Int. J. Open Problems Compt. Math., Vol. 3, No. 4, December 2010
ISSN 1998-6262; Copyright ©ICSRS Publication, 2010
www.i-csrs.org

Fuzzy Quasidifferential Equations in Connection With the Control Problems

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Abstract

In this article conception of the fuzzy quasidifferential equations was introduced, existence theorems for the solutions and some its properties was proved. Also, we proved the close of the attainable set for control fuzzy quasidifferential equations.

Keywords: *Approximation Equations, Fuzzy Quasidifferential Equations, Fuzzy Differential Equations, Fuzzy Differential Inclusions, Control Problems, Attainable Set.*

1 Introduction

The first study of differential equations with multivalued right-hand sides was performed by A.Marchaud [31] and S.C.Zaremba [78]. In early sixties, T.Wazewski [77], A.F.Filippov [15] obtained fundamental results on existence and properties of the differential equations with multivalued right-hand sides (differential inclusions). One of the most important results of these articles was an establishment of the relation between differential inclusions and optimal control problems, that promoted the developing of the differential inclusion theory [5, 10].

Considering of the differential inclusions required to study properties of multivalued functions, i.e. an elaboration of the whole tool of mathematical analysis for multivalued functions [6, 11, 14].

Simultaneously papers [12, 24, 25, 48, 55] appeared which used Hukuhara derivative [16] of multivalued function for investigation of differential equations with multivalued right-hand sides and solutions.

In works [38] annotate of an $R-$solution for differential inclusion is introduced as an absolutely continuous multivalued function. Various problems for the $R-$solution theory were regarded in [36, 38, 39, 62, 64, 76].

The basic idea for a development of an equation for R-solutions (integral funnels) is contained in [35].

Here the equation was considered as a particular case of an approximated equation in a metric space. Approximated equations make possible to exclude a differentiation operation and there by to avoid linearity requirement for a solution space and to consider differential equations in linear metric spaces, equations with multivalued solutions and dynamical systems in nonlinear metric spaces by unified positions [35, 37, 63, 64, 67, 68, 70].

Therefore, an approximated equation is a quasidifferential equation for determination of the dynamical system in a metric space.

The theory of mutational equations in metric spaces, which deals with multivalued trajectories (pipers) and trajectories in nonlinear spaces has been developed in [4].

As well as in [35] an approach is taken that does not use a derivative in explicit form for description in nonlinear metric spaces, while in [4] analogous results are obtained by construction "differential calculus" in nonlinear metric spaces.

Moreover in [35, 68, 69] quasidifferential equations were considered for locally compact spaces, while in [37, 64, 67, 70] for complete metric space.

In the last years a new approach to control problems of dynamic systems has been forming, which foundation on analysis of trajectory bundle but not separate trajectories. The section of this bundle in any instant is some set and it is necessary to describe the evolution of this set. Obtaining and research of dynamic equations of sets are important problems in this case. The metric space of sets with the Hausdorff metric is natural space for description dynamic of sets. In theory of multivalued maps definitions on derivative as for single-valued maps is impossible because the space of sets is nonlinear. This bound possibility description dynamic sets by differential equations. Therefore, the control differential equations with set of initial conditions [79, 33], the control differential inclusions [22, 23, 34, 46, 47, 49, 50, 51, 52, 56, 57, 58], the control differential equations with Hukuhara derivative [45, 64] and the control quasidifferential equations [53, 54, 64, 66] use for it.

In recent years, the fuzzy set theory introduced by Zadeh has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations, and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations [2, 18, 19, 20, 21, 27, 40, 41, 59, 60, 73, 74] and inclusions [3, 8, 9, 17, 28, 29, 60] as well as in the theory of control fuzzy differential equations [42, 43, 44] and inclusions [32, 72].

In the given article we introduce the fuzzy quasidifferential equation to similarly how it has been made in the theory of the differential equations with a multivalued right-hand side [35, 64]. It gives the chance to avoid some complexities which arise at researches in the theory of the fuzzy differential equations and inclusions. Also we receive some properties of solutions of the control fuzzy quasidifferential equations.

