta}{\Gamma(\beta+1)} \\ \\ u_2(x,t) = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1(x,t) + A_1\right)\right] \\ = L_\beta^{-1}\left[\frac{1}{s^\beta}L_\beta\left(\left(2(2-\alpha)(1-\alpha)x^{2-2\alpha} + 16x^{4-\alpha} + 2x^6\right)\frac{t^\beta}{\Gamma(\beta+1)}\right)\right] \\ = \left(2(2-\alpha)(1-\alpha)x^{2-2\alpha} + 16x^{4-\alpha} + 2x^6\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)} \\ . \\ . \\ . \end{array} \tag{4.12}$$

Thus,

$$\begin{array}{c} u(x,t) = x^2 + \left(2x^{2-\alpha} + x^4\right)\frac{t^\beta}{\Gamma(\beta+1)} \\ + \left(2(2-\alpha)(1-\alpha)x^{2-2\alpha} + 16x^{4-\alpha} + 2x^6\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + ... \end{array} \tag{4.13}$$

Figure 1.2 is a graph of the solution (4.13), for α, β equal to 1.25, 1.5, 1.75, and 2.

$itbpF2.29in1.702in0inFigure$	$itbpF2.3004in1.7097in0inFigure$
(a)	(b)
$itbpF2.2174in1.6483in0inFigure$	$itbpF2.2174in1.6483in0inFigure$
(c)	(d)

Fig1.2 Graph of Eq.(4.13) with different α, β values
(a) $\alpha = \beta = 1.25$, (b) $\alpha = \beta = 1.5$, (c) $\alpha = \beta = 1.75$, (d) $\alpha = \beta = 2$,

4.2 Example 2

We consider the nonlinear fractional equation

$$T_t^{\beta}u(x,t) - T_x^{\alpha}u(x,t) = u(x,t) + u^2(x,t), \quad 1 < \alpha, \beta \leq 2. \tag{4.14}$$

with the initial conditions:

$$u(x,0) = 1 + x, \quad u_t(x,0) = 0. \tag{4.15}$$

When the Laplace transform is applied to both sides of Equation (4.14), we can write

$$\begin{aligned} u(x,s) = {} & \tfrac{1}{s}u(x,0) + \tfrac{1}{s^2}u_t(x,0) + \tfrac{1}{s^{\beta}}L_{\beta}\left[T_x^{\alpha}u(x,t)\right] \\ & + \tfrac{1}{s^{\beta}}L_{\beta}\left[u^2(x,t)\right]\tfrac{1}{s^{\beta}}L_{\beta}\left[u(x,t)\right] \end{aligned} \tag{4.16}$$

Hence,

$$u(x,s) = \frac{x^2}{s} + \frac{1}{s^{\beta}}L_{\beta}\left(T_x^{\alpha}u(x,t) + u^2(x,s) + u(x,s)\right) \tag{4.17}$$

Applying the inverse Laplace transform, to both sides of Equation (4.17) yields:

$$u(x,t) = x^2 + L_{\beta}^{-1}\left[\frac{1}{s^{\beta}}L_{\beta}\left(T_x^{\alpha}u(x,t) + u^2(x,t) + u(x,t)\right)\right] \tag{4.18}$$

Hence,

$$\sum_{n=0}^{\infty}u_n(x,t) = x^2 + L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left[\sum_{n=0}^{\infty}T_x^{\alpha}u_n(x,t) + \sum_{n=0}^{\infty}B_n + u_n(x,t)\right]\right] \tag{4.19}$$

On the other hand, we have

$$\begin{aligned} B_0 &= u_0^2 \\ B_1 &= 2u_0u_1 \\ B_2 &= 2u_0u_2 + u_1^2 \\ B_3 &= 2u_0u_3 + 2u_1u_2 \end{aligned}$$

