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Eigenvalue Equations For Nonnull Curve in Minkowski Plane

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Abstract

In this paper, we study the evolute and involute curves for nonnull curves with positive curvature κ and spherical parametrization σ such that their radius of curvature κ^{-1} satisfies an eigenvalue equation in Minkowski plane.

Keywords: *curvature, eigenvalue equations, evolute curve, involute curve, Minkowski plane.*

1 Introduction

In mathematical and physical study of relativity theory, a material particle in a spacetime is understood as a future-pointing timelike curve of unit speed in Minkowski spacetime. It is the mathematical setting in which Einstein's theory of special relativity is most conveniently formulated. In this setting, three dimension and time form a 4-dimensional manifold which is representing a Minkowski spacetime. The unit speed parameter is called as the proper time of a material particle. The fact that relativity theory is expressed in terms of Lorentzian geometry is important for geometers. They can thus penetrate surprisingly quickly in the cosmology (redshift, expanding universe and big bang) and the gravitation of a single star (perihelion procession, bending of light and black holes),that is a topic no less interesting geometrically.Therefore, timelike curves , spacelike and null (lightlike, isotropic) curves in Minkowski space have been studied extensively by both physitists and differential geometers [1, 2, 3, 4, 5, 6]. For applications of null curve theory to general relativity, we refer to [7].

On the other hand , eigenvalue equations play an important role in differential geometry , physics and engineering. There are many applications of eigenvalue equations in the theory of submanifolds. In [8, 9], the authors characterized hyperellipsoids in terms of eigenvalue equations for some functions of geometric importance, namely support functions and certain curvature functions. Later they proved analogous local and global characterizations of conic sections in terms of eigenvalue equations for the curvature and support function in the plane Euclidean curve theory. In [10], they continued the investigations from [9] about characterizations of plane curves in term of certain Euclidean curvature properties and stated the classification relating eigenvalues to the geometry of curves.

Due to the importance of Minkowski space and evolute-involute curves of nonnull curves(timelike or spacelike) which satisfy eigenvalue equations in mathematical physics and engineering , in this study, making use of method in [10], we defined the evolute and involute curve for nonnull curves in Minkowski plane and gave the Evolute Lemma and Involute Lemma to them. However the eigenvalue equations for nonnull curves on the Minkowski geometry has not been studied yet. Other purpose of this paper is to investigate some results for curves which are the solution of the eigenvalue equations in the Minkowski plane.

We hope that this letter will contribute to the study of kinematics of engines, the construction of toothwheel gears and physics applications.

2 Preliminaries

Let L^2 be the Minkowski plane with metric $g(+,-)$. A vector X of L^2 is said to be *spacelike* if $g(X,X)>0$ or $X=0$, *timelike* if $g(X,X)<0$ and *null* if $g(X,X)=0$ and $X\neq 0$.

A curve c is a smooth mapping $c: I \to L^2$ from an open interval I in to L^2. Let t be a parameter of c. By $c(t)=(x(t),y(t))$ we denote the orthogonal coordinate represantation of $c(t)$. The vector field $\frac{dc}{dt}=(\frac{dx}{dt},\frac{dy}{dt})=: X$ is called the tangent vector field of the curve c. If the tangent vector field X of the c is a spacelike, timelike, or null, then the curve c is called spacelike, timelike, or null, respectively. Recall that when c is a nonnull curve in L^2, then the Frenet formulas

$$X=\|\dot{c}\|^{-1}.\dot{c}, \quad \frac{dX}{dt}=\kappa\|\dot{c}\|Y, \quad \frac{dY}{dt}=\kappa\|\dot{c}\|X, \tag{2.1}$$

where κ is the curvature of c. The vector field Y is called the normal vector field of the curve c. [11].

We use a spherical parametrization; it is an important advantage that the spherical parametrization is the same for the curve and also for its evolute and its involutes; that makes our calculations simple and elegant. Moreover, the spherical parametrization makes visible the interplay between eigenvalue equations and geometric properties of the classes of curves considered.