2 Notations and definitions

Let $(conv(R^n), h)$ denote the family of all nonempty compact convex subsets of R^n. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$h(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(conv(R^n), h)$ becomes a metric space.

Denotes $E^n = \{\, u : R^n \to [0,1] \mid u \,\text{satisfies (i)–(iv) below}\,\}$, where
(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \, cl\{x \in R^n \mid u(x) > 0\,\}$ is compact.

For $0 < \alpha \le 0$ denote $[u]^\alpha = \{x \in R^n \mid u(x) \ge \alpha\,\}$. Then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in conv(R^n)$ for all $0 \le \alpha \le 1$.

Let $\hat{o}$ be the fuzzy set defined by $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \ne 0$.

Define $D : E^n \times E^n \to R^+ \bigcup\{0\}$ by the equations $D(u,v) = \sup\limits_{0\le\alpha\le1} h([u]^\alpha, [v]^\alpha)$. Then it is easy to show that D is a metric in E^n.

Using the result in [71], we know that
(1) (E^n, D) is a complete metric space,
(2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,
(3) $D(ku, kv) = |k|D(u, v)$ for all $u, v \in E^n$, $k \in R$.

Definition 2.1 *A mapping $f : [0,T] \times E^n \to E^n$ is called continuous at point $(t_0, u_0) \in (0,T) \times E^n$ provided for any $\varepsilon > 0$, there exists an $\delta > 0$ such that $D(f(t,u), f(t_0,u_0)) < \varepsilon$ whenever $|t - t_0| < \delta$ and $D(u, u_0) < \delta$.*

If $f : [0,T] \times E^n \to E^n$ is continuous, then we denote by $f \in C([0,T] \times E^n, E^n)$.

Definition 2.2 *A mapping $f : [0,T] \times E^n \to E^n$ is called bounded if there exists a constant $\lambda > 0$ such that $\|y\| \leq \lambda$ for all $y \in [f(t,x)]^0$.*

Definition 2.3 *A mapping $\phi : [0,\sigma) \times [0,T] \times E^n \to E^n$ is called fuzzy local quasimotion if*
1) $\phi(0,t,x) = x$;
2) $D(\phi(h,0,x_0), \phi(h_m, t_{m-1}, x_{m-1})) = o(h)$;
3) a mapping $\phi(h,t,x)$ is continuous,
where $h = \sum\limits_{i=1}^{m} h_i,\ h_i \geq 0,\ t_i = \sum\limits_{s=0}^{i} h_s,\ x_i = \phi(h_i, t_{i-1}, x_{i-1}),\ \lim\limits_{h\to 0} \frac{o(h)}{h} = 0$.

The approximation equation

$$D(x(t+h), \phi(h,t,x(t))) = o(h),\ x(0) = x_0 \tag{1}$$

is said fuzzy quasidifferential equation.

Definition 2.4 *A continuous mapping $x : [0,T] \to E^n$ is a solution of the problem (1) if it satisfies the equation (1) for all $t \in [0,T]$.*

3 Existence of solutions

In this section we prove the existence of solutions for (1).

Theorem 3.1 *Let $\phi : R^1_+ \times [0,\sigma] \times E^n \to E^n$ is fuzzy local quasimotion and for any τ_1, τ_2 there exists a constant $\lambda > 0$ such that*

$$D(\phi(\tau_1,t,x), \phi(\tau_2,t,x)) \leq \lambda|\tau_1 - \tau_2|.$$

Then there exist constant $\eta > 0$ such that the problem (1) has a solution on $[0,\eta]$.

Proof. Fix $r > 0$. Let $S_r(x_0) = \{x \,|\, D(x,x_0) \leq r\}$ and $\eta = \min\{\sigma_0, \frac{r}{\lambda}\}$. To do this divide the interval $[0,\eta]$ on m-subintervals $[t_i, t_{i+1}]$, where $t_i = \frac{i\eta}{m},\ i = 0,1,\ldots,m-1$ and write $y^m(t)$ in the form

$$y^m(t) = \phi(t - t_i, t_i, y^m(t_i)),\ t \in [t_i, t_{i+1}],\ i = \overline{0, m-1},\ y^m(0) = x_0.$$

By Arzela theorem, it follows that the sequence $\{y^m(\cdot)\}_{m=1}^{\infty}$ is precompact in $C(R^1_+ \times [0,\sigma] \times E^n, E^n)$. Then there is the subsequence $\{y^{m_k}(\cdot)\}_{k=1}^{\infty}$ such that $y^{m_k}(t)$ uniformly converges to $y(t)$ on $[0,\eta]$. This completes the proof.