Then

$$\begin{aligned} & u_0(x,t) = 1 + x^2, \\ & u_{n+1}(x,t) = L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left[\sum_{n=0}^{\infty}T_x^{\alpha}u_n(x,t) + \sum_{n=0}^{\infty}B_n + u_n(x,t)\right]\right] \end{aligned} \tag{4.20}$$

Consequently

$$\begin{aligned} u_1(x,t) &= L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left(T_x^{\alpha}u_0(x,t) + B_0 + u_0(x,t)\right)\right] \\ &= L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left(T_x^{\alpha}u_0 + u_0^2 + u_0\right)\right] \\ &= L_{\beta}^{-1}\left[\tfrac{1}{s^{\beta}}L_{\beta}\left(2x^{2-\alpha} + x^4 + 2 + 3x^2\right)\right] \\ &= \left(2x^{2-\alpha} + x^4 + 3x^2 + 2\right)\tfrac{t^{\beta}}{\Gamma(\beta+1)} \end{aligned}$$

$$u_2(x,t) = L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1(x,t) + B_1 + u_1(x,t)\right)\right]$$
$$= L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left(T_x^\alpha u_1 + 2u_0u_1 + u_1\right)\right]$$
$$= L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left(\begin{array}{c} T_x^\alpha\left(\left(2x^{2-\alpha}+x^4+3x^2+2\right)\frac{t^\beta}{\Gamma(\beta+1)}\right) \\ +2\left(1+x^2\right)\left(\left(2x^{2-\alpha}+x^4+3x^2+2\right)\frac{t^\beta}{\Gamma(\beta+1)}\right) \\ +\left(2x^{2-\alpha}+x^4+3x^2+2\right)\frac{t^\beta}{\Gamma(\beta+1)} \end{array}\right)\right]$$
$$= L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left(\begin{array}{c} T_x^\alpha\left(\left(2x^{2-\alpha}+x^4+3x^2+2\right)\frac{t^\beta}{\Gamma(\beta+1)}\right) \\ +\left(6x^{2-\alpha}+4x^{4-\alpha}+2x^6+9x^4+13x^2+6\right)\frac{t^\beta}{\Gamma(\beta+1)} \end{array}\right)\right]$$
$$= \left(2\left(2-\alpha\right)x^{2-2\alpha}+8x^{4-\alpha}+12x^{2-\alpha}+2x^6+9x^4+13x^2+6\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}$$
.
.
.

Thus,

$$u(x,t) = 1 + x^2 + \left(2x^{2-\alpha}+x^4+3x^2+2\right)\frac{t^\beta}{\Gamma(\beta+1)}$$
$$+\left(2\left(2-\alpha\right)x^{2-2\alpha}+8x^{4-\alpha}+12x^{2-\alpha}+2x^6+9x^4+13x^2+6\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\cdots \tag{4.21}$$

Figure 1.3 is a graph of the solution (4.21), for α, β equal to 1.25, 1.5, 1.75, and 2.

*itbpF*2.3627*in*1.7582*in*0*inFigure* *itbpF*2.373*in*1.7651*in*0*inFigure*
(*a*) (*b*)
*itbpF*2.4146*in*1.7953*in*0*inFigure* *itbpF*2.4362*in*1.8118*in*0*inFigure*
(*c*) (*d*)

Fig1.3 Graph of Eq.(4.21) with different α, β values
$(a)\ \alpha=\beta=1.25,\ \ (b)\ \alpha=\beta=1.5,\ \ (c)\ \alpha=\beta=1.75,\ \ (d)\ \alpha=\beta=2,$

