

Ramanujan's Entries and Solutions of Incomplete Elliptic Integrals in Terms of Hypergeometric Functions

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Abstract

The main object of this note is to obtain some necessary conditions (in terms of truncated Gaussian hypergeometric forms) associated with four results of entry 7 of chapter 17 of second note book of Ramanujan. Unsolved - incomplete elliptic integrals of first and second kinds are also solved and generalized in terms of hypergeometric functions.

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2000 AMS Classifications: 33-Special Functions.

1 Introduction and Preliminary Results

- (i) Pochhammer Symbol - If a is neither zero nor negative integer then

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} = \begin{cases} 1 & ; r = 0 \\ a(a+1)\cdots(a+r-1) & ; r = 1, 2, 3, \dots \end{cases} \quad (1.1)$$

where the notation (Γ) stands for Gamma function.

(ii) Gauss ordinary hypergeometric function ${}_2F_1$ is defined as

$${}_2F_1 \left[\begin{matrix} a, b; \\ c \end{matrix}; z \right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r r!} z^r$$

where $|z| < 1$ and $c \neq 0, -1, -2, -3, \dots$ (1.2)

(iii) Truncated ordinary hypergeometric function [21] is defined as

$${}_2F_1 \left[\begin{matrix} a, b; \\ c \end{matrix}; z \right] \text{ to } m \text{ terms} = {}_2F_1 \left[\begin{matrix} a, b; \\ c \end{matrix}; z \right]_{m-1} = \sum_{r=0}^{m-1} \frac{(a)_r (b)_r}{(c)_r r!} z^r \quad (1.3)$$

when $m \rightarrow \infty$, then (1.3) reduces to (1.2).

(f) Complete Elliptic Integral of First and Second Kind

$$K(\sqrt{x}) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} 1/2, 1/2; \\ 1 \end{matrix}; x \right]$$

$$E(\sqrt{x}) = \int_0^{\pi/2} \sqrt{(1-x \sin^2 \theta)} d\theta = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} 1/2, -1/2; \\ 1 \end{matrix}; x \right]$$

where $|x| < 1$ (1.4)

(v). Kampé de Fériet's ordinary double hypergeometric function [22; 23]:

(In the modified notation of *Srivastava and Panda* [24; 25; 26])

$${}^F A : B; D \left[\begin{matrix} (a_A) : (b_B); (d_D); \\ (e_E) : (g_G); (h_H); \end{matrix}; x, y \right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n} [(b_B)]_m [(d_D)]_n x^m y^n}{[(e_E)]_{m+n} [(g_G)]_m [(h_H)]_n m! n!} \quad (1.10)$$

where above all notations have their standard usual meanings. For convergence of above double power series, any one of the following conditions should be satisfied

$$\text{Either, (i) } A+B < E+G+1, A+D < E+H+1, \text{ then } \max \{|x|, |y|\} < \infty \quad (1.6)$$

$$\text{Or, (ii) } A+B = E + G + 1, A + D = E + H + 1 \text{ and } A > E$$

$$\text{then } |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 \quad (1.7)$$

$$\text{Or, (iii) If } A+B = E+G+1, A+D = E + H + 1 \text{ and } A \leq E$$

$$\text{then } \max\{|x|, |y|\} < 1 \quad (1.8)$$

(vi) Reduction formula for Clausenian hypergeometric function ${}_3F_2$

$${}_3F_2 \left[\begin{matrix} a, b, 1 \\ c, 2 \end{matrix} ; z \right] = \frac{(c-1)}{(a-1)(b-1)z} \left\{ {}_2F_1 \left[\begin{matrix} a-1, b-1 \\ c-1 \end{matrix} ; z \right] - 1 \right\} \quad (1.9)$$

The equation (1.14) can be derived easily from the following method

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} a-1, b-1 \\ c-1 \end{matrix} ; z \right] &= \sum_{m=0}^{\infty} \frac{(a-1)_m (b-1)_m z^m}{(c-1)_m m!} \\ &= 1 + \sum_{m=1}^{\infty} \frac{(a-1)_m (b-1)_m z^m}{(c-1)_m m!} = 1 + \sum_{m=0}^{\infty} \frac{(a-1)_{m+1} (b-1)_{m+1} z^{m+1}}{(c-1)_{m+1} (m+1)!} \\ &= 1 + \frac{(a-1)(b-1)z}{(c-1)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m (2)_m} \quad \square \end{aligned}$$