Theorem 3.2 *Let $\phi : R^1_+ \times [0,\sigma] \times E^n \to E^n$ is fuzzy local quasimotion, for any τ_1, τ_2 there exists a constant $\lambda > 0$ such that*

$$D(\phi(\tau_1, t, x), \phi(\tau_2, t, x)) \leq \lambda|\tau_1 - \tau_2|$$

and satisfy the following condition:

$$|D(x, y) - D(\phi(\tau, t, x), \phi(\tau, t, y))| \leq \mu(\tau)D(x, y),$$

where $\lim\limits_{\tau\to 0} \frac{\mu(\tau)}{\tau} = \gamma$, $\gamma > 0$. Then, the problem (1) has a unique solution on $[0, \eta]$.

Proof. By theorem 3.1 a fuzzy solution $x(t)$ of (1) exists on $[0, \eta]$. Let $y(t)$ be another solution of (1) on $[0, \eta]$ and let $r(t) = D(x(t), y(t))$. Then $r(0) = 0$ and

$$|r(t + \Delta) - r(t)| = |D(x(t + \Delta), y(t + \Delta)) - D(x(t), y(t))| =$$
$$= |D(\phi(\Delta, t, x(t)), \phi(\Delta, t, y(t))| \leq \mu(\Delta)r(t) + 0(\Delta).$$

Since $r(t)$ is a continuous function, then $r(t) \equiv 0$ on $[0, \eta]$. The proof is complete.

Remark 3.3 *Let $\phi(\Delta, t, x) = x + \Delta \cdot f(t, x)$, where $f \in C([0, T] \times E^n, E^n)$ and satisfies the Lipschitz condition of the form $D(f(t, x), f(t, y)) \leq \lambda D(x, y)$, then the problem (1) has a unique solution on $[0, \eta]$ and it is solution of the fuzzy differential equation $x' = f(t, x)$, $x(0) = x_0$, considered in the papers [18, 19, 20, 21, 40, 41, 59].*

Remark 3.4 *Let $\phi(\Delta, t, x)$ such that $[\phi(\Delta, t, x)]^\alpha = \bigcup\limits_{z\in[x]^\alpha} (z+\Delta\cdot[F(t, z)]^\alpha)$, where $F \in C([0, T] \times R^n, E^n)$ and satisfies the Lipschitz condition of the form $D(F(t, x), F(t, y)) \leq \lambda\|x - y\|$, then the problem (1) has a unique solution on $[0, \eta]$ and it is fuzzy R-solution of the fuzzy differential inclusion $x' \in F(t, x)$, $x(0) = x_0$, considered in the papers [3, 8, 9, 17, 28, 29].*

Theorem 3.5 *Let $\phi : R^1_+ \times [0,\sigma] \times E^n \to E^n$ is fuzzy local quasimotion, for any τ_1, τ_2 there exists a constant $\lambda > 0$ such that*

$$D(\phi(\tau_1, t, x), \phi(\tau_2, t, x)) \leq \lambda|\tau_1 - \tau_2|$$

and satisfy the following condition:

$$|D(x, y) - D(\phi(\Delta, t, x), \phi(\Delta, t, y))| \leq \Delta \cdot \gamma \cdot D(x, y).$$

Then

$$D(x(t), y(t)) \leq e^{\gamma t}\delta_0,$$

where $x(t)$ and $y(t)$ are solutions problem (1), which $x(0) = x_0$, $y(0) = y_0$ and $\delta_0 = D(x_0, y_0)$.