Definition 2.1 *For $t_0 \in I$ the arc length of the spherical image is defined by [12]*

$$\sigma(t) := \sigma(t_0, t) := \int_{t_0}^{t} \left\| \dot{X}(\tau) \right\| d\tau = \int_{t_0}^{t} |\kappa(\tau)| \, . \, \|\dot{c}(\tau)\| \, d\tau. \tag{2.2}$$

In the following we restrict curves with curvature $\kappa \neq 0$, thus we may assume $\kappa > 0$ (otherwise we change the orientation of the curve). Then the curve admits a representation in terms of a spherical arc length parameter σ. By a prime $'$ we denote the differentation with respect to σ. Then

$$' = \frac{d}{d\sigma} = \frac{1}{\kappa \|\dot{c}\|} \frac{d}{dt}, \quad s' = \frac{ds}{d\sigma} = \|c'\| = \frac{1}{\kappa}, \tag{2.3}$$

The Frenet formulas in terms of a spherical parametrization are

$$c' = \frac{1}{\kappa} X, \quad X' = Y, \quad Y' = X. \tag{2.4}$$

Definition 2.2 *Let c be a nonnull curve in Minkowski plane. Then the support function of c with respect to a fixed point $p_0 \in L^2$ is defined by*

$$\rho(p_0) := g(Y, \ p_0 - c). \tag{2.5}$$

Differentiating (2.5) with respect to spherical arc length parameter σ and using (2.4), we have

$$\rho(p_0)' = g(X, \ p_0 - c) \tag{2.6}$$

and we get a represantation of c in terms of the support function:

$$c - p_0 = -\varepsilon_1 \rho(p_0)' X - \varepsilon_2 \rho(p_0) Y, \tag{2.7}$$

where $\varepsilon_1 = g(X, X) = \mp 1$, $\varepsilon_2 = g(Y, Y) = \pm 1$.

Differentiating (2.7), we get

$$\frac{1}{\kappa} = -\varepsilon_1 \rho(p_0)'' - \varepsilon_2 \rho(p_0). \tag{2.8}$$

Equations (2.7) and (2.8) give another modification of the local fundamental theorem of plane curve theory in Minkowski geometry:

Theorem 2.3 ***Fundamental Theorem of the Local Theory of Curves in Minkowski Space:*** *Let $\rho : I \to \mathbb{R}$ be a sufficiently differentiable function with parameter σ and $-\varepsilon_1\rho(p_0)'' - \varepsilon_2\rho(p_0) > 0$. Then there exist a differentiable, regular curve with spherical parameter σ and positive curvature κ such that $\rho(\sigma) = \rho(p_0)(\sigma)$ is its support function with respect to some fixed point $p_0 \in L^2$ satisfying $\frac{1}{\kappa} = -\varepsilon_1\rho(p_0)'' - \varepsilon_2\rho(p_0)$; moreover, any other curve c^* satisfying the same conditions, different from c by a rigid motion [13].*

3 Evolute and Involute Curves in Minkowski Plane

Definition 3.1 *Let $c : I \to L^2$ be a nonnull curve with positive curvature κ. The locus of the centre of curvature of a plane curve c is called the evolute of the curve c and given by*

$$E_c(t) := c(t) - \kappa^{-1}(t)Y(t). \tag{3.1}$$

From (3.1), we get

$$\dot{E}_c(t) := \kappa^{-1}(t)^{\cdot}\, Y(t); \tag{3.2}$$

thus the evolute is regular if and only if c has no vertex ($\dot{\kappa} \neq 0$); in the following, whenever we discuss evolutes, we exclude vertices of c. If $\{X, Y\}$ is an oriented orthonormal frame of c then

$$\widetilde{X} = -sign(\kappa^{-1}(t)^{\cdot})Y, \qquad \widetilde{Y} = +sign(\kappa^{-1}(t)^{\cdot})X \tag{3.3}$$

defines an oriented orthonormal frame of the evolute curve E_c.