4.3 Example 3

Consider the nonlinear fractional wave equation [16, 17]

$$T_t^\alpha u(x,t) + aT_x^\alpha u(x,t) + bu(t,x) + cu^3(t,x) = 0,$$
$$0 < x < 1,\ t > 0,\ 1 < \alpha, \beta \le 2. \tag{4.22}$$

with the initial conditions:

$$u(x,0) = h_1(x), \quad u_t(x,0) = h_2(x). \tag{4.23}$$

where a, b, c are constants.
We have

$$\begin{aligned} & u_0(x,t) = h_1(x) + t \times h_2(x) \\ & u_{n+1}(x,t) = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[a\sum_{n=0}^{\infty}T_x^\alpha u_n(x,t) + b\sum_{n=0}^{\infty}u_n(x,t) + c\sum_{n=0}^{\infty}C_n\right]\right] \end{aligned} \tag{4.24}$$

where C_n are given by

$$C_n = \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^3\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{4.25}$$

Hence,

$$\begin{aligned} C_0 &= \tfrac{1}{0!}\tfrac{d^0}{d\lambda^0}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^3\right]_{\lambda=0} = (u_0)^3 \\ C_1 &= \tfrac{1}{1!}\tfrac{d^1}{d\lambda^1}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^3\right]_{\lambda=0} = 3u_1(u_0)^2 \\ C_2 &= \tfrac{1}{2!}\tfrac{d^2}{d\lambda^2}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^3\right]_{\lambda=0} = 3(u_1)^2(u_0) + 3u_2(u_0)^2 \\ C_3 &= \tfrac{1}{3!}\tfrac{d^3}{d\lambda^3}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^3\right]_{\lambda=0} = 3u_3(u_0)^2 + 6(u_0)(u_1)(u_2) + (u_1)^3 \end{aligned}$$

For $\alpha = 2, h_1(x) = x$ and $h_2(x) = 0$, we obtain the following recurrence relations,

$$\begin{aligned} & u_0(x,t) = x \\ & u_{n+1}(x,t) = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[aT_x^\alpha u_{nxx}(x,t) + bu_n(x,t) + cC_n\right]\right] \end{aligned}$$

Consequently

$$\begin{aligned} & u_0(x,t) = x, \\ & u_1(x,t) = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[au_{0xx}(x,t) + bu_0(x,t) + c(u_0)^3\right]\right] \\ & = -(bx + cx^3)\tfrac{t^\beta}{\Gamma(\beta+1)} \\ & u_2(x,t) = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[au_{1xx}(x,t) + bu_1(x,t) + 3cu_1(u_0)^2\right]\right] \\ & = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[-6acx - b(bx + cx^3) - 3c(bx + cx^3)(x)^2\right]\right] \\ & \left[(6ac + b^2)x + 4bcx^3 + 3c^2x^5\right]\tfrac{t^{2\beta}}{\Gamma(2\beta+1)} \\ & u_3(x,t) = -L_\beta^{-1}\left[\tfrac{1}{s^\beta}L_\beta\left[au_{2xx}(x,t) + bu_2(x,t) + 3c(u_1)^2(u_0) + 3cu_2(u_0)^2\right]\right] \end{aligned}$$

Thus,

$$\begin{aligned} u(x,t) = x - (bx + cx^3)\tfrac{t^\beta}{\Gamma(\beta+1)} \\ + ((6ac + b^2)x + 4bcx^3 + 3c^2x^5)\tfrac{t^{2\beta}}{\Gamma(2\beta+1)} + ... \end{aligned} \tag{4.26}$$

Figure 1.4 is a graph of the solution (4.21), for $a = b = c = 1$ and α, β equal to $1.25, 1.5, 1.75$, and 2.

$itbpF2.6126in1.9424in0inFigure$ $itbpF2.6022in1.9346in0inFigure$
(a) (b)
$itbpF2.4984in1.8568in0inFigure$ $itbpF2.4984in1.8576in0inFigure$
(c) (d)

Fig1.4 Graph of Eq.(4.26) with different α, β values
(a) $\alpha = \beta = 1.25$, (b) $\alpha = \beta = 1.5$, (c) $\alpha = \beta = 1.75$, (d) $\alpha = \beta = 2$,

5 Conclusion and Open Problem

The LDM method is applied for some nonlinear conformable fractional differential problems. It is shown that this method is a powerful device to solve not only linear problems but it is also valid for nonlinear ones.
At the end of this paper, we shall propose the following open question:
We think it is important to address a comparative study with other numerical methods for the above general conformable fractional evolution problem.