Proof. To do this divide the interval $[0,\eta]$ on m-subintervals $[t_i,t_{i+1}]$, where $t_i=\frac{i\eta}{m}$, $i=0,1,\dots,m-1$. Let $t\in[t_k,\, t_{k+1}]$. Then

$$D(x(t),y(t))\le D(\phi(t-t_k,t_k,x(t_k)),\phi(t-t_k,t_k,y(t_k)))+o(\Delta)\le$$

$$\le(1+\Delta\cdot\gamma)D(x(t_k),y(t_k))+o(\Delta)\le$$

$$\le(1+\Delta\cdot\gamma)D(\phi(\Delta,t_{k-1},x(t_{k-1})),\phi(\Delta,t_{k-1},y(t_{k-1})))+(1+\Delta\cdot\gamma)o(\Delta)+o(\Delta)\le$$

$$\le(1+\Delta\cdot\gamma)^2D(x(t_{k-1}),y(t_{k-1}))+o(\Delta)(1+\Delta\cdot\gamma)+o(\Delta).$$

Hence, we obtain

$$D(x(t),y(t))\le(1+\Delta\cdot\gamma)^{k+1}\delta_0+[(1+\Delta\cdot\gamma)^k+\ldots+(1+\Delta\cdot\gamma)+1]o(\Delta)\le$$

$$\le(1+\frac{\eta}{m}\gamma)^m\delta_0+\frac{(1+\Delta\cdot\gamma)^{k+1}-1}{\Delta\cdot\gamma}o(\Delta)\le e^{\gamma\eta}\delta_0+\frac{o(\Delta)}{\Delta}.$$

This ends the proof.

4 Control fuzzy quasidifferential systems

The approximation equation

$$D(x(t+h,u),\phi(h,t,x(t,u),u))=o(h),\ x(0,u)=x_0 \tag{2}$$

is said to be the control fuzzy quasidifferential equation, where $u\in U$ is a control; U is a subset in the complete space Y; a map $\phi(\cdot,\cdot,\cdot,\cdot):R^1_+\times R^1_+\times E^n\times Y\to E^n$ is fuzzy local quasimotion.

Remark 4.1 *Let $Y=C(R^1_+), U=\{u(\cdot)|u\in Y, u(t)\in P, P\in conv(R^m)\}$, $\phi(\Delta,t,x,u)=x+\Delta\times f(t,x,u)$, the map $f(\cdot,\cdot,\cdot):R^1_+\times E^n\times R^m\to E^n$ such that*

1) $f(\cdot,x,u)$ is continuous on R^1_+ ;

2) $f(t,\cdot,u)$ satisfies the Lipschitz condition on E^n;

3) $f(t,x,\cdot)$ is continuous on R^m,

then all solutions of the control fuzzy differential equation

$$x'=f(t,x,u),\quad x(0)=x_0$$

are the solutions of the problem (2) and vice-versa.

Remark 4.2 *Let*

$$\phi(\Delta,t,x,u)=x+\int\limits_t^{t+\Delta}f(s,x,u(s))ds,$$

such that $f(\cdot,\cdot,\cdot)$ is measurable on R^1_+, satisfies the Lipschitz condition on E^n, is continuous on R^m and $D(f(t,x,u),0) \le \beta(t)$ for a.e. $t \in [0,T]$, where $\beta(\cdot) \in L[t_0,T]$, $Y = L(R^1_+)$, then all solutions of the control fuzzy differential equation

$$x' = f(t,x,u), \quad x(0) = x_0$$

are the solutions of the problem (2) and vice-versa.

Remark 4.3 *Let $\phi(\Delta,t,x,u)$ such that*

$$[\phi(\Delta,t,x,u)]^\alpha = \bigcup_{z\in[x]^\alpha} (z + \Delta \cdot [F(t,z,u)]^\alpha),$$

where $F(\cdot,\cdot,\cdot) : R^1 \times R^n \times R^m \to E^n$ is measurable on R^1_+, satisfies the Lipschitz condition on R^n, is continuous on R^m and $D(F(t,x,u),0) \le \beta(t)$ for a.e. $t \in [0,T]$, $\beta(\cdot) \in L[t_0,T]$, $Y = L(R^1_+)$, then all fuzzy R-solutions of the control fuzzy differential equation

$$x' \in F(t,x,u), \quad x(0) = x_0$$

are the solutions of the problem (2) and vice-versa.