A spherical parametrization simplifies the relations and gives the following Evolute Lemma.

Lemma 3.2 *(Evolute Lemma)* *Let $c : I \to L^2$ be a nonnull curve with positive curvature κ. Then the evolute E_c of c satisfies the following properties:*

(i) $\{\widetilde{X}, \widetilde{Y}\} = \{\pm Y, \mp X\}$,
(ii) the spherical parametrizations coincide: $\sigma_{E_c} = \sigma_c$,
(iii) the curvature functions are related by:

$$\kappa_{E_c}^{-1}(\sigma) = sign((\kappa_c^{-1})').(\kappa_c^{-1}(\sigma))';$$

(iv) for $p_0 \in L^2$, the associated support functions are related by:

$$\rho_{E_c}(p_0)(\sigma) = sign((\kappa_c^{-1})').\rho_c(p_0)'(\sigma).$$

Proof.

(i) It is clear from (3.3).

(ii) If we consider the equation (2.2) for the evolute curve E_c we get $\sigma_E = \sigma_c$.

(iii) Differentiating (3.1) with respect to σ, we have $\kappa_{E_c}^{-1}(\sigma) = sign((\kappa_c^{-1})').(\kappa_c^{-1}(\sigma))'$.

(iv) If we consider the equation (2.5) for the evolute curve E_c, we get $\rho_{E_c}(p_0)(\sigma) = sign((\kappa_c^{-1})').\rho_c(p_0)'(\sigma)$.

Definition 3.3 *Let $c : I \to L^2$ be a nonnull curve with arc length parametrization. For a fixed value $s_1 \in \mathbb{R}$, the involute of the curve c is defined by;*

$$\mathcal{I}_{c,s_1}(s) := c(s) - (s + s_1).X(s). \tag{3.4}$$

The condition $(s + s_1) \neq 0$ is equivalent to the regularity of the involute $\mathcal{I}_{c,s_1}$; in the following we suppose all involutes to be regular.

If s_1 varies, one obtains a one-parameter family of involutes.

Similar to Lemma 3.1 , we have the following Involute Lemma:

Lemma 3.4 *(Involute Lemma) Let $c : I \to L^2$ be a nonnull curve with positive curvature κ. Then the involute $\mathcal{I}_{c,s_1}$ of c satisfies the following properties:*

(i) If $\{X, Y\}$ is an oriented orthonormal frame of c then $\{\overline{X} = -sign(s + s_1)Y, \overline{Y} = sign(s + s_1)X\ \}$ is an oriented orthonormal frame of $\mathcal{I}_{c,s_1}$

(ii) the spherical parametrizations coincide $\sigma_{\mathcal{I}_{c,s_1}} = \sigma_c$

(iii) the curvature function $\kappa_{\mathcal{I}_{c,s_1}}$ of the involute $\mathcal{I}_{c,s_1}$ satisfies:

$$\kappa_{\mathcal{I}_{c,s_1}} = |s + s_1|^{-1}$$

and thus $\kappa_{\mathcal{I}_{c,s_1}}$ *is* positive and we can write the equation (3.4) by

$$\mathcal{I}_{c,s_1} := c(s) - \frac{sign(s + s_1)}{\kappa_{\mathcal{I}_{c,s_1}}} X(s)$$

(iv) Using a spherical parametrization, the curvature functions κ_c of c and $\kappa_{\mathcal{I}_{c,s_1}}$ of the involute curve $\mathcal{I}_{c,s_1}$ are related by;

$$\left((\kappa_{\mathcal{I}_{c,s_1}}(\sigma))^{-1}\right)' = sign(s(\sigma) + s_1).\kappa_c(\sigma)^{-1}$$

(v) for $p_0 \in L^2$, the associated support functions are related by:

$$\rho_{\mathcal{I}_{c,s_1}}(p_0)(\sigma) = sign(s(\sigma) + s_1)[\rho_c(p_0)'(\sigma) - \varepsilon_1(s(\sigma) + s_1)]$$

and

$$\left(\rho_{\mathcal{I}_{c,s_1}}(p_0)(\sigma)\right)' = -sign(s(\sigma)+s_1)\rho_c(p_0).$$

Proof. Its proof is routine similar to the proof of Lemma 3.2.