2 Generalized Form of Incomplete Elliptic Integrals of First and Second Kind

The following reduction formula is familiar in the literature

$$(a) \quad I_n = \int \sin^n \theta d\theta = \frac{-\sin^{n-1} \theta \cos \theta}{n} + \frac{n-1}{n} I_{n-2}; \quad n \geq 2 \quad (2.1)$$

where $I_0 = \theta$, $I_1 = -\cos \theta$

By successive application of (2.1), we can find

$$(b) \quad I_{2m} = \int \sin^{2m} \theta d\theta \\ = \left(\frac{-(1/2)_m \sin \theta \cos \theta}{m!} \sum_{r=0}^{m-1} \frac{(1)_r (\sin^2 \theta)^r}{(3/2)_r} \right) + \left(\frac{(1/2)_m \theta}{m!} \right); \text{ where } m \geq 0 \quad (2.2)$$

Here $I_0 = \theta$ and empty sum $\sum_{r=0}^{-1}$ is treated as zero. The integral (2.2) can be verified for $m = 0, 1, 2, 3, 4, 5 \dots$

In this note, the following incomplete elliptic integral of first kind, will play an important role, to decide the necessary conditions.

$$F(\sqrt{x}, \alpha) = \int_0^\alpha \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} = \alpha {}_2F_1 \left[\begin{matrix} 1/2, 1/2; \\ 1 \end{matrix}; x \right] - \frac{x \sin \alpha \cos \alpha}{4} \times \\ \times F \left[\begin{matrix} 2:1; 2 \\ 2:0; 1 \end{matrix}; \begin{matrix} 3/2, 3/2:1; 1, 1 \\ 2, 2 \quad :-; 3/2 \end{matrix}; x, x \sin^2 \alpha \right] \quad (2.3)$$

where $\max \{ |x|, |x \sin^2 \alpha| \} < 1$

Proof of (2.3):

$$\begin{aligned} F(\sqrt{x}, \alpha) &= \int_0^\alpha \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = \int_0^\alpha (1-x\sin^2\theta)^{-1/2} d\theta \\ &= \int_0^\alpha {}_1F_0\left[\begin{matrix} 1/2; \\ -; \end{matrix}; x\sin^2\theta\right] d\theta = \int_0^\alpha \left(\sum_{m=0}^{\infty} \frac{(1/2)_m x^m \sin^{2m}\theta}{m!}\right) d\theta \end{aligned}$$

Since change the order of integration and summation, is permissible under suitable convergence condition [28, p.45], we get

$$\begin{aligned} F(\sqrt{x}, \alpha) &= \sum_{m=0}^{\infty} \frac{(1/2)_m x^m}{m!} \int_0^\alpha \sin^{2m}\theta d\theta \\ &= \sum_{m=0}^{\infty} \frac{(1/2)_m x^m}{m!} \left\{ \left(\frac{-(1/2)_m \sin\alpha \cos\alpha}{m!} {}_2F_1\left[\begin{matrix} 1, 1; \\ 3/2; \end{matrix}; \sin^2\alpha\right]_{m-1} \right) + \left(\frac{(1/2)_m \alpha}{m!} \right) \right\} \end{aligned} \quad (2.4)$$

$$\begin{aligned} &= \alpha + \sum_{m=1}^{\infty} \frac{(1/2)_m x^m}{m!} \left[\left(-\sin\alpha \cos\alpha \frac{(1/2)_m}{m!} \sum_{r=0}^{m-1} \frac{r!(\sin^2\alpha)^r}{(3/2)_r} \right) + \left(\frac{(1/2)_m \alpha}{m!} \right) \right] \end{aligned} \quad (2.5)$$

Replacing m by $m+1$ in equation (2.5) and expressing single and double power series in hypergeometric form, we get

$$\begin{aligned} F(\sqrt{x}, \alpha) &= \alpha - \frac{x \sin\alpha \cos\alpha}{4} F \begin{matrix} 2:1; 2 \\ 2:0; 1 \end{matrix} \left[\begin{matrix} 3/2, 3/2 : 1; 1, 1; \\ 2, 2 \quad \quad \quad : -; 3/2; \end{matrix} x, x \sin^2\alpha \right] + \\ &\quad + \frac{x\alpha}{4} {}_3F_2 \left[\begin{matrix} 1, 3/2, 3/2; \\ 2, 2 \quad \quad \quad ; \end{matrix} x \right] \end{aligned} \quad (2.6)$$

Now using (1.14) in equation (2.6), we get

$$F(\sqrt{x}, \alpha) = \alpha {}_2F_1 \left[\begin{matrix} 1/2, 1/2; \\ 1 \end{matrix} ; x \right] - \frac{x \sin \alpha \cos \alpha}{4} F \begin{matrix} 2:1;2 \\ 2:0;1 \end{matrix} \left[\begin{matrix} 3/2, 3/2:1 \\ 2, 2 \end{matrix} ; 1, 1 ; x, x \sin^2 \alpha \right] \quad (2.7)$$

In view of the convergence conditions given in equations (1.2) and (1.13), we can say that right hand side of (2.7) is convergent. For $\alpha = \pi/2$, equation (2.7) will reduce to equation (1.9).