Definition 4.4 *The set $Z(T) = \{x(T,u)|u \in U\}$ is said the attainable set of the problem (2).*

Theorem 4.5 *Let the map $\phi(\cdot,\cdot,\cdot,\cdot)$ is fuzzy local quasimotion and satisfies*
1) $\phi(\tau,\cdot,x,u)$ is continuous on $[0,T]$;
2) $\phi(\cdot,t,x,u)$ satisfies the Lipschitz condition on R^1_+, i.e.,

$$D(\phi(\tau_1,t,x,u),\phi(\tau_2,t,x,u)) \le \mu|\tau_1 - \tau_2|;$$

3) $\phi(\tau,t,\cdot,u)$ satisfies the Lipschitz condition on E^n, i.e.,

$$D(\phi(\tau,t,x,u),\phi(\tau,t,y,u)) \le \lambda D(x,y);$$

4) exists the constant $\nu > 0$ such, that

$$D(\phi(\tau,t,x,u_1),\phi(\tau,t,x,u_2)) \le \nu\alpha(u_1,u_2)$$

for all $u_1,u_2 \in U$, where $\alpha(\cdot,\cdot)$ is weakly metric in the space Y;
5) set U is the weakly compactness in the space Y;
6) exists one solution $x(\cdot,u)$ of the problem (2) on $[0,T]$ for all $u \in U$.
Then the attainable set $Z(T) = \{x(T,u)|u \in U\}$ of the problem (2) is closed and bounded in the space E^n.

Remark 4.6 *If the map $\phi(\tau,t,\cdot,u)$ satisfies conditions of theorem 4.5 and*

$$|D(x,y)-D(\phi(\tau,t,x,u),\phi(\tau,t,y,u))|\le\mu(\tau)D(x,y),$$

then the problem (2) has one solution for all $u\in U$, where $\lim\limits_{\tau\to 0_+}\mu(\tau)/\tau=\chi,\chi>0$ (see theorem 3.2).

Proof. Obviously the set $Z(T)$ is boundedness in the space E^n. We show that the set $Z(T)$ is closed. Let $\{x_i\}_{i=1}^{\infty},\ x_i\in Z(T), i=\overline{1,\infty}$ is any sequence decreasing to x_*, i.e. $x_*=\lim\limits_{i\to\infty}x_i$. Let $x(\cdot,u_i),\ u_i\in U,\ i=\overline{1,\infty}$ are the solutions of the problem (2) such that $x(T,u_i)=x_i,\ i=\overline{1,\infty}$.

By weekly compactness of U, there is a subsequence $\{u_k\}_{k=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ which weekly converges to any $u_*\in U$.

Fix any $\varepsilon>0$ and consider $D(x_*,x(T,u_*))$. Choose $N=N(\varepsilon)$ such that $k>N$ implies $D(x(T,u_k),x_*)<\varepsilon/2$.

Then

$$D(x_*,x(T,u_*))\le D(x_*,x(T,u_k))+D(x(T,u_k),x(T,u_*))< \tag{3}$$

$$<\varepsilon/2+D(x(T,u_k),x(T,u_*))$$

for any $k>N(\varepsilon)$.

To do this divide the interval $[0,T]$ on m-subintervals $[t_i,t_{i+1}]$, where $t_i=\frac{i\eta}{m},\ i=0,1,\ldots,m-1$. Let

$$\begin{aligned}&x^m(t,u_*)=\phi(t-t_i,t_i,x^m(t_i,u_*),u_*),\ t\in[t_i,t_{i+1}],\\&x^m(0,u_*)=x_0,\ i=\overline{0,m-1},\\&x^m(t,u_k)=\phi(t-t_i,t_i,x^m(t_i,u_k),u_k),\ t\in[t_i,t_{i+1}],\\&x^m(0,u_k)=x_0,\ i=\overline{0,m-1}.\end{aligned}$$

We have

$$D(x^m(t_i,u_k),x(t_i,u_k))= \tag{4}$$

$$=D(\phi(\Delta,t_{i-1},x^m(t_{i-1},u_k),u_k),\phi(\Delta,t_{i-1},x(t_{i-1},u_k),u_k))+\mathrm{o}(\Delta)\le$$

$$\le\frac{\mathrm{o}(\Delta)}{\Delta}(\exp(\lambda(T-t_0))-1),$$

and

$$D(x^m(t_i,u_*),x(t_i,u_*))\le\frac{\mathrm{o}(\Delta)}{\Delta}(\exp(\lambda(T-t_0))-1). \tag{5}$$

for all $i=\overline{0,m-1}$.