As a consequence of the Lemma 3.2 and Lemma 3.4 we get:

Corollary 3.5 *For a given nonnull curve with positive curvature in the Minkowski plane, but without vertices, there is a joint spherical parametrization for the curve, its evolute and its involutes.*

Definition 3.6 *Let $c : I \to L^2$ be a nonnull curve with positive curvature κ and spherical parametrization σ.*

(i) We call $E : c \to E_c$ the evolution operator, iterating the operator m times, we write

$$E^m := E(E^{m-1}), \quad E^1 := E, \quad E^0 := id, \ m \in \mathbb{N};$$

(ii) for $s_1 \in \mathbb{R}$, we call $\mathcal{I}_{s_1} : c \to \mathcal{I}_{c,s_1}$ an involution operator;
(iii) We call $N : c \to Y$ the normal spherical operator
[14].

Lemma 3.7 *(Operator Lemma) We have:*

(i) $\mathcal{I}_{s_1}.E = id + d.N$ for an appropriate $d \in \mathbb{R}$; thus, for any $s_1 \in \mathbb{R}$, the curve $\mathcal{I}_{s_1}.E(c)$ is a curve parallel to c.
(ii) $E.\mathcal{I}_{s_1} = id$ for any $s_1 \in \mathbb{R}$;
(iii) $\mathcal{I}_{s_1}.E - E.\mathcal{I}_{s_2} = d.N$ for given s_1, s_2 and appropriate $d \in \mathbb{R}$
[14].

In this study, we denote by $\mathbb{E}_0$ the class of sufficiently differentiable, nonnull curves with positive curvature κ and spherical parametrization satisfying the homogeneous eigenvalue equation

$$(\kappa^{-1})'' + \lambda.\kappa^{-1} = 0 \tag{3.5}$$

for some real λ, in anology, by $\mathbb{E}_f$ we denote the class satisfying the inhomogeneous eigenvalue equation

$$(\kappa^{-1})'' + \lambda.\kappa^{-1} = f \tag{3.6}$$

with a sufficiently differentiable function $f : I \to \mathbb{R}$.

Theorem 3.8 *(Evolute Theorem) Let $c : I \to L^2$ be a nonnull curve with positive curvature κ and spherical parameter σ, but withuot vertices. let the radius of curvature*

$$(\kappa^{-1})''(\sigma) + \lambda.\kappa^{-1}(\sigma) = f(\sigma)$$

for some differentiable function f. Then $\kappa_{E_c}^{-1}$ of the evolute E_c satisfies the equation

$$(\kappa_{E_c}^{-1})''(\sigma) + \lambda.\kappa_{E_c}^{-1}(\sigma) = \mp f'(\sigma). \tag{3.7}$$

Proof. According to the Lemma 3.2, if we use the joint spherical parametrization for c and E_c, we have

$$(\kappa_{E_c}^{-1})''(\sigma) + \lambda.\kappa_{E_c}^{-1}(\sigma) = sign((\kappa_c^{-1})'.f'(\sigma)$$

which means that (3.7).

Corollary 3.9 *All evolutes of nonnull curves in Minkowski plane in $\mathbb{E}_0$ are again in $\mathbb{E}_0$, i.e., the class $\mathbb{E}_0$ is invariant under the operator E.*

Theorem 3.10 *(Involute Theorem) Let $c : I \to L^2$ be a nonnull curve with positive curvature κ_c and spherical parameter σ, but withuot vertices. We fix $\sigma_1 \in I$. Let the radius of curvature κ^{-1} of c satisfy the inhomogeneous eigenvalue equation (3.6). Then the radius of curvature*

$$(\kappa_{\mathcal{I}_{c,s_1}}^{-1})''(\sigma) + \lambda.\kappa_{\mathcal{I}_{c,s_1}}^{-1}(\sigma) = \mp \int_{\sigma_1}^{\sigma} f(t)dt + \gamma \tag{3.8}$$

for some real γ.