References

[1] T. Abdeljawad: *On conformable fractional calculus*, J. Comput. Appl. Math. 279(1), pp. 57-66. (2015).

[2] B.C. Agbata, B.N. Ani, M.M. Shior, I.G. Ezugorie, R.V. Paul, P.K. Meseda: *Analysis of Adomian decomposition method and its application for solving linear and nonlinear differential equations.* Asian Research Journal of Mathematics. Vol. 18(2), pp. 56-70. (2022).

[3] Z. Al-zhour, F. Alrawajeh, N. Al-mutairi and R. Alkhasawneh: *New results on the conformable fractional Sumudu transforms: Theories and applications*, International Journal of Analysis and Applications. 17(6), 1019-1033. (2019).

[4] A. Anber and Z. Dahmani: *Solutions of the reaction-diffusion Brusselator with fractional derivatives.* Journal of Interdisciplinary Mathematics. Vol. 17(5 & 6), pp. 451–460. (2014).

[5] A. Anber, I. Jebril, Z. Dahmani, N. Bedjaoui, A. Lamamri: *The Tanh method and the (G' /G)-expantion method for solving the space -time conformable FZK and FZZ evolution equations.* International Journal of Innovative Computing, Information and Control. Vol. 20(2), pp.557-573. (2024).

[6] A. Anber and Z. Dahmani: *The SGEM method for solving some time and space-conformable fractional evolution problems*, Int. J. Open Problems Compt. Math. Vol.16(1), pp.33-43, (2023).

[7] H. O. Bakodah, M. Al-Mazmumy, S. O. Almuhalbedi, Lazim Abdullah: *Laplace discrete Adomian decomposition method for solving nonlinear integro differential equations*. Journal of Applied Mathematics and Physics. Vol. 7, pp. 1388-1407. (2019).

[8] M. Bezziou, I. Jebril and Z. Dahmani: *A new nonlinear duffing system with sequential fractional derivatives*. Chaos, Solitons and Fractals. Vol. 151. (2021)

[9] A. Bouchenak: *Generalization of fractional Laplace transform for higher order and its application*. J. Innov. Appl. Math. Comput. Sci. 1(1), pp. 79–92. (2021).

[10] Z. Dahmani, A. Anber and I. Jebril: *Solving conformable evolution equations by an extended numerical method*. Jordan Journal of Mathematics and Statistics (JJMS), Vol.15(2). pp.363-280. (2021)

[11] Z. Dahmani, A. Anber, Y. Gouari, M. Kaid and I, Jebril: *Extension of a method for solving nonlinear evolution equations via conformable fractional approach*. International Conference on Information Technology. 38-42. (2021).

[12] H. Demir, I.C. Sungu, I. Keles: *Investigating the Laplace transform method's efficiency to a simple engineering problem*. Turk. J. Math. Comput. Sci. 15(2). pp. 326–333. (2023).

[13] O.G. Gaxiola: *The Laplace-Adomian decomposition method applied tothe Kundu-Eckhaus equation*. International Journal of Mathematics And its Applications. Vol. 5(1-A), pp. 1–12. (2017).

[14] Y. Gouari, Z. Dahmani and I. Jebril: *Application of fractional calculus on a new differential problem of duffing type*. Advances in Mathematics: Scientific Journal .9 (12). pp. 10989–11002. (2020).

[15] S. S. Handibag: *Laplace decomposition method for the system of non Linear PDEs*. Open Access Library Journal, 6. (2019).

[16] H. Jafari, C.M. Khalique, M. Nazari: *Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion–wave equations*. Applied Mathematics Letters 24, 1799–1805. (2011).