Using same technique, the equation (2.3) can be generalized in the following form

$$\int_0^\alpha F_B \left[\begin{matrix} a_1, a_2, \dots, a_A; \\ b_1, b_2, \dots, b_B; \end{matrix} ; x \sin^2 \theta \right] d\theta = \alpha {}_{A+1}F_{B+1} \left[\begin{matrix} 1/2, (a_A); \\ 1, (b_B) \end{matrix} ; x \right] - \frac{x \sin \alpha \cos \alpha \prod_{i=1}^A (a_i)}{2 \prod_{j=1}^B (b_j)} \times \\ \times F \begin{matrix} A+1:1;2 \\ B+2:0;1 \end{matrix} \left[\begin{matrix} 1+(a_A), 3/2:1;1,1 \\ 1+(b_B), 2, 2:-;3/2 \end{matrix} ; x, x \sin^2 \alpha \right] \quad (2.8)$$

Put $A=1, B=0, a_1 = 1/2$ in equation (2.8), we get equation (2.3) or (2.7), which is the exact solution of incomplete elliptic integral of first kind and solution is not found in the literature [2;3;4;8;9;18;19].

In equation (2.8), put $A=B=0$, we get

$$\int_0^\alpha e^{x \sin^2 \theta} d\theta = \alpha {}_1F_1 \left[\begin{matrix} 1/2; \\ 1 \end{matrix} ; x \right] - \frac{x \sin \alpha \cos \alpha}{2} \times \\ \times F \begin{matrix} 1:1;2 \\ 2:0;1 \end{matrix} \left[\begin{matrix} 3/2:1;1,1 \\ 2, 2:-;3/2 \end{matrix} ; x, x \sin^2 \alpha \right] \quad (2.9)$$

In (2.8) put $A=1, B=0, a_1 = -1/2$, we get

$$E(\sqrt{x}, \alpha) = \int_0^\alpha \sqrt{1 - x \sin^2 \theta} d\theta = \alpha {}_2F_1 \left[\begin{matrix} 1/2, -1/2; \\ 1 \end{matrix}; x \right] + \frac{x \sin \alpha \cos \alpha}{4} \times \\ \times F \left[\begin{matrix} 2:1; 2 \\ 2:0; 1 \end{matrix}; \begin{matrix} 1/2, 3/2:1 \\ 2, 2 \end{matrix}; 1; 1, 1; x, x \sin^2 \alpha \right] \quad (2.10)$$

Which is the incomplete elliptic integral of second kind and solution is not found in literature.

Abramowitz and Stegun [1], Byrd and Friedman [5; 6], Cayley [7], Heuman [11], Jahnke – Emde [12], Jahnke, Emde and Lösche [13], King [14], Lawden [15], Neville [16], Pearson [17], Roberts [20], and Thomson [27] etc, have calculated approximately the complete and incomplete elliptic integrals of first kind and second kind for different values of x and α , using Trapezoidal rule, Simpson-1/3 rule, Simpson-3/8 rule, , Boole's rule, Weddle's rule etc.

3 Some entries of Ramanujan

[3; 8; 19] are given below

Entry 7(ii): If $\tan \alpha = \sqrt{1-x} \tan \beta$

$$\text{then } \int_0^\alpha \frac{d\theta}{\sqrt{(1-x \cos^2 \theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}}$$

Entry 7(v): If $\cot \alpha \cot \beta = \sqrt{1-x}$

$$\text{then } \int_0^\alpha \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} + \int_0^\beta \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} 1/2, 1/2; \\ 1 \end{matrix}; x \right]$$

Entry 7(vi): If $\cot \alpha \tan \left(\frac{\beta}{2} \right) = \sqrt{(1-x \sin^2 \alpha)}$

then
$$2 \int_0^\alpha \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}}$$

Entry 7(viii): If $\int_0^\alpha \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} + \int_0^\beta \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}} = \int_0^\gamma \frac{d\theta}{\sqrt{(1-x \sin^2 \theta)}}$

then

(a)
$$\tan \left(\frac{\gamma}{2} \right) = \frac{\sin \alpha \cos \beta \sqrt{(1-x \sin^2 \beta)} + \sin \beta \cos \alpha \sqrt{(1-x \sin^2 \alpha)}}{\cos \alpha + \cos \beta}$$