Proof. If we use Lemma 3.4-(iv), we get (3.8).

Corollary 3.11 *All involutes of nonnull curves in Minkowski plane in $\mathbb{E}_0$ are again in $\mathbb{E}_0$.*

Theorem 3.12 *(First Equivalence Theorem For Minkowski Plane Curves) Let $c : I \to L^2$ be a sufficiently differentiable, nonnull curve with positive curvature κ, with spherical parameter σ and Frenet frame $\{X, Y\}$. We state equivalences in the following three cases:*

Case I: $\lambda = -1$.

In this case the following properties are equivalent:

(i) The radius of curvature κ^{-1} of c satisfies the eigenvalue equation

$$(\kappa^{-1})'' - \kappa^{-1} = 0,$$

(ii) There exists $s_1 \in \mathbb{R}$ such that the evolute

$$E_c(\sigma) := c(\sigma) - \kappa^{-1}(\sigma)Y(\sigma)$$

of c and the involute $\mathcal{I}_{c,s_1}$ of c associated to s_1,

$$\mathcal{I}_{c,s_1}(\sigma) := c(\sigma) - (s(\sigma) + s_1).X(\sigma),$$

satisfy the relation

$$E_c = \mathcal{I}_{c,s_1} + const.,$$

that means the evolute E_c and the involute $\mathcal{I}_{c,s_1}$ differ by a parallel translation.

Case II: $-1, 0 \neq \lambda \in \mathbb{R}$.

In this case the following properties are equivalent:

(i) The radius of curvature κ^{-1} of c satisfies the homogeneous eigenvalue equation

$$(\kappa^{-1})'' + \lambda\kappa^{-1} = 0,$$

(ii) There exists $s_1 \in \mathbb{R}$ such that the evolute E_c of c and the involute $\mathcal{I}_{c,s_1}$ of c associated to $s_1$satisfies the relation

$$E_c = -\lambda\mathcal{I}_{c,s_1} + const.,$$

(iii) There exists $s_1 \in \mathbb{R}$ and $p_0 \in L^2$ such that

$$\lambda(\mathcal{I}_{c,s_1} - p_0) = -(E_c - p_0);$$

in particular that means that E_c and $\mathcal{I}_{c,s_1}$ are homothetic(without translation), p_0 as center; if $\lambda > 0$ the homothety includes a reflection,

(iv) There exists $p_0 \in L^2$ such that the support function $\rho(p_0)$ with respect to p_0 satisfies

$$\rho(p_0) = \varepsilon_1 \frac{1}{1+\lambda}.\kappa^{-1},$$

(v) There exists $p_0 \in L^2$ such that

$$\rho(p_0)'' + \lambda\rho(p_0) = 0.$$

CaseIII: $\lambda = 0$.

In this case the following properties are equivalent:

(i) The radius of curvature κ^{-1} of c satisfies

$$(\kappa^{-1})'' = 0,$$

(ii) There exists $p_0 \in L^2$ such that the support function $\rho(p_0)$ with respect to p_0 satisfies

$$\rho(p_0)'' = 0,$$

(iii) $\rho(p_0) = \varepsilon_1 \kappa^{-1}$.