Therefore

$$D(x^m(t,u_k),x^m(t_i,u_k))=$$

$$=D(\phi(t-t_i,t_i,x^m(t_i,u_k),u_k),\phi(0,t_i,x^m(t_i,u_k),u_k))\le$$

$$\leq \mu(t - t_i) \leq \mu T/m \tag{6}$$

$$D(x^m(t, u_*), x^m(t_i, u_*)) \leq \mu(t - t_i) \leq \mu T/m, \tag{7}$$

for all $t \in [t_i, t_{i+1}]$.

By (4)–(7), there exist constant m_0 such that

$$D(x^m(t, u_k), x(t, u_k)) < \varepsilon/6 \tag{8}$$

$$D(x^m(t, u_*), x(t, u_*)) < \varepsilon/6$$

for all $m \geq m_0$ and $t \in [0, T]$.

Next note that

$$D(x^m(T, u_k), x^m(T, u_*)) = \tag{9}$$

$$= D(\phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_k), u_k), \phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_*), u_*)) \leq$$
$$\leq D(\phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_k), u_k), \phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_*), u_k)) +$$
$$+ D(\phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_*), u_k), \phi(\Delta, t_{m-1}, x^m(t_{m-1}, u_*), u_*)) \leq$$

(by conditions 3) and 4) of theorem, we have)

$$\leq \lambda D(x^m(t_{m-1}, u_k), x^m(t_{m-1}, u_*)) + \nu\alpha(u_k, u_*) \leq$$
$$\leq \lambda\left[\lambda D(x^m(t_{m-2}, u_k), x^m(t_{m-2}, u_*)) + \nu\alpha(u_k, u_*)\right] + \nu\alpha(u_k, u_*) \leq \ldots \leq$$
$$\leq \lambda[\lambda[\ldots[\lambda D(x^m(t_0, u_k), x^m(t_0, u_*)) + \nu\alpha(u_k, u_*)]\ldots] + \nu\alpha(u_k, u_*)] +$$
$$+ \nu\alpha(u_k, u_*) = \lambda[\lambda[\ldots[\lambda\alpha(u_k, u_*)]\ldots] + \nu\alpha(u_k, u_*)] + \nu\alpha(u_k, u_*).$$

Since subsequence $\{u_k\}_{k=1}^{\infty}$ weekly converges to u_*, there exist $M > N$, such that $D(x^m(T, u_k), x^m(T, u_*)) < \varepsilon/6$ for all $k > M$.

Hence and by (8), we have

$$D(x(T, u_k), x(T, u_*)) < \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2.$$

Therefore $D(x(T, u_*), x_*) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, i.e., $x(T, u_*) = x_*$ and the set $Z(T)$ is closed. This ends the proof.

5 Conclusion

In this article we introduced the fuzzy quasidifferential equations which summarize the fuzzy differential equations and inclusions, by analogy with the theory of the ordinary differential equations with the multivalued right-hand sides. These approximation equations from one side are unobviously used for the decision of the fuzzy differential equations and the differential inclusions [1, 30, 59], and with the other they enable to examine the mixed systems with the single point of view, when part of variables described by the fuzzy differential equations, and others are the R-solutions of the fuzzy differential inclusions.

Also in the article we introduced the control fuzzy quasidifferential equation and was proved that the attainable set is closed.

6 Open Problem

In recent years, the fuzzy set theory introduced by Zadeh has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations, and engineering sciences. Obviously, that quasidifferential equations can describe many of these processes and research of their property is important.

ACKNOWLEDGEMENTS. Authors would like to express the gratitude of the referee and to the editor-in-chief, Prof. Dr. Iqbal H. Jebril.

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