(b)
$$\gamma = \tan^{-1} \left(\tan \alpha \sqrt{(1-x \sin^2 \beta)} \right) + \tan^{-1} \left(\tan \beta \sqrt{(1-x \sin^2 \alpha)} \right)$$

(c)
$$\cot \alpha \cot \beta = \frac{\cos \gamma}{\sin \alpha \sin \beta} + \sqrt{(1-x \sin^2 \gamma)}$$

(d)
$$\frac{\sqrt{x}}{2} = \frac{\sqrt{\{\sin(s) \sin(s-\alpha) \sin(s-\beta) \sin(s-\gamma)\}}}{\sin \alpha \sin \beta \sin \gamma}; \text{ where } s = \frac{\alpha + \beta + \gamma}{2}$$

If x is independent of α, β, γ and other parameters then we have the following necessary conditions provided that associated power series is convergent.

If $\sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} c_m x^m$ then $a_m + b_m = c_m$

and if $p \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} c_m x^m$ then $pa_m = c_m$; where p is a finite number

B.C.Berndt [3, pp 104 -113] and R.Y.Denis *et al.* [8, pp 115 – 117] have derived and verified the above entries.

4 Necessary Conditions

If x is independent of α , β , γ and other parameters then we have the following necessary conditions provided that associated integrals are convergent.

$$\text{If } \int_0^\alpha \frac{d\theta}{\sqrt{(1-x\cos^2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \quad (4.1)$$

then

$$\sin \alpha \cos \alpha {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \cos^2 \alpha \right]_{m-1} + \alpha = -\sin \beta \cos \beta {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \beta \right]_{m-1} + \beta$$

$$\text{where } m = 0, 1, 2, 3, \dots \quad (4.2)$$

$$\text{If } \int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix}; x \right] \quad (4.3)$$

then

$$-\sin \alpha \cos \alpha {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \alpha \right]_{m-1} - \sin \beta \cos \beta {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \beta \right]_{m-1} + \alpha + \beta = \frac{\pi}{2}$$

$$\text{where } m = 0, 1, 2, 3, \dots \quad (4.4)$$

$$\text{If } p \int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}, \quad (4.5)$$

then

$$-p \sin \alpha \cos \alpha {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \alpha \right]_{m-1} + p\alpha = -\sin \beta \cos \beta {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \beta \right]_{m-1} + \beta$$

where $m = 0, 1, 2, 3, \dots$ (4.6)

$$\text{If } \int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^\beta \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^\gamma \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \quad (4.7)$$

then

$$\begin{aligned} & -\sin \alpha \cos \alpha {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \alpha \right]_{m-1} - \sin \beta \cos \beta {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \beta \right]_{m-1} + \alpha + \beta \\ & = -\sin \gamma \cos \gamma {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \gamma \right]_{m-1} + \gamma, \text{ where } m = 0, 1, 2, 3, \dots \end{aligned} \quad (4.8)$$

Derivation for Necessary Conditions

Using (2.4) in (4.3), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{(1/2)_m x^m}{m!} \left\{ \left(\frac{-(1/2)_m \sin \alpha \cos \alpha}{m!} {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \alpha \right]_{m-1} \right) + \left(\frac{(1/2)_m \alpha}{m!} \right) \right\} \\ & + \sum_{m=0}^{\infty} \frac{(1/2)_m x^m}{m!} \left\{ \left(\frac{-(1/2)_m \sin \beta \cos \beta}{m!} {}_2F_1 \left[\begin{matrix} 1, 1 \\ 3/2 \end{matrix}; \sin^2 \beta \right]_{m-1} \right) + \left(\frac{(1/2)_m \beta}{m!} \right) \right\} \\ & = \sum_{m=0}^{\infty} \frac{(1/2)_m x^m}{m!} \left\{ \frac{\pi(1/2)_m}{2(m!)} \right\} \end{aligned} \quad (4.9)$$

Since above three power series are convergent, then equate the coefficients of x^m on both sides and we get necessary condition (4.4). Similarly we can derive the conditions (4.6) and (4.8) for (4.5) and (4.7) respectively.

5 Open Problem(s)

In the study of space problems we need to calculate several distances / lengths mostly are elliptic paths / curves in nature, which involved

incomplete elliptic integrals and their exact solutions, most of such solutions are still unknown.

In this research paper we have presented solutions of some incomplete elliptic integrals. Now we can search various elliptic paths in space and try to calculate their exact lengths (Not approximate, which we are knowing yet) with the help of solutions of incomplete integrals and use them in our space study.

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