Proof. If we use the equations (3.1) and (3.4), we have

$$E_c + \lambda \mathcal{I}_{c,s_1} = (1+\lambda)c - (s+s_1)\lambda X - \kappa^{-1} Y \tag{3.9}$$

and using (2.3) and (2.4), we get

$$(E_c + \lambda \mathcal{I}_{c,s_1})' = -[(\kappa^{-1})' + \lambda(s+s_1)]Y \tag{3.10}$$

Now we prove the asserted equivalences in the First Equivalence Theorem:
Case I: $\lambda = -1$.
(i)⇒(ii): If $(\kappa^{-1})'' - \kappa^{-1} = 0$, then we have $((\kappa^{-1})' - s)' = 0$. Thus

$$(\kappa^{-1})' - s = const. =: s_1$$

for an appropriate $s_1 \in \mathbb{R}$; for this s_1 consider $\mathcal{I}_{c,s_1}$; with equation (3.10) we get

$$E_c = \mathcal{I}_{c,s_1} + const.,$$

that means the evolute E_c and the involute $\mathcal{I}_{c,s_1}$ differ by a parallel translation. Similarly, we can prove *(ii)⇒(i)*.
Case II: $-1, 0 \neq \lambda \in \mathbb{R}$.
(i)⇒(ii): If $(\kappa^{-1})'' + \lambda\kappa^{-1} = 0$, then

$$((\kappa^{-1})' + \lambda s)' = 0,$$

thus we have

$$(\kappa^{-1})' + \lambda s = const. =: -\lambda s_1$$

for an appropriate $s_1 \in \mathbb{R}$; for this s_1 consider $\mathcal{I}_{c,s_1}$; with equation (3.10) we get

$$(E_c + \lambda \mathcal{I}_{c,s_1})' = -(-\lambda s_1 + \lambda s_1)Y = 0.$$

Similarly, we can prove *(ii)⇒(i)*.

The implication *(ii)⇒(i)* yields for any λ. The equivalence *(i)⇒(ii)* yields for all $\lambda \neq 0$. If $\lambda = 0$, then *(ii)* implies that $\kappa = const.$, thus c is a Lorentzian circle with center p_0 in the Minkowski plane.

To realize that the implication *(i)⇒(ii)* is not true for $\lambda = 0$ consider a nonnull curve c that is an involute of a circle in the Minkowski plane.

(ii)⇒(iii): Let $E_c + \lambda \mathcal{I}_{c,s_1} = const =: (1+\lambda)p_0$ for an appropriate p_0. Then we have

$$\lambda(\mathcal{I}_{c,s_1} - p_0) = -E_c + p_0.$$

(iii)⇒(iv): If we use (2.7), we get

$$\{-\varepsilon_1(1+\lambda)\rho'(p_0) - \lambda(s+s_1)\}X + \{\varepsilon_1(1+\lambda)\rho(p_0) - \kappa^{-1}\}Y = 0.$$

The coefficients of the frame $\{X, Y\}$ must vanish, this means that

$$\rho(p_0) = \varepsilon_1 \frac{1}{1+\lambda}\kappa^{-1}.$$

To realize that *(iii)* can not be satisfied for $\lambda = -1$. Because the implication *(iii)⇒(iv)* gives

$$\varepsilon_1(1+\lambda)\rho(p_0) = \kappa^{-1},$$

thus $\lambda \neq -1$.

(iv)⇒(v): If we consider (2.8), we get

$$\begin{aligned} \rho(p_0) &= \varepsilon_1 \frac{1}{1+\lambda}\kappa^{-1} \Leftrightarrow \varepsilon_1(1+\lambda)\rho(p_0) \\ &= \kappa^{-1} = -\varepsilon_1\rho(p_0)'' - \varepsilon_2\rho(p_0) \\ \Leftrightarrow& \ \rho(p_0)'' + \lambda\rho(p_0) = 0. \end{aligned}$$

(v)⇒(i): The assumption and the equation (2.8) give

$$\kappa^{-1} = \varepsilon_1(1+\lambda)\rho(p_0),$$

this means that κ^{-1} satisfies the homogeneous eigenvalue equation.

Case III: $\lambda = 0$.

In this case the proof is similar to proofs of the other cases.