[17] H. Jafari, S. Momani: *Solving fractional diffusion and wave equations by modified homotopy perturbation method*. Phys. Lett. A 370, 388–396. (2007).

[18] R. Khalil, M. Al Horani, A. Yousef. and M. Sababheh: *A new definition of fractional derivative*. Journal of Computational and Applied Mathematics, 264, 65-70. (2014).

[19] H. Khan, R. Shah, P. Kumam, D. Baleanu and M. Arif: *Laplace decomposition for solving nonlinear system of fractional order partial differential equations*. Advances in Difference Equations, 375, (2020).

[20] N. A. Khan, O. A. Razzaq, and M. Ayaz: *Some properties and applications of conformable fractional Laplace transform (CFLT)*. J. Fract. Calc. Appl. 9(1), 72–81. (2018).

[21] N. Kokulan and C.H. Lai: *A Laplace transform method for the image inpainting*. Journal of Algorithms & Computational Technology. Vol. 9 (1), pp. 95-104. (2014).

[22] W. Li and Y. Pang: *Application of Adomian decomposition method to nonlinear systems*. Advances in Difference Equations. Vol. 67. (2020).

[23] P.M. Makila: *A note on the Laplace transform method for initial value problems*. International Journal of Control Vol. 79(1), pp. 36-41. (2006).

[24] W. Malfliet, W. Hereman: *The tanh method: I. exact solutions of nonlinear evolution and wave equations.* Phys. Scripta, 54, pp. 563-568. (1996).

[25] F. Martinez, I. Martinez, M.K.A. Kaabar, R.O. Munuera, S. Paredes: *Note on the conformable fractional derivatives and integrals of complex-valued functions of a real variable*. IAENG International Journal of Applied Mathematics, 50(3), 18. (2020).

[26] D.A. Maturi, H. M. Malaikah: *The Adomian decomposition method for solving nonlinear partial differential equation using Maple*. Advances in Pure Mathematics. Vol. 11(6). pp. 595-603. (2021).

[27] M.Z. Mohamed, T.M. Elzaki: *Comparison between the Laplace decomposition method and Adomian decomposition in time-space fractional nonlinear fractional differential equations*, Applied Mathematics, 9, 448-458. (2018).

[28] M. Rakah, A. Anber, Z. Dahmani, I. Jebril: *An Analytic and Numerical study for two classes of differential equation of fractional order involving Caputo and Khalil derivative*. An. Stiint. Univ. Al. I. Cuza Iasi. Mat.(NS), Tomul LXIX, f. 1, pp 29-48. (2023).

[29] M. Sambath and K. Balachandran: *Laplace Adomian decomposition method for solving a fish farm model,* Nonauton. Dyn. Syst. 3, p.p 104–111. (2016)

[30] A.S. Shazad, B.M.A. Miran, A.H. Shokhan: *Laplace Adomian decomposition and modify Laplace Adomian decomposition methods for solving linear Volterra integro-fractional differential equations with constant multi-time retarded delay*, Journal of University of Babylon for Pure and Applied Sciences. Vol. 27 (5), 35-53. (2019).

[31] S.A. Shazad, A.H.S. Shokhan, R.A. Mariwan: *Laplace Adomian and Laplace modified Adomian decomposition methods for solving nonlinear integro-fractional differential equations of the Volterra-Hammerstein type.* Iraqi Journal of Science, Vol.60 (10), pp: 2207-2222. (2019).

[32] F.S. Silva, D.M. Moreira, M.A. Moret: *Conformable Laplace transform of fractional differential equations.* J. axioms, 7(3), 50. (2018).

[33] A.M. Wazwaz: *New solitary wave solutions to modified forms of Degasperis-Procesi and Camassa-Holm equations.* Applied Mathematics and Computations. (2007).

[34] Z.H. Xing, A.H. Li and C. Yong: *A Laplace decomposition method for nonlinear partial differential equations with nonlinear term of any order.* Commun. Theor. Phys. 61 23. (2018).