Corollary 3.13 $-1 \neq \lambda \in \mathbb{R}$ *is an eigenvalue for* κ^{-1} *in the equation (3.5) if and only if it is an eigenvalue for* $\rho(p_0)$ *with* p_0 *appropriate.*

Lemma 3.14 *Let* $c : I \to L^2$ *be a nonnull curve with positive curvature* κ *and spherical parameter* σ. *Let* $f : I \to \mathbb{R}$ *be sufficiently differentiable and* $-1 \neq \lambda \in \mathbb{R}$. *If the support function* $\rho(p_0)$ *for* $p_0 \in L^2$ *satisfies the inhomogeneous eigenvalue equation*

$$\rho(p_0)'' + \lambda\rho(p_0) = f\,; \tag{3.11}$$

then the radius of curvature κ^{-1} satisfies the inhomogeneous eigenvalue equation

$$(\kappa^{-1})'' + \lambda\kappa^{-1} = \varepsilon_1(f - f'') \tag{3.12}$$

Proof. From (2.8), we have

$$\rho(p_0)'' = -\varepsilon_1\kappa^{-1} + \rho(p_0). \tag{3.13}$$

Using the (3.13), we get

$$\rho(p_0) = \frac{1}{1+\lambda}(f + \varepsilon_1\kappa^{-1}) \tag{3.14}$$

Differentiation of the (3.14) gives

$$\rho(p_0)'' = \frac{1}{1+\lambda}(f + \varepsilon_1\kappa^{-1})''. \tag{3.15}$$

If we consider (3.14) and (3.15) into (3.11), we have

$$(\kappa^{-1})'' + \lambda\kappa^{-1} = \varepsilon_1(f - f'').$$

Theorem 3.15 *(Second Equivalence Theorem For Minkowski Plane Curves) Let $c : I \to L^2$ be a nonnull curve with positive curvature κ and spherical parameter σ. Let $f : I \to \mathbb{R}$ be sufficiently differentiable and $-1 \neq \lambda \in \mathbb{R}$. Assume that, for a fixed point $p_0 \in L^2$ and a C^2-function f , the support function $\rho(p_0)$ satisfies the following inhomogeneous eigenvalue equation*

$$\rho(p_0)'' + \lambda\rho(p_0) = f\,,$$

then we have the equivalence of *(i)* and *(ii)*:

(i) The function f is linear, $f(\sigma) = \gamma_1\sigma + \gamma_2$; $\gamma_1, \gamma_2 \in \mathbb{R}$;

(ii) the radius of curvature κ^{-1} satisfies the inhomogeneous eigenvalue equation

$$(\kappa^{-1})'' + \lambda\kappa^{-1} = \varepsilon_1 f.$$

Proof. The proof is a direct consequence of the Lemma 3.14.

Corollary 3.16 *Let c be a spacelike curve in Minkowski plane. In the Theorem 3.15, if f is a polynomial of order $m \in \mathbb{N}$, then all iterated evolutes $E^{m+k}(c)$ for $k \in \mathbb{N}$ are in the class $\mathbb{E}_0$;in particular, the support function $\rho_{E^{m+k}}(p_0)$ and the radius of curvature $\kappa^{-1}_{E^{m+k}}$ satisfy the same eigenvalue equation.*

Corollary 3.17 *Let c be a nonnull curve in Minkowski plane. and f be a linear function. If c is a spacelike curve , then its radius of curvature κ^{-1} and support function $\rho(p_0)$ are in the class $\mathbb{E}_f$.*

Corollary 3.18 *Let c be a nonnull curve in Minkowski plane. and f be a linear function. If c is a timelike curve , then its radius of curvature κ^{-1} and support function $\rho(p_0)$ are not in the class $\mathbb{E}_f$.*

Corollary 3.19 *Let c be a spacelike curve in Minkowski plane. If the radius of curvature κ satisfies*

$$(\kappa^{-1})'' + \lambda\kappa^{-1} = \gamma_1 = const.,$$

then the curvature function $\kappa^{-1}_{\mathcal{I}_{c,s_1}}$ and the support function $\rho_{\mathcal{I}_{c,s_1}}$ of an involute both satisfy the same inhomogeneous eigenvalue equation:

$$(\kappa^{-1}_{\mathcal{I}_{c,s_1}})'' + \lambda\kappa^{-1}_{\mathcal{I}_{c,s_1}} = \pm(\gamma_1\sigma + \gamma_2)$$

and

$$(\rho_{\mathcal{I}_{c,s_1}})'' + \lambda\rho_{\mathcal{I}_{c,s_1}} = \pm(\gamma_1\sigma + \gamma_2).$$

Proof. The proof follows from the the Theorem 3.10 and Theorem 3.15.

4 Solutions of Eigenvalue Equations for the Support Function

In this section we determine the Minkowski plane curves satisfying the homogeneous equation

$$\rho(p_0)'' + \lambda\rho(p_0) = 0, \tag{4.1}$$

where $p_0 \in L^2$.

CaseI: $\lambda \neq -1$. and $-\lambda = \omega^2 > 0$.

In this case; we can write the general solution of the homogeneous equation (4.1) as follow:

$$\rho(p_0)(\sigma) = A\cosh\omega\sigma + B\sinh\omega\sigma,$$

where $A, B \in \mathbb{R}$.

By an apropriate rotation of c, we may asssume that

$$X(\sigma) = (\cosh\omega\sigma, \sinh\omega\sigma), \quad Y(\sigma) = (-\sinh\omega\sigma, \cosh\omega\sigma).$$

Thus; from the equation (2.7)

$$c(\sigma) - p_0 = -\rho(p_0)'X(\sigma) + \rho(p_0)Y(\sigma)$$

$$c(\sigma)-p_0 = -(A\omega\sinh\omega\sigma+B\omega\cosh\omega\sigma)\begin{pmatrix}\cosh\omega\sigma\\ \sinh\omega\sigma\end{pmatrix}+(A\cosh\omega\sigma+B\sinh\omega\sigma)\begin{pmatrix}-\sinh\omega\sigma\\ \cosh\omega\sigma\end{pmatrix}.$$

Case II: $\lambda \neq -1$. and $-\lambda = \omega^2 < 0$.

In this case; we can write the general solution of the homogeneous equation (4.1) as follows:

$$\rho(p_0)(\sigma) = A\cos\omega\sigma + B\sin\omega\sigma,$$

where $A, B \in \mathbb{R}$.

By an appropriate rotation of c, we may asssume that

$$X(\sigma) = (1,0), \quad Y(\sigma) = (0,-1).$$

Thus; from the equation (2.7)

$$c(\sigma) - p_0 = (A\omega\sin\omega\sigma - B\omega\cos\omega\sigma)\begin{pmatrix}1\\0\end{pmatrix} + (A\cos\omega\sigma + B\sin\omega\sigma)\begin{pmatrix}0\\-1\end{pmatrix}.$$

Case III: $\lambda = 0$.

In this case, a curve satisfying the homogeneous equation (4.1) is part of a Lorentzian circle or of one of its involutes.

Case IV: To determine the curves satisfying the inhomogeneous equation

$$\rho(p_0)'' + \lambda\rho(p_0) = f, \tag{4.2}$$

for a linear function f.

In this case , from the linearity of (4.2) the general solution ρ of the inhomogeneous equation is the superposition of the general homogeneous solution and the following particular inhomogeneous solution

$$\rho_{inh} = \lambda^{-1}.f(\sigma) = \lambda^{-1}.(\gamma_1\sigma + \gamma_2), \quad \lambda \neq 0, \tag{4.3}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$. The general homogeneous solutions are given in *Case I, II, III.* The solution (4.3) generates a circle or involute of a circle.

5 Open Problem

In this paper, we investigate some results for curves which are the solution of the eigenvalue equations in the Minkowski plane. We hope that this study will contribute to the study of kinematics of engines, the construction of toothwheel gears and physics applications. Additionally, problems such as; investigation of eigenvalue equations for curves in Minkowski 3-space or higher dimensional Euclidean or Lorentzian spaces can be presented as further researches